

CHAPTER ONE

INTRODUCTION

1.1 Introduction

Lifetime distributions are essential statistical tools utilized in a variety of fields, including reliability engineering, survival analysis, actuarial science, and risk management. These distributions mathematically characterize the time until a specified event of interest occurs, this could include the failure of machinery, the mortality of living organisms, or the intervals between occurrences of events, such as natural disasters or mechanical breakdowns. A thorough understanding of the properties and applications of these distributions can yield invaluable insights for effectively modeling and analyzing phenomena encountered in the real world.

1.2 Background of the Study

This study provides an in-depth examination of three prominent lifetime distributions: the Exponential, Weibull, and Gamma distributions. Each of these distributions is widely recognized for their adaptability, simplicity, and broad applicability across various domains. The Exponential distribution, often associated with memoryless processes, is commonly used to model the time until an event occurs when the event rates are constant over time. The Weibull distribution offers increased flexibility, allowing for the modeling of increasing or decreasing failure rates, and making it particularly useful in reliability analysis. The Gamma distribution provides a versatile framework for modeling skewed data, extending the capability of the other two distributions through its shape parameters.

In addition to these three key distributions, the Birnbaum-Saunders distribution warrants discussion. Originally developed to model the lifetimes of materials subjected to cyclic loading, it caters to scenarios where materials experience repeated strain and stress, thereby presenting real-life implications in fields like materials science and engineering. Moreover, the Extreme Value Distribution is pivotal for modeling lifetimes of events characterized by extreme conditions, such as the occurrence of earthquakes or hurricanes, where only rare but significant occurrences are of interest.

Another important distribution in the context of product lifetimes is the Lognormal distribution. It is frequently used to model the duration of life for products ranging from aircraft to household appliances, as it adequately accommodates the multiplicative effects of small, random variables typically influencing product degradation over time.

Lifetime distribution models serve as comprehensive descriptions of component or unit lifetimes. These models aim to encompass all possible lifetimes for units produced under a specific design, material composition, and manufacturing methodology. Consequently, lifetime distributions become integral to reliability analysis, which focuses on determining failure times and analyzing time-to-event data. In this context, an "event" can refer to various measurable occurrences, including death, disease, mechanical failure, or any significant threshold breach.

Furthermore, numerous lifetime distributions are intricately linked to the study of extreme values. For example, in a system arranged in series configuration, the

overall system ceases to operate upon the failure of its first component. Conversely, in a parallel connection, it continues to function until the last component fails. This relationship highlights the critical nature of understanding different lifetime distributions, as they influence both design decisions and risk assessments in engineering and beyond.

1.2 Objectives of the Study

The objectives of this study include:

1. To explore the fundamental properties of the Exponential, Weibull, and Gamma distributions.
2. To analyze their applications in real-world scenarios such as reliability, survival analysis, and risk assessment.
3. To compare their characteristics to identify suitable use cases for each distribution.

1.3 Scope of the Study

The study focuses on the mathematical properties and real-world applications of the three distributions. Key aspects such as probability density function, cumulative distribution function, moments, and hazard rate are examined for each distribution. Practical applications are drawn from reliability engineering, healthcare, and financial modeling literature.

1.4 Significance of the Study

Understanding lifetime distributions aids in making informed decisions in critical areas such as maintenance scheduling, healthcare planning, and risk management. By exploring these distributions, this study aims to enhance the practical application of statistical tools in solving real-world problems.

1.5 Organization of the Study

This study is organized as follows:

Chapter 1 introduces the study, highlighting its background, objectives, scope, significance, and organization.

- Chapter 2 reviews the properties and applications of the Exponential, Weibull, and Gamma distributions in detail.
- Chapter 3 discusses the comparative analysis of the distributions.
- Chapter 4 presents case studies to illustrate practical applications.
- Chapter 5 concludes the study and provides recommendations for future research.

1.6 DEFINITION OF BASIC IMPORTANT TERMS

1. LIFETIME DATA AND DISTRIBUTION

Life data can be lifetimes of products such as the time the product operated successfully or the time the product operated before it failed. Lifetime distributions

are important statistical tools used to model the different characteristics of lifetime data set.

2. PROBABILITY DENSITY FUNCTION (PDF)

The probability density function of a continuous random variable is the value the function $f(x)$ gives us at the probability density at point x . It is also referred to a function that will give us the probability that a continuous random variable, say X , will take a value exactly equal to x .

3. CUMULATIVE DENSITY FUNCTION (CDF)

The cumulative density function of a continuous random variable say X , which is evaluated at x , is the probability that X takes a value less than or equal to x .

4. SURVIVAL (RELIABILITY) FUNCTION

The survival function gives the probability that an individual, device or any other object of interest will survive longer than any certain or specified time, t .

5. HAZARD FUNCTION

The hazard function is the probability of an event occurring during any given time point. It is the probability that a person, machine or business dies in the next instant, given that it survived to time t .

6. GAMMA DISTRIBUTION

A two-parameter family of continuous distributions often used to model waiting times or processes involving multiple stages.

7. MOMENTS

Quantities that describe various characteristics of a distribution, such as mean (first moment) and variance (second moment).

8. RELIABILITY ENGINEERING

A field focused on the study of the performance and failure of systems over time.

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

Probability distributions are fundamental tools in survival analysis and reliability engineering, as they provide insights into the nature of various parameters and functions, particularly the failure rate (or hazard) function (Ross, 2014). Over the past decades, significant research has been devoted to developing lifetime models that go beyond the classical increasing and decreasing hazard rates. This trend has been motivated by the need to describe complex real-world scenarios with greater accuracy and flexibility (Chahkandi & Ganjali, 2009).

This chapter provides a comprehensive review of the theoretical foundations of three key lifetime distributions: the Exponential, Weibull, and Gamma distributions. These distributions are central to the study of reliability and survival analysis, as they model the time until the occurrence of specific events, such as system failures or survival times.

The discussion begins with an overview of each distribution's fundamental characteristics, including their probability density functions (PDFs), cumulative distribution functions (CDFs), and moments.

By examining the existing literature, this chapter aims to elucidate the distinctive features and versatile uses of these lifetime distributions. It also seeks to establish a strong foundation for the comparative analysis and case studies presented in subsequent chapters, demonstrating their relevance to solving real-world problems.

2.2 SOME LIFETIME DISTRIBUTIONS

The choice of a lifetime distribution is critical because these distributions exhibit different properties, particularly in their left and right tails. The simplest model for lifetimes is the Exponential distribution, which assumes a constant hazard rate. The Weibull distribution, with its two parameters, offers greater flexibility and can model increasing, decreasing, or constant hazard rates. The Gamma distribution is another versatile model, particularly useful for skewed data and processes with multiple stages. This project focuses on these three lifetime distributions due to their widespread applicability and theoretical importance.

2.2.1 Exponential Distribution and Related work in the Literature

The Exponential distribution is one of the simplest and most used lifetime distributions. It is characterized by a constant hazard rate, which makes it suitable for modeling memoryless processes. It is a special case of the Weibull distribution when $k=1$, has been extensively studied for its simplicity and unique properties. Cox (1962) emphasized its memoryless property, which states that the probability of an event occurring in the next time interval is independent of how much time has already elapsed. This property makes the Exponential distribution particularly useful in modeling the lifetimes of electronic components and radioactive decay, where failure rates are constant over time.

In queueing theory, Gross and Harris (1998) applied the Exponential distribution to model inter-arrival times and service times in systems such as call centers and telecommunications networks. Similarly, Lawless (2003) highlighted its use in

reliability engineering for systems with constant failure rates, such as certain types of electrical equipment.

However, researchers like Lee and Wang (2003) have cautioned that the Exponential distribution's simplicity can be a limitation. For example, it cannot model data with increasing or decreasing failure rates, which are common in many real-world scenarios. Meeker and Escobar (1998) suggested that the Exponential distribution should only be used when there is strong evidence of a constant hazard rate, as its assumptions may not hold in more complex situations.

The probability density function (PDF) of the Exponential distribution is defined as follows

$$f(x ;\lambda)= \lambda e^{-\lambda x}, \quad x \geq 0, \lambda > 0 \quad (2.1)$$

where λ is the rate parameter (the inverse of the mean).

The CDF of an Exponential Distribution is given as:

$$F(x ;\lambda)= 1 - e^{-\lambda x}, \quad x > 0 \quad (2.2)$$

Just as the normal distribution plays a pivotal role in classical statistics because of the central limit theorem, the exponential distribution plays a pivotal role in reliability and lifetime modeling because it is the only continuous distribution with a constant hazard function. The exponential distribution has often been used to model the lifetimes of electronic components and is appropriate when a used component that has not failed is statistically as good as a new component in terms of its remaining time to failure. This is a rather restrictive assumption. Moreover, the exponential distribution is presented first because of its simplicity (William &

Mary). The exponential distribution has a single positive scale parameter λ , often called the failure rate, measured in failures per unit time.

The exponential distribution should be judiciously applied because the memoryless property restricts its applicability. It can easily be misapplied for the sake of simplicity because the statistical techniques for the exponential distribution are particularly tractable, or because small sample sizes do not support more than a one-parameter distribution.

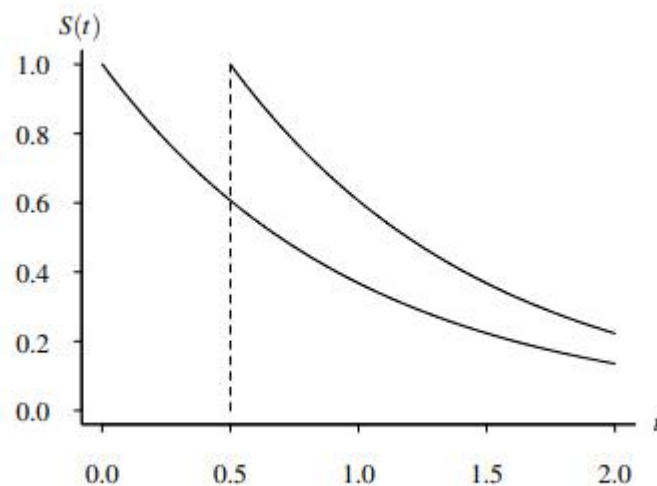


Figure 4.4: The memoryless property of the exponential distribution.

2.2.2 Weibull Distribution and Related Work in the Literature

Weibull (1951) asserted in his work that the Weibull distribution could rightly be applied to a wide range of problems. This he showed with several examples ranging from the strength of steel to the strength of adult males in the then British

Isles. It was shown that Weibull distribution is appropriate in both of these statements on the basis of the findings of recent and past researchers in the field.

Jiang et al. (1999) is the first to study dealing with the inverse Weibull mixture model. They deal with multi components system where the individual components have inverse Weibull shape parameter and showed that the system failure distribution is given by an inverse Weibull mixture allowing negative weights.

Tang (2004) studied the extended Weibull distribution in reliability analysis employed the comparison of estimation of graphical method and maximum likelihood estimation for the Weibull extension distribution. Lee et al. (2007) proposed the Beta-Weibull distribution which can be applied to censored data sets on bus-motor failures and a censored data set on head-and-neck-cancer clinical trial. The cumulative distribution function for beta-Weibull random variable is defined as

$$G(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)} \int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1} dt, \quad 0 < \alpha, \beta < \infty \quad (2.3)$$

The corresponding probability density function for the beta-Weibull distribution is given by

$$g(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)} \frac{c}{\gamma} \left(\frac{x}{\gamma}\right)^{c-1} \left[1 - e^{-(x/\gamma)^c}\right]^{\alpha-1} e^{-\beta(x/\gamma)^c} \quad (2.4)$$

A simulation was conducted to compare the beta-Weibull distribution with the exponentiated Weibull distribution. It was observed that the biases of the mle from the beta- Weibull distribution are smaller than the biases of the mle from exponentiated Weibull model.

Basu et al. (2009) conducted a study to determine if Weibull distribution was the most appropriate statistical strength distribution for brittle materials. It was discovered that many structural applications are usually characterized on Weibull strength distribution function. This is validated using the strength data, measured with a number of structural ceramic and glass materials.

Osemwenkhae and Osagie (2010) studied the mathematical modelling of the gradual aging of systems using the Weibull hazard function and used a mathematical expression for obtaining the shape and scale parameters. The implication of their work is to obtain the aging coefficient of any system which is gradually aging. Using the two parameter Weibull hazard function, they concluded that the nature of aging property depends on the aging coefficient of the system which makes the system to gradually age with time.

The Weibull distribution is a versatile lifetime distribution that generalizes the Exponential distribution. Its probability density function (PDF) is:

$$f(x: \lambda, k) = \left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} \ell^{-\left(\frac{x}{\lambda}\right)^k} \quad (2.5)$$

where λ is the scale parameter and k is the shape parameter.

Where the CDF is given as

$$F(x: \lambda, k) = 1 - \ell^{-\left(\frac{x}{\lambda}\right)^k}, \quad x > 0 \quad (2.6)$$

2.2.3 Gamma Distribution and Related Work in the Literature

Researchers have praised the Gamma distribution for its ability to model skewed data and its flexibility in representing various shapes of hazard functions. According to Stuart and Ord (1994), the Gamma distribution is defined by a shape parameter (α) and a rate parameter (β), which allow it to take on different forms depending on the values of these parameters. When $\alpha=1$, the Gamma distribution reduces to the Exponential distribution, but for $\alpha>1$, it can model increasing hazard rates, and for $\alpha<1$, it can model decreasing hazard rates.

Gupta and Kundu (1999) compared the Gamma distribution to the Weibull and Exponential distributions and concluded that it is particularly useful for modeling data with high skewness, such as insurance claims and rainfall amounts. In medical research, Klein and Moeschberger (2003) applied the Gamma distribution to model patient survival times and hospital stay durations, noting its ability to capture the long tails often observed in such data.

In Bayesian statistics, Gelman et al. (2013) highlighted the Gamma distribution's role as a conjugate prior for various likelihood functions, such as the Poisson and Exponential distributions. This property makes it a popular choice in Bayesian inference for modeling parameters that are strictly positive, such as rates and scales.

The Gamma distribution has been extensively studied and applied in diverse fields. According to Johnson, Kotz, and Balakrishnan (1994), it is a cornerstone in the study of waiting times and aggregate phenomena. The Gamma function, central to this distribution, offers analytical tractability, making it ideal for mathematical and applied statistics. In insurance and risk management, Klugman et al. (2012) highlight its utility in modeling loss distributions. Meanwhile, in hydrology, the Gamma distribution has proven instrumental in predicting extreme events such as droughts and floods (Hogg et al., 2019).

The Gamma distribution is a two-parameter family of continuous distributions that generalizes the Exponential distribution.

The probability density function (PDF) is:

$$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad x \geq 0, \alpha, \beta > 0 \quad (2.7)$$

Where:

x The random variable.

α : Shape parameter (sometimes denoted as k).

β : Rate parameter (inverse of the scale parameter)

$\Gamma(\alpha)$: Gamma function,

The CDF is defined as

$$F(x; \alpha, \beta) = \int_0^x f(t; \alpha, \beta) dt = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)} \quad (2.8)$$

Where $\gamma(\alpha, \beta x)$ is the incomplete gamma function.

2.3 Comparative Analysis of the Distributions

Several researchers have conducted comparative studies to evaluate the performance of the Weibull, Exponential, and Gamma distributions in different applications. Lawless (2003) argued that the Weibull distribution is more flexible than the Exponential distribution due to its additional shape parameter, making it better suited for modeling complex failure data. However, Gupta and Kundu

(1999) found that the Gamma distribution often outperforms the Weibull distribution in scenarios where the data exhibits a high degree of skewness.

Meeker and Escobar (1998) emphasized that the choice of distribution depends on the specific characteristics of the data and the context of the application. For example, in reliability engineering, the Weibull distribution is often preferred for its ability to model varying hazard rates, while the Exponential distribution is used when the hazard rate is constant. The Gamma distribution, on the other hand, is favored for modeling highly skewed data and in Bayesian inference.

Hence, the implication is that each distribution is suited for specific scenarios based on its properties as. The Exponential distribution is ideal for memoryless processes.

The Weibull distribution's flexibility makes it useful for a broad range of reliability and survival scenarios. The Gamma distribution's versatility is advantageous in modeling processes with multiple stages or failure modes.

2.4 Representation of lifetimes

Researchers have demonstrated the practical applications of these distributions across various fields. In reliability engineering, Nelson (1982) used the Weibull distribution to model the lifetimes of mechanical components, while Cox (1962) applied the Exponential distribution to analyze failure data in electronic systems. In environmental science, Seguro and Lambert (2000) utilized the Gamma distribution to model precipitation data, noting its ability to capture the variability in rainfall amounts.

In healthcare, Klein and Moeschberger (2003) applied both the Weibull and Gamma distributions to analyze patient survival times, highlighting the importance of selecting the appropriate distribution based on the underlying hazard function. In finance, Klugman et al. (2012) used the Gamma distribution to model insurance claims, taking advantage of its ability to handle skewed data.

These representations are mathematically equivalent: which one you use depends on whether the representation has a [tractable](#) (i.e., easily solvable) form or if a plot of the representation gives you valuable information compared to an equation Leemis, L (1986). There are five main ways to represent the distribution of a continuous nonnegative random variable T in reliability. They have been widely used since the 1950s Leemis, L.:

- [Probability density function \(PDF\)](#): gives the probability that an item's lifetime will be a certain value.
- [Survival function](#): gives the probability an item will survive beyond a certain time.
- [Hazard rate](#): gives the instantaneous rate of failure of an item: can be interpreted as the probability an item that has survived up to a certain point in time will fail in the next instant.
- Cumulative hazard function: gives the total amount of failure that has occurred up to a certain point in time.
- Mean residual life function: gives the expected remaining lifetime of an item at a certain time.
- The [cumulative distribution function \(CDF\)](#) is another valuable tool for assessing failure probabilities because it gives us the probability a randomly selected unit will fail by time t . Other less common ways to represent distributions in survival analysis are [moment generating functions](#)

[\(MGF\)](#), [characteristic functions](#), Mellin transformation, density quantile function and the total time on test transform

Summary

The Weibull, Exponential, and Gamma distributions have been widely studied and applied in various fields due to their unique properties and flexibility. Researchers agree that the Weibull distribution is particularly useful for modeling failure times with varying hazard rates, while the Exponential distribution is ideal for scenarios requiring a memoryless property. The Gamma distribution, on the other hand, is preferred for modeling highly skewed data and in Bayesian inference. The choice of distribution ultimately depends on the specific characteristics of the data and the context of the application, as highlighted by numerous studies in the literature.

CHAPTER THREE

METHODOLOGY

3.1 INTRODUCTION

This chapter describes the methodology employed to investigate the properties and applications of the Exponential, Weibull, and Gamma distributions. The study aims to compare these distributions in terms of their ability to model lifetime data, with a focus on their flexibility, accuracy, and suitability for different real-world scenarios. The methodology includes data collection, parameter estimation, model fitting, and validation procedures.

3.2 RESEARCH DESIGN

The research design is a comparative analytical study that evaluates the performance of the Exponential, Weibull, and Gamma distributions in modeling lifetime data. It adopts a quantitative approach, using both simulated and real-world datasets to assess the flexibility, accuracy, and applicability of these distributions.

This study focuses on statistical modeling and estimation techniques to analyze lifetime distributions. It involves applying probability distributions to real-world lifetime data to identify the most suitable model for various scenarios, including survival analysis, reliability testing, and risk assessment.

3.3 DATA COLLECTION

The study relies entirely on real-world data obtained from published research studies and publicly available sources. The selected datasets include:

3.3.1 Real-World Data

Real-world datasets are obtained from reputable research studies and academic sources (Rama and Shukla, 2016). These datasets include:

1. Survival Times of Head and Neck Cancer Patients

- Reported by Efron (1988) in the *Journal of the American Statistical Association*.
- Consists of survival times of patients treated with radiotherapy.
- Used to model and compare lifetime distributions in a medical context.

2. Waiting Times of Bank Customers

- Provided by Ghitany et al. (2008) in *Mathematics and Computers in Simulation*.
- Contains waiting times (in minutes) before service for 100 bank customers.

- Used to analyze lifetime distributions in queueing theory and service systems.

Each dataset is preprocessed to handle missing values, remove inconsistencies, and standardize time units for accurate analysis. These datasets provide a practical foundation for evaluating the applicability of different lifetime distributions in survival analysis and reliability modeling.

3.4 Statistical Methods for Lifetime Distribution Analysis

The following statistical techniques are applied in the study:

3.4.1 Descriptive Statistics

Descriptive analysis is used to summarize the lifetime data, including measures such as:

- **Mean and Median Lifetime** – Central tendency of survival/failure times.
- **Variance and Standard Deviation** – Measures of dispersion.
- **Skewness and Kurtosis** – Understanding the shape of the distribution.

Survival Function S(t)

The Survival Function represents the probability that a system (engine) survives beyond time t .

$$S(t) = P(T > t) = 1 - F(t) \quad (3.1)$$

where $F(t)$ is the cumulative distribution function (CDF) of failure times.

Hazard Function $h(t)$

The Hazard Function gives the instantaneous failure rate at time t , given survival up to time t .

$$h(t) = \frac{S(t)}{f(t)} \quad (3.2)$$

where $f(t)$ is the probability density function (PDF) of failure times.

3.5 Probability Distribution Models

To model lifetime data, different probability distributions are tested:

1. Exponential Distribution:

The PDF of an Exponential Distribution is given as

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0, \lambda > 0 \quad (3.3)$$

The CDF of an Exponential Distribution is given as:

$$f(x; \lambda) = 1 - e^{-\lambda x}, \quad x > 0 \quad (3.4)$$

$$\text{Mean: } \frac{1}{\lambda} \quad (3.5)$$

$$\text{Variance: } \frac{1}{\lambda^2} \quad (3.6)$$

The Survival Function of an Exponential Distribution is given as

$$S(t) = e^{-\lambda t} \quad (3.7)$$

The Hazard Function of an Exponential Distribution is given as

$$h(t) = \lambda \quad (3.8)$$

where λ is the rate parameter (the inverse of the mean). Assumes a constant failure rate (memoryless property).

2. Weibull Distribution:

The PDF of a Weibull Distribution is given as

$$f(x: \lambda, k) = \left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} \ell^{-\left(\frac{x}{\lambda}\right)^k} \quad (3.9)$$

where λ is the scale parameter and k is the shape parameter.

Where the CDF is given as

$$f(x: \lambda, k) = 1 - \ell^{-\left(\frac{x}{\lambda}\right)^k}, \quad x > 0 \quad (3.10)$$

Mean: $\lambda\Gamma(1+1/k)$, where Γ is the gamma function.

Variance: Depends on both shape and scale parameters

The Survival Function of Weibull Distribution is given as

$$S(t) = \ell^{-\left(\frac{x}{\lambda}\right)^k} \quad 3.11$$

The Hazard Function of an Weibull Distribution is given as

$$h(t) = \left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} \quad 3.12$$

Flexible shape parameter (β) allows for increasing, constant, or decreasing failure rates.

3. Gamma Distribution:

The PDF of a Gamma Distribution is given as

$$f(x;\alpha,\beta) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad x \geq 0, \alpha, \beta > 0 \quad (3.13)$$

The CDF is defined as

$$F(x;\alpha,\beta) = \int_0^x f(t; \alpha, \beta) dt = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)} \quad (3.14)$$

$$\text{Mean: } \frac{\alpha}{\beta}$$

(3.15)

$$\text{Variance: } \frac{\alpha}{\beta^2} \quad (3.16)$$

The Survival Function of an Gamma Distribution is given as

$$S(t) = \frac{\gamma(\alpha, \beta t)}{\Gamma(\alpha)} \quad (3.17)$$

The Hazard Function of an Gamma Distribution is given as

$$h(t) = \frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t} / \Gamma(\alpha)}{\Gamma(\alpha, \lambda t) / \Gamma(\alpha)} \quad (3.18)$$

Can model different failure rate patterns.

3.6 Model Properties and Application

3.6.1 Exponential Distribution:

Properties

1. Memoryless Property: The Exponential distribution is the only continuous distribution with the memoryless property, implying that the probability of an event occurring in the future is independent of the past.
2. Mean and Variance: The mean is $\frac{1}{\lambda}$, and the variance is $1/\lambda^2$.
3. Hazard Rate: The hazard rate is constant and equal to λ .

Applications

- In reliability engineering, the Exponential distribution is used to model systems with constant failure rates (Nelson, 1982).
- In queueing theory, it is applied to model inter-arrival times in Poisson processes (Gross & Harris, 1998).
- In healthcare, it is used to model survival times for diseases or treatments with constant hazard rates (Klein & Moeschberger, 2003).

- In environmental studies, it is applied to model the time until natural disasters (Ross, 2014).

Limitations

The assumption of a constant hazard rate limits its applicability in situations where failure rates change over time.

3.6.2 Weibull Distribution

Properties

1. Hazard Rate: The hazard rate can increase, decrease, or remain constant, depending on the value of k :

$k < 1$: Decreasing hazard rate.

$k = 1$: Constant hazard rate (reduces to the Exponential distribution).

$k > 1$: Increasing hazard rate.

2. Flexibility: The Weibull distribution can model a wide range of behaviors by adjusting k and λ .

3. Mean and Variance : The mean and variance depend on the shape and scale parameters and can be computed using the gamma function:

$$\mu = \lambda \Gamma\left(1 + \frac{1}{k}\right) \quad (3.19)$$

$$\sigma^2 = \lambda^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2 \right] \quad (3.20)$$

Applications

- In reliability engineering, the Weibull distribution is used to model systems with increasing or decreasing failure rates (Murthy et al., 2004).
- In survival analysis, it is applied to model patient survival times with aging effects (Lee & Wang, 2003).
- In climatology, it is used to model extreme weather events, such as wind speeds (Seguro & Lambert, 2000).
- In industrial applications, it is used to analyze material fatigue and failure rates (Meeker & Escobar, 1998).

Limitations

Requires estimation of both shape and scale parameters, which may be challenging in small sample sizes.

3.6.3 Gamma Distribution

Properties

1. Shape and Flexibility: The shape parameter α and rate parameter β allow the Gamma distribution to model a wide variety of data.
2. Mean and Variance: The mean is α/β , and the variance is α/β^2 .
3. Special Cases: When $\alpha = 1$, the Gamma distribution reduces to the Exponential distribution.

Applications

- In reliability engineering, the Gamma distribution is suitable for systems with multiple failure modes (Meeker & Escobar, 1998).
- In insurance and finance, it is used to model claim sizes and aggregate losses (Klugman et al., 2012).
- In biology, the Gamma distribution models waiting times for biological processes, such as enzyme kinetics (Bailey, 1990).
- In meteorology, it is applied in rainfall modeling and other environmental studies (Wilks, 2011).

Limitations

Parameter estimation can be computationally intensive for complex datasets.

3.7 MODEL FITTING AND VALIDATION

A goodness of fit test is a statistical method used to determine whether a random sample comes from a specified distribution. The goodness of fit of the Exponential, Weibull, and Gamma distributions can be assessed using the following statistical measures.

3.5.1 KOLMOGOROV-SMIRNOFF TEST (KS TEST)

The KS test is a non-parametric statistical test used to evaluate the similarity between two probability distributions. It can determine whether two samples come from the same continuous distribution or whether a sample differs significantly from a known distribution. The test calculates the maximum

difference between cumulative distribution functions (CDFs) and rejects the null hypothesis if the difference is greater than a critical value. The KS test is useful in fields like finance, biology, and engineering.

With the formula

$$D_n = \max_x |F_n - F(x)| \quad (3.21)$$

Where \max_x is the maximum distance between the distributions.

3.5.2 ANDERSON DARLING (A*) TEST

The Anderson-Darling test is a statistical test used to assess whether a given sample of data follows a particular distribution. It's particularly useful when you want to test for goodness-of-fit against a specified distribution.

This test employs a specific distribution to calculate critical values.

$$A^* = -N - S$$

$$\text{Where } S = \sum_{i=1}^n \frac{2i-1}{n} [\ln F(Y_i) + \ln (1 - F(Y_{N+1-i}))] \quad (3.22)$$

3.5.3 CRAMÉR-VON MISES (W*) TEST.

The Cramér-von Mises test is another statistical test commonly used for goodness-of-fit testing. It's similar to the Anderson-Darling test but places less weight on the

tails of the distribution. The test statistic denoted as W^* , or cumulative distribution function F^* compared to the empirical distribution F_n . It is defined as

$$W^* = \int_{-\infty}^{\infty} [F_n(x) - F^*(x)]^2 df(x) \quad (3.23)$$

Where F_n is the empirically observed distribution. When comparing distributions, the one with the smallest value of the test statistic W^* is the best.

3.6 INFORMATION CRITERIA

Information Criteria are statistical tools used for model selection. They provide a means to compare different statistical models based on their goodness of fit to the observed data while accounting for model complexity. The main goal is to identify the model that best explains the data without overfitting.

3.6.1 AKAIKE INFORMATION CRITERION (AIC)

AIC is an estimator of prediction error and thereby a relative quality of statistical distributions for a given set of data.

Given a collection of distributions for the data, AIC provides a means for distribution selection. It is given by:

$$AIC = 2k - 2\ln(L) \quad (3.24)$$

Where k is the number of estimated parameters in the distribution and

L maximum value of the likelihood function for the distribution.

3.6.2 BAYESIAN INFORMATION CRITERION (BIC)

BIC is a criterion for distribution selection among a finite set of distributions; distributions with lower BIC are generally preferred.

$$BIC = k \ln(n) - 2 \ln(L) \quad (3.25)$$

Where n is the number of observations

3.6.3 HANNAN-QUINN INFORMATION CRITERION (HQIC)

HQIC is an alternative to AIC and BIC. It is given as:

$$HQIC = -2L(\ln(n))_{max} \quad (3.26)$$

Where L_{max} is the value of the maximum likelihood estimate

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3.7 MAXIMUM LIKELIHOOD ESTIMATE

We apply the Maximum Likelihood Estimation (MLE) method to determine the best-fit parameters for these distributions. The likelihood function is maximized to obtain estimates for shape, scale, and rate parameters.

The likelihood Function L is defined as

$$L = \prod_{i=1}^n [f(x_i; \theta)] \quad (3.27)$$

where $f(x_i; \theta)$ is the probability density function (PDF) of the chosen distribution, and θ represents the parameters to be estimated.

For Maximum Likelihood Estimation (MLE), we typically work with the log-likelihood function:

$$\ln L = \prod_{i=1}^n [\ln f(x_i; \theta)]$$

Exponential distribution

The **Exponential distribution** is a special case of the Weibull distribution where $k=1$, and it is given by:

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0, \lambda > 0$$

where λ is the **rate parameter**.

The **log-likelihood function** is:

$$\ln L(\lambda) = n \ln(\lambda) - (\lambda) \sum_{i=1}^n x_i$$

Setting the derivative to zero gives:

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n x_i} \quad 3.28$$

Weibull distribution

The Weibull distribution is defined as:

$$f(x; \lambda, k) = \left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}$$

where:

- k is the **shape parameter**,
- λ is the **scale parameter**.

The **log-likelihood function** for $X = \{x_1, x_2, \dots, x_n\}$ is:

$$\ln L(k, \lambda) = n \ln k - nk \ln(\lambda) + (k, \lambda) \sum_{i=1}^n \ln x_i - \sum_{i=1}^n \left(\frac{x_i}{\lambda}\right)^k$$

The MLE estimates are found by solving:

$$\frac{n}{k} - n \ln \lambda + \sum_{i=1}^n \ln x_i = 0$$

$$\lambda = \left(\frac{1}{n} \sum_{i=1}^n x_i^k\right)^{\frac{1}{k}} \quad 3.29$$

Gamma distribution

The **Gamma distribution** is defined as:

$$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$$

where:

- α is the **shape parameter**,
- β is the **rate parameter**,
- x represents the failure times,
- $\Gamma(\alpha)$ is the **Gamma function**.

The **log-likelihood function** for a dataset $X=\{x_1,x_2,\dots,x_n\}$ is:

$$\ln L(\alpha, \beta) = n\alpha \ln \beta - n \ln \Gamma(\alpha) + (\alpha - 1) \sum_{i=1}^n \ln x_i - \beta \sum_{i=1}^n x_i$$

The MLE estimates of α and β are found by solving:

$$\hat{\beta} = \frac{n\alpha}{\sum_{i=1}^n x_i}$$

$$\ln \hat{\alpha} - \frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} = \frac{1}{n} \sum_{i=1}^n \ln x_i - \ln \left(\frac{1}{n} \sum_{i=1}^n x_i^k \right) \quad 3.30$$

3.8 SUMMARY OF METHODOLOGY

This chapter outlined the methodology for comparing the Exponential, Weibull, and Gamma distributions. The study employs a quantitative approach, utilizing real world datasets. Parameter estimation is performed using maximum likelihood estimation and the method of moments, while model fitting and validation are assessed using goodness of fit tests, information criteria, and cross validation techniques. The next chapter presents the results and discussion of the analysis.

CHAPTER FOUR

DATA ANALYSIS AND RESULTS

4.1 Introduction

This chapter presents a comprehensive analysis of two datasets using Maximum Likelihood Estimation (MLE) for three probability distributions: Exponential, Weibull, and Gamma. The analysis involves estimating distribution parameters, evaluating model fit using Log Likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC), and identifying the best fitting model for each dataset.

4.2 APPLICATION USING LIFETIME DATA

For data analysis in this chapter, three-lifetime distributions will be used: the Exponential distribution, the Weibull distribution, and the Gamma distribution. Their respective PDFs are given below. (Just for emphasis)

- **Exponential Distribution:**

The PDF of an Exponential Distribution is given as

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0, \lambda > 0 \quad (4.1)$$

- **Weibull Distribution:**

The PDF of a Weibull Distribution is given as

$$f(x; \lambda, k) = \left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} \ell^{-\left(\frac{x}{\lambda}\right)^k} \quad (4.2)$$

- **Gamma Distribution:**

The PDF of a Gamma Distribution is given as

$$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad x \geq 0, \alpha, \beta > 0 \quad (4.3)$$

Data Set 1: The data set reported by Efron (1988) represents the survival times of a group of patients suffering from Head and Neck cancer disease and treated using radiotherapy (RT).

6.53 7 10.42 14.48 16.10 22.70 34 41.55 42 45.28 49.40 53.62 63 64 83
 84 91 108 112 129 133 133 139 140 140 146 149 154 157 160 160 165
 146 149 154 157 160 160 165 173 176 218 225 241 248 273 277 297
 405 417 420 440 523 583 594 1101 1146 1417

Data Set 2: This data set represents the waiting times (in minutes) before service of 100 Bank customers and examined and analyzed by Ghitany et al.(2008)

0.8 0.8 1.3 1.5 1.8 1.9 1.9 2.1 2.6 2.7 2.9 3.1 3.2 3.3 3.5 3.6 4
 4.1 4.2 4.2 4.3 4.3 4.4 4.4 4.6 4.7 4.7 4.8 4.9 4.9 5 5.3 5.5 5.7 5.7 6.1
 6.2 6.2 6.2 6.3 6.7 6.9 7.1 7.1 7.1 7.1 7.4 7.6 7.7 8 8.2 8.6 8.6 8.6 8.8

8.8 8.9 8.9 9.5 9.6 9.7 9.8 10.7 10.9 11 11 11.1 11.2 11.2 11.5 11.9 12.4
 12.5 12.9 13 13.1 13.3 13.6 13.7 13.9 14.1 15.4 15.4 17.3 17.3 18.1
 18.2 18.4 18.9 19 19.9 20.6 21.3 21.4 21.9 23 27 31.6 33.1 38.5

The results obtained from this analysis will provide insights into the underlying data distribution and help in making reliable statistical inferences.

Applying the three distributions to the two datasets, we obtain the following results.

4.3 Descriptive Statistics

In this section, we present the summary statistics for the two datasets analyzed. Table 4.1 below provides an overview of key descriptive measures, including the **mean, median, standard deviation, minimum, maximum, and skewness**.

Table 4.1: Descriptive Statistics of Dataset 1 and Dataset 2

Category	Mean	Median	Standard Deviation	min	Max	Skewness
Dataset 1	226.17	151.5	271.5715	34	1417	2.796341
Dataset 2	9.877	8.1	7.2370	0.8	38.5	1.4953

Interpretation

The mean and median of Dataset 1 are 226.17 and 151.5, respectively, while Dataset 2 has a mean of 9.877 and median of 8.1. The difference between the mean and median in Dataset 1 suggests positive skewness due to extreme values, as indicated by its skewness value of 2.80. Dataset 2 also exhibits positive skewness, though to a lesser extent (1.4953). The standard deviation of Dataset 1 (271.57) is

significantly higher than that of Dataset 2 (7.237), indicating greater variability in Dataset 1.

4.4 Model Comparison Results

MLE is a widely used statistical technique for estimating the parameters of a probability distribution by maximizing the likelihood function. The estimation process is performed for the Exponential, Weibull, and Gamma distributions, which are commonly applied in reliability and survival analysis.

To determine the best fitting model, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) were computed for each distribution. These criteria help in model selection by penalizing model complexity. Lower values indicate a better model fit

4.4.1 Cancer Survival Times (Dataset 1) Analysis Results

MLE was applied to estimate the parameters of the three distributions fitted to the cancer survival dataset. The results are summarized in Table 4.2 and Table 4.4.

Table 4.2: MLE Results for Cancer Survival Times

Distribution	Log-Likelihood	Mean	Variance	Parameter Estimates	Std. Error
Exponential	-372.436	221.98	49,275.10	$\lambda = 221.98$	22.198
Weibull	-372.416	222.219	20,120.10	$k = 247.2$	16.2861,

				$\lambda = 1.6044$	0.121865
Gamma	-372.395	221.98	22,010.70	$\alpha = 265.24$ $\beta = 0.6268$	0.29605, 14.6922

Table 4.4: Model Comparison for Cancer Survival Times

Distribution	Log-Likelihood	AIC	BIC
Exponential	-372.436	746.87	748.93
Weibull	-372.395	748.79	752.95
Gamma	-372.416	748.83	752.86

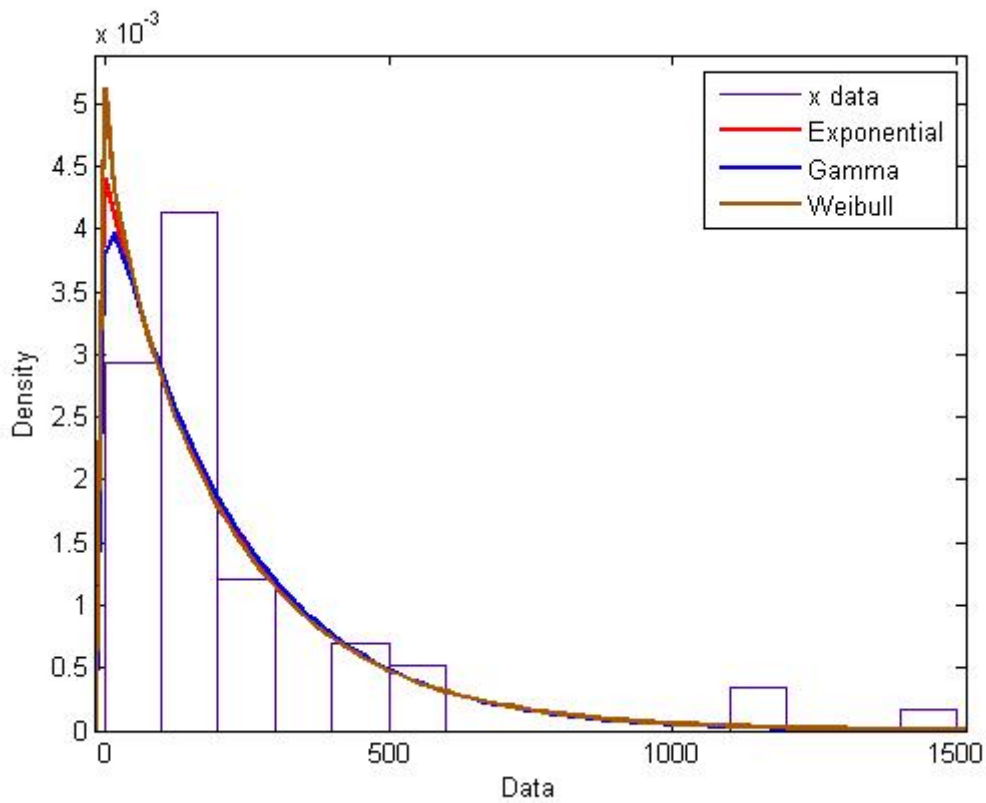


Fig.4.1 Fitted Model for Cancer Survival Times

Interpretation:

- The **Exponential distribution** has the lowest log-likelihood (-372.436) and the lowest AIC (746.87) and BIC (748.93), suggesting it is the better model among the three in terms of simplicity.
- The **Weibull distribution** has a slightly better log-likelihood (-372.395) but a higher AIC (748.79) and BIC (752.95), making it a slightly less preferable choice.
- The **Gamma distribution** has a log-likelihood of -372.416, but its AIC (748.83) and BIC (752.86) are also higher than those of the Exponential model, indicating it is not the best fit.

4.4.2 Bank Waiting Times (Dataset 2) Analysis Results

The model comparison was also applied to the bank waiting times dataset. Table 4.3 and Table 4.4 presents the estimated parameters for each distribution.

Table 4.3: MLE Results for Bank Waiting Times

Distribution	Log-Likelihood	Mean	Variance	Parameter Estimates	Std. Error
Exponential	-329.021	9.877	97.5551	$\lambda = 9.877$	0.9877
Weibull	-318.731	9.9255	47.8308	k= 10.955 $\lambda=$	0.794236, 0.109787

				1.45849	
Gamma	-317.3	9.877	48.5637	$\alpha = 2.0088$ $\beta = 4.9168$	0.263903, 0.733202

Table 4.5: Model Comparison for Bank Waiting Times

Distribution	Log-Likelihood	AIC	BIC
Exponential	-329.021	660.4	662.65
Weibull	-318.731	641.42	646.63
Gamma	-317.3	638.6	642.81

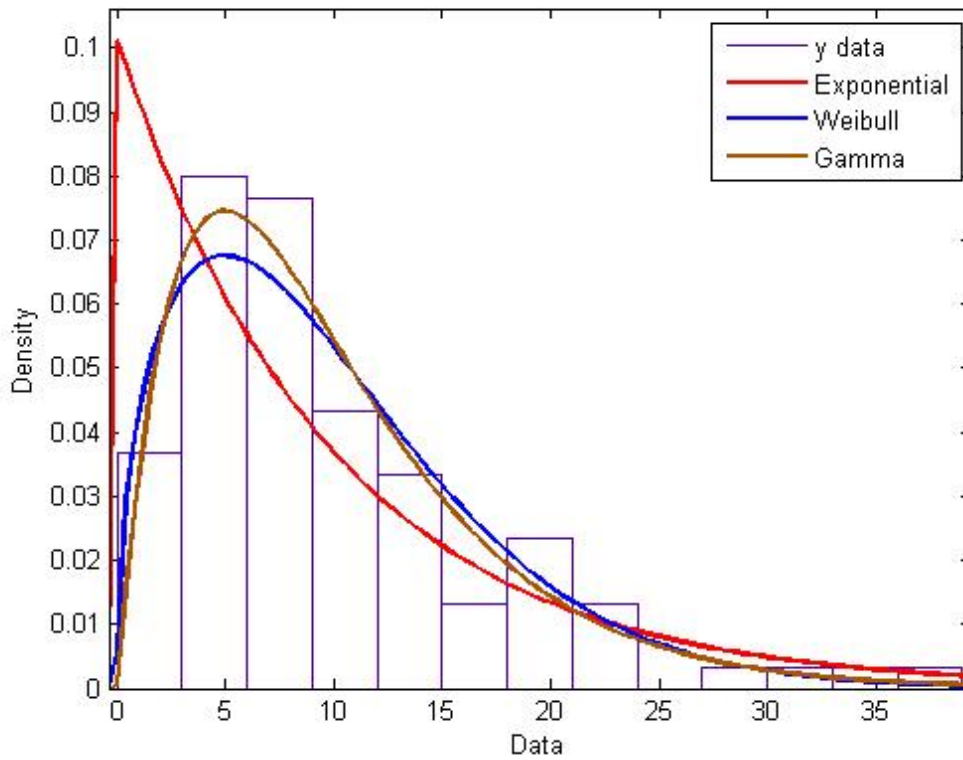


Fig.4.2 Fitted Model for Bank Waiting Times

Interpretation:

- The **Exponential distribution** has the lowest log-likelihood (-329.021) and the highest AIC (660.40) and BIC (662.65), indicating that it is the least suitable model for bank waiting times.
- The **Weibull distribution** performs better than the Exponential model, with a higher log-likelihood (-318.731) and lower AIC (641.42) and BIC (646.63), making it a more suitable fit.
- The **Gamma distribution** has the highest log-likelihood (-317.300), lowest AIC (638.60), and lowest BIC (642.81), confirming it as the best-fitting model for bank waiting times.

4.7 Summary

This chapter presented a detailed statistical analysis using MLE to estimate the parameters of Exponential, Weibull, and Gamma distributions.

CHAPTER FIVE

DISCUSSION AND CONCLUSION

5.1 Introduction

This chapter provides an in-depth discussion of the findings from the Maximum Likelihood Estimation (MLE) analysis presented in Chapter Four. It evaluates the implications of selecting different probability distributions for modeling survival and waiting times. Additionally, the limitations of the study are outlined, and recommendations for future studies are suggested.

5.2 Discussion of Findings

5.2.1 Cancer Survival Times (Dataset 1)

The analysis of the cancer survival dataset revealed that the Exponential distribution provides the best model based on simplicity, as it has the lowest Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values. However, while the Exponential model is preferred for its simplicity, the Gamma distribution remains widely used in medical research and survival analysis due to its flexibility in capturing different hazard functions as it has the highest Log likelihood.

- **Key findings**

The Exponential distribution base in its simplicity proof to be best fit because is simpler (with only one parameter) compared to the more complex Gamma and Weibull models. These simpler models tend to perform better in AIC and BIC, even if their log-likelihoods are lower, due to their lower complexity.

The Gamma distribution had lower AIC and BIC values compared to the Exponential model, but it also had the highest log-likelihood, confirming its superior fit. However, the choice between these models if depends on model simplicity favors the Exponential model but on best fit favors the Gamma distribution.

The Weibull distribution performed worse compared to its AIC and BIC criteria but still showed a better fit than the Exponential model base on its log likelihood.

- **Implications:**

The preference for the Exponential model in terms of AIC and BIC highlights the trade-off between simplicity and fit quality. While it provides an easier interpretation and requires fewer assumptions, it may not fully capture the complexity of survival times.

The superior log-likelihood of the Gamma model suggests that cancer survival times follow a nonconstant hazard function, aligning with real-world clinical data where survival rates change over time.

The choice of distribution directly impacts medical prognosis models and treatment evaluations, influencing how survival probabilities are estimated and interpreted.

5.2.2 Bank Waiting Times (Dataset 2)

For the bank waiting times dataset, the analysis indicated that the Gamma distribution provided the best fit among the three models. This is an important finding as it supports the notion that waiting times are influenced by multiple factors and do not follow a simple memoryless process as assumed by the Exponential distribution.

- **Key findings:**

The Exponential distribution had the lowest loglikelihood, indicating a poor fit.

The Gamma distribution had the highest loglikelihood and the lowest AIC and BIC values, making it the most appropriate model.

The Weibull distribution was a close contender but had slightly higher AIC and BIC values than the Gamma model.

- **Implications:**

The better fit of the Gamma distribution suggests that bank waiting times have a more complex underlying structure that includes variability in service rates.

This finding can help optimize queue management strategies in banks, leading to improved customer satisfaction and resource allocation.

5.3 Limitations of the Study

While the findings provide valuable insights, certain limitations should be acknowledged:

1. **Sample Size:** The datasets analyzed were of limited size, which may affect the generalizability of the results.
2. **Assumption of Distributional Forms:** The study only considered Exponential, Weibull, and Gamma distributions. Other potential distributions such as Lognormal or Generalized Gamma were not explored.
3. **Real World Variability:** Factors such as external interventions, human behavior, and environmental changes were not accounted for in the data modeling process.

5.4 Recommendations for Future Research

To build on the findings of this study, future research should consider:

1. **Expanding Dataset Size:** Analyzing larger datasets to improve the robustness and generalizability of the findings.
2. **Exploring Additional Distributions:** Investigating other probability distributions to determine whether they offer a better fit for survival and waiting time data.
3. **Applying Machine Learning Techniques:** Incorporating machine learning models such as neural networks or decision trees to predict survival and waiting times more accurately.
4. **Assessing External Factors:** Evaluating the impact of external factors such as demographic variations, economic conditions, and treatment differences on survival and waiting times.

5.5 Conclusion

This study applied MLE to estimate parameters for Exponential, Weibull, and Gamma distributions to model cancer survival and bank waiting times.

These results contribute to a better understanding of survival and waiting time phenomena, offering valuable insights for applications in healthcare and service industries. Future studies should consider additional distributions and larger datasets to refine these conclusions further.

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