A NEW CORRELATED BIVARIATE EXPONENTIAL DISTRIBUTION WITH APPLICATIONS.

BY

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A THESIS WRITTEN IN THE DEPARTMENT OF STATISTICS AND SUBMITTED TO THE COLLEGE OF POSTGRADUATE STUDIES IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF PHILOSOPHY (MPhil.) STATISTICS

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CERTIFICATION

This dissertation was accomplished by me, ONUKWUBE Obioma Gertrude in the Department of Statistics, Faculty of Physical Sciences, University of Benin, Benin City, Nigeria, under the supervision of Prof J.I. Mbegbu. I have not duplicated or copied any author(s) rather I acknowledged and cited all works used.

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DECLARATION

UNIVERSITY OF BENIN

SCHOOL OF POSTGRADUATE STUDIES

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DEDICATION

This thesis is dedicated to Almighty God for His grace and favor upon me. To Him alone be all the glory and honor, Amen.

I express my gratitude to my husband Sir. C.I. Onukwube and our wonderful children. The pursuit of this dream would not have been possible without you. I also treasure the confidence you inspired in me.

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CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

The exponential distribution became the cornerstone of survival analysis and reliability engineering throughout the latter half of the 20th century and it is imperative to mention that the successive times in exponential distribution are assumed to occur independently and randomly over time with a constant rate. The aim of the study was to develop a generalized and flexible bivariate exponential distribution that will incorporate correlation parameter ρ, extending the domain to a positive real-line using the framework of linear regression.

A secondary dataset from Federal Road Safety Corps on road accidents in Imo State from 2020 to 2024 was used in the study and it was obtained from the Head office of Federal Road Safety Corps, off Egbu road Owerri, Imo state. In the study, we developed a generalized and bivariate exponential model that incorporates a correlation parameter, while preserving analytical simplicity. The proposed model, referred to as the New Correlated Bivariate Exponential Distribution (NCBED).

The consistency of the NCBED was assessed using Kolmogorov Smirnov and Cramer Von Mises tests, in comparison with the baseline Grine model (2018). The Federal Road Safety (FRSC) dataset demonstrates that both injury and fatality data follow heavy-tailed exponential-type distributions and the NCBED provided a superior fit compared to the baseline model, capturing real-world correlations between crash outcomes. The findings indicate that the Maximum Likelihood estimates of the proposed model are consistent with the nature of the model.

CHAPTER ONE

INTRODUCTION

1.1 Background of Study

Exponential distribution is one of the most useful distributions in statistics. Typically, it is used to model times between events or arrival times. It is a skewed distribution. The earliest form of exponential distribution had single-parameter, for example, λ, but in recent times, modifications have been made to the single-parameter exponential distribution such that the distribution can now take more than one parameter (multi-parameter) for example, Forson et al., (2019). The exponential distribution is known to have a relationship with the Poisson and Gamma distributions. Whereas the Poisson distribution is used to model the number of events occurring in a given time interval, the exponential distribution models the time between those events. On the other hand, the exponential distribution is considered a special case of the gamma distribution if the shape parameter $\alpha = 1$,(see Gupta et al.,(1999)).

The successive times in exponential distribution are assumed to occur independently and randomly over time with a constant rate. The exponential distribution has also been successfully applied in the modeling of some physical phenomena such as populations, interest rates, radioactive decay, and the amount of medicine in the bloodstream, (see Shapiro et al.,(1972)).

The probability density function(pdf) of an exponential distribution is a decreasing function on the positive half of the real line with a constant hazard function which is key for extending to bivariate like the one prosed here. It has a simple analytical structure, and it is possible to obtain exact inferential results in many different situations like, reliability engineering and survival analysis and it is distinctly well behaved. The exponential distribution became the cornerstone of survival analysis and reliability engineering throughout the latter half of the 20th century, see Barlow et al.,(1966). In a Poisson process, where events happen continuously, independently, and at a constant average rate, it is mostly used to simulate the intervals between occurrences. The distribution is characterized by its simplicity; in its most basic form, it has a single parameter that represents the process's scale or rate. The conventional single-parameter exponential distribution has undergone adaptations and alterations over time, enabling it to handle more complicated events through multi-parameter versions.

It has been frequently used in medical studies to model the survival times and the time until component failure. The theory and construction of bivariate exponential distributions have advanced significantly in recent decades. In current finding by Balakrishnan and Shiji (2014),on classes of bivariate exponential distribution improved the flexibility of model dependency. Bladt and Nielsen (2008) presented a new method based on phase-type distributions, which are the distribution of the time until absorption in a finite-state Markov process. Their construction permits bivariate exponential distributions with arbitrary correlation coefficients, both positive and negative, and their model has the important feature that any linear combination of the correlated exponential variables remains a phase-type distribution, making statistical inference and hypothesis testing easier.

The exponential distribution is being used in Bayesian statistics in the recent past, essentially for modeling Poisson process rate parameters. The gamma distribution, of which the exponential is a particular case, has a conjugate prior characteristic which makes it an asset for Bayesian inference (Gelman et al., (2013)). With a pdf that falls exponentially, the exponential distribution is naturally skewed and is defined on the positive real line. The instantaneous failure rate or event occurrence rate is represented by its hazard function, which remains constant across time. Since the chance of an event occurring in the next instant is independent of the past, this fact makes the exponential distribution an ideal choice for representing the lives of memoryless systems.

The study on modifications and application of the exponential distribution have occupied a dominating and leading position in distribution theory. For example, Alotaibi et al., (2021) developed and studied the mixture of a new bivariate absolutely continuous distribution using a mixture of two independent two-parameter Exponentiated Exponential distributions (EE). They constructed their model under two mechanisms. In the first mechanism, the two random variables $X \, and \, Y$ were assumed independent and distributed as EE with their respective parameters $(\lambda_1, \theta_1) \, and \, (\lambda_2, \theta_2)$. In the second mechanism, a Bivariate mixture exponentiated Exponential distribution (BMEE) was constructed using the bivariate Gaussian Copula Approach.

Alotaibi et al.,(2021)developed a bivariate mixture exponentiated exponential(BMEE) distribution using copula methods, the observed marginal distribution of $X \, and \, Y$ can be obtained from the joint distribution of $\theta_1 and \, \theta_2$ as follows:

Ogunwale et al., (2022) proposed a new probability distribution called Exponential -Exponential distribution and provided a comprehensive study of its theory and derived appropriate expressions for its statistical properties. Their approach was based on the method proposed in Alzaatreh et al., (2013).

Bladt and Nielsen (2010) discussed the construction of bivariate exponential-exponential distributions that are arbitrarily correlated. Their distribution has the property that any linear combination of the correlated exponential distribution is a phase-type distribution.

Marshall and Olkin (2012) applied different methods to obtain the derivations of a multivariate exponential distribution and indicated the conditions under which the distribution is appropriate.

Frechet (1951) state that for a given marginal distribution, there exist infinitely many bivariate distributions with the marginals. Mckenzie (1982) proposed a product form of an autoregressive model of order one (PAR (1)) to generate a gamma sequence and prove a characterizing autocorrelation property but the model was not absolutely continuous.A useful account of this density function is given in Johnson and Kotz (1972). The works of researchers in this field continued to be a fundamental resource.

In spite of these developments, there are still some limitations associated with efficiency of the existing bivariate exponential distributions. Most of the existing correlated bivariate exponential models have restrictive assumptions and complex parameterizations that hamper their practical usefulness.

1.2 Statement of Problem

Despite the substantial developments in the construction of bivariate exponential distributions, most existing models remain limited due to restrictive assumptions, and complex parameter structures. These limitations hinder their applicability in capturing the wide range of dependence structures observed in real-world data, especially those involving negative correlations or domain extending to the entire positive real line.

Specifically, the model proposed by Grine (2018), which utilizes a two-parameter exponential conditional framework, is constrained to a finite interval and does not account for correlation using a parameter such as ρ. This restricts its generalizability and applicability in modeling interdependent exponential problem.

Therefore, this study aims to develop a generalized and flexible bivariate exponential model that incorporates a correlation parameter, extends the domain to positive real line. The propose model refers to as the New Correlated Bivariate Exponential Distribution (NCBED) will be constructed using the framework of linear regression, with analysis of its statistical properties, applicability to real-life data, and empirical validation of the model.

1.3 Aim and Objectives of Study

The aim of this study is to develop a generalized and flexible bivariate exponential distribution that will incorporate correlation parameter ρ, extending the domain to a positive real-line using the framework of linear regression.

The specific objectives are to;

i. construct a theoretical framework for the New Correlated Bivariate Exponential Distribution (NCBED) using the framework of linear regression to determine the strength of the relationship between the random variables X and Y.
ii. derive the statistical properties of the NCBED.
iii. determine the global usefulness of the model through simulation study.
iv. compare the proposed NCBED with the Baseline Grine model using real life data.
v. discuss the applicability of the propose distribution, to specific real-life data.

1.4 Significance of the Study

The understanding and accurately modeling of the dependency between two exponentially distributed random variables has significant implications across numerous domains. For instance, in reliability engineering, the lifespan of components might be interdependent, and a better model could improve maintenance schedules and safety measures. In finance, correlated exponential distributions can enhance risk management strategies by providing a more realistic assessment of joint risks.

The study is justified by the ability of the proposed distribution to describe some specific real-life situation appropriately. Also, this study will contribute to the theoretical foundation of statistical modeling and provide practical tools for better decision-making in applied settings. Moreso, the simplification of the estimation of the parameters of the distribution justifies its study.

1.5 Definitions of Terms

The definitions of the following important concepts are provided to guarantee uniformity and clarity throughout this study:

Definition 1. Feller, (1949), The Exponential distribution is a continuous probability distribution that models the time between events in a Poisson process (where events occur independently at a constant average rate). It is characterized by its memoryless property, meaning the probability of an event occurring in the next interval is independent of how much time has already elapsed.

Definition 2. Johnson, et al., (1997), A bivariate distribution describes the joint probability distribution of two random variables, $X \text{ and } Y$. It characterizes how probabilities are assigned to pairs (x, y) of outcomes, accounting for the relationship (e.g., dependence or independence) between the variables. Bivariate distributions can be discrete or continuous, depending on whether the variables are discrete or continuous.

Definition 3. Sklar (1959) and Nelsen, (2006). A copula is a mathematical tool that models the dependence structure between random variables, independent of their marginal distributions. It couples marginal distributions to form a joint multivariate distribution, enabling flexible modeling of complex dependencies (e.g., nonlinear or tail dependence).

Definition 4. According to Neut, (1981), the Phase-Type distribution is a continuous probability distribution that models the time until absorption in a finite-state continuous-time Markov chain (CTMC) with transient states and a single absorbing state. It generalizes distributions like the exponential and Erlang by representing processes with multiple sequential or parallel "phases." PH distributions are widely used due to their mathematical tractability and ability to approximate complex systems.

It is denoted by $tuple = (\alpha, T)\ where$:

- α is a row vector of initial probabilities for starting in each transient state.
- T is the sub-generator matrix (transition rates between transient states).

Definition 5: (Kendall, & Stuart,(1973), Rodgers, & Nicewander, (1988)**)** The correlation coefficient ρ is a statistical measure that quantifies the strength and direction of the linear relationship between two random variables and It ranges from −1 to +1, where:

- +1 implies perfect positive linear relationship.
- −1 implies perfect negative linear relationship.
- 0 means No linear relationship**.**

The Pearson's ρ correlation measures linear dependence between two continuous variables X and Y.

Definition 6: (Fisher, (1922), Casella, & Berger, (2002), Lehmann, & Casella, (1998)**)** Maximum Likelihood Estimation (MLE) is a statistical method used to estimate the parameters of a probability distribution by maximizing the likelihood function, which measures how well the model explains the observed data. The estimated parameters are those that make the observed data most probable under the assumed model.

Definition 7; (Pearson, (1894), Casella, & Berger, (2002)**),** The Method of Moments (MoM) is a classical statistical technique used to estimate the parameters of a probability distribution by equating sample moments (e.g., mean, variance) to their theoretical population counterparts. The estimates are obtained by solving the resulting system of equations.

Definition 8:(Kalbfleisch, & Prentice,(2002), Barlow, & Proschan,(1975)), The hazard function or hazard rate is a fundamental concept in survival analysis and reliability engineering. It describes the instantaneous risk of an event (e.g., failure, death) occurring at time t, given that the event has not occurred before time t.

Definition 9: Billingsley, (1995)**,** The characteristic function (CF) of a random variable X is a complex-valued function that uniquely characterizes its probability distribution. The characteristic function always exists for any random variable, even when moments (e.g., mean, variance) does not, because $|e^{itX}| = 1$, ensuring the integral converges

Definition 10: Shannon Entropy, introduced by Shannon in (1948), quantifies the uncertainty or randomness inherent in a discrete probability distribution. It measures the average amount of information produced by a stochastic source of data.

Definition 11: The Bayesian Information Criterion (BIC) and the Akaike Information Criterion (AIC) are statistical metrics that are used to evaluate the relative merits of various statistical models for a certain collection of data. They penalize models with additional parameters by balancing model fit and complexity.

The AIC is defined as: $AIC = -2\log(L) + 2k$

Where; L is the value of the likelihood function.

k is the number of the estimated parameters in the model.

Also,

the BIC is the defined as: $BIC = -2\log(L) + klog(n)$

Where; L is the value of the likelihood function.

k is the number of the estimated parameters in the model.

n is the number of observations

CHAPTER TWO

LITERATURE REVIEW

One of the most useful distributions in statistics is an Exponential distribution. It is used to model time between events. The univariate model proposed by (Johnson, Kotz and Balakrishnan (1994)) provided a simple and tractable model for real life phenomena. The extension from the univariate exponential distribution to its bivariate counterpart represents a crucial advancement in the modeling of joint behaviour of continuous random variables. This transformation is particularly relevant in applications involving reliability, survival analysis, and risk modelling, where the assumption of independence between variables is often unrealistic as external stresses in reliability engineering and medical research often induce dependence.

This has necessitated the development of bivariate and multivariate exponential models that retain the marginal exponential properties while incorporating dependence structures via copula functions. Marshal and Olkin (1967) proposed the shock-based model which has the ability modeling dependent lifetimes and it became the bedrock in constructing multivariate exponential distribution. Also, (Rameshwar, Gupta and Kundu (1999)) developed a generalized exponential distribution with greater flexibility.

One of the prominent strategies for incorporating dependence has been the use of copula functions, which allow the separation of marginal behaviour from the dependence structure. Alotaibi et al. (2021) exemplified this by employing the Gaussian copula to construct the Bivariate Mixture Exponentiated Exponential (BMEE) distribution, see equations below;

$$h_{x,y}(x,y) = \int\int h(x,y,\theta_1,\theta_2)d\theta_1 d\theta_2 \tag{2.1}$$

$$f(x,\theta,\lambda) = \theta\lambda e^{-\lambda}\left(1-e^{-\lambda x}\right)^{\theta-1}, x > 0 \tag{2.2}$$

where $\theta > 0$ and $\lambda > 0$ are the shape and scale parameters. Also, X and Y are two random variables with parameters θ_1 and θ_2 respectively.

$$g(y,\varphi\beta) = \varphi\beta e^{-\beta_i y}\left(1-e^{-\beta_i y}\right)^{\theta-1}, \beta > 0, y > 0 \tag{2.3}$$

Where β is a random scale parameter.

$$g(x) = \lambda^2 e^{\lambda^2 x}, x > 0, \lambda > 0 \tag{2.4}$$

This offers closed-form expressions for both joint and marginal distributions. Their model demonstrated practical efficacy through real data fitting, highlighting its value for modeling positively correlated non-negative random variables

Al-Saadi and Young (1980) have discussed the small sample properties of the estimators for the correlation coefficient ρ in the bivariate exponential density of the form,

$$f(x_1,x_2) = \frac{\mu_1\mu_2}{1-\rho} exp\left\{-\frac{\mu_1 x_1 - \mu_2 x_2}{1-\rho}\right\} \mathrm{I}_0\left(\frac{\sqrt[2]{\rho\mu_1\mu_2 x_1 x_2}}{1-\rho}\right) \tag{2.5}$$

$where\ \mu_1\mu_2 > 0, x_1x_2 > 0, with\ 0 \le \rho < 1\ and\ \mathrm{I}_0(z)$

$$= \sum_{j=0}^{\infty}\left(\frac{z}{2j!}\right)^{2j}$$, is the modified Bessel function of first kind of order zero

In a related effort, Ogunwale et al. (2022) introduced the Exponential-Exponential (EE) distribution, an extension grounded in the work of Alzaatreh et al. (2013). Their model enriched the class of bivariate exponential distributions with theoretical robustness and tractable statistical formulations. Though, it has drawbacks of limited flexibility in modeling hazard function and parameter estimation when sample size is small.

Kundu and Gupta (2009) made a significant contribution by generalizing the exponential distribution to a bivariate form, referred to as the Bivariate Generalized Exponential Distribution (BVGE). This model maintained exponential marginals and was analytically expressed through compact forms of the joint density and survival functions. Using the Expectation-Maximization (EM algorithm), they derived maximum likelihood estimators and Fisher information matrices, demonstrating that the BVGE distribution provided superior fit compared to traditional bivariate exponential models. Also, it has some drawbacks of complex joint density function and may not capture adequately strong tail dependence.

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}\left(1-e^{-\lambda x}\right)^{\alpha_1-1}\left(1-e^{-\lambda y}\right)^{\alpha_2-1} \times \left(1-e^{-\lambda \min(x,y)}\right)^{\alpha_3-1}\big[\alpha_3(\alpha_1+\alpha_2+\alpha_3+1)e^{-\lambda \min(xy)} + (\alpha_1\alpha_2)\delta(x-y)\big] \tag{2.6}$$

where $\delta(x-y)$ is the Dirac delta function, accounting for the diagonal case where $x = y$.

A notable copula-based innovation was the Bivariate Beta-Exponential (BBE**)** distribution proposed by Abd Elaal (2017). By integrating various copula forms, the study offered a flexible modeling approach for bivariate lifetimes. Estimation was performed using both parametric and semi-parametric methods, further strengthening its statistical applicability.

In terms of 'goodness-of-fit' Alba-Fernández et al. (2014) proposed a test specifically for the Moran-Downton distribution, leveraging its characteristic function. The authors analyzed large-sample properties and applied the distribution to hydrological data, illustrating its robustness and relevance in environmental studies.

From a structural perspective, Bladt and Nielsen (2010) introduced a class of bivariate exponential distributions grounded in multivariate phase-type distributions. Their model could accommodate any correlation including negative and exhibited the useful property that linear combinations of marginals remained phase-type, which is critical for hypothesis testing in linear models.

Historically, Gumbel (1960) provided foundational insights by analyzing two bivariate exponential distributions with differing conditional expectations and correlation structures. This early work highlighted how dependence could manifest differently under varying configurations of the joint distribution.

More recent developments include Mirhosseini et al. (2016), who proposed a mixture of bivariate exponential distributions incorporating covariates using linear relationships, inspired by the Marshall-Olkin structure. This type of mixture model is well-suited for capturing heterogeneity in survival or reliability data.

The Exponential-Exponential (EE) model offered further flexibility through a novel PDF formulation, enriching the modeling of interdependent exponential phenomena.

Censoring mechanisms have also been integrated into recent models. Fayomi et al. (2024) studied bivariate lifetime data under Type II progressive censoring with random removals, using a length-biased exponential distribution. Their methodological innovations included both likelihood and Bayesian parameter estimation, with confidence intervals evaluated via asymptotic and bootstrap approaches. The study's simulation-based comparison of censoring schemes enhanced the practical relevance of their model.

David et.al., (2003) contributed by focusing on expectations of order statistics in correlated exponential pairs, which has implications for reliability and risk analysis. Their foundational work across three studies bolstered the theoretical underpinnings of bivariate exponential structures.

An extension of the Kundu and Gupta (2009) model was proposed by George et al. (2024), who developed the Bivariate Generalized Exponential Distribution with Positive probability at instantaneous failure (BVGEDP**)**. This model addressed scenarios where immediate system failures are non-negligible. Through iterative estimation and simulation studies, the authors established the model's effectiveness using real data.

Theoretical discussions on constructing bivariate exponential models have emphasized the importance of selecting appropriate dependence structures. As Frèchet (1951) demonstrated, an infinite number of joint distributions can share the same marginal distributions, underscoring the critical role of copula or other dependency modeling. Also, Lin et al. (2016) proposed a model to treat all the bivariate lack-of-memory (BLM) distributions in a unified approach and some new general properties of the BLM distributions were developed.

Additionally, McKenzie (1982) and Al-Saadi and Young (1980) contributed to the literature on autoregressive models and small-sample estimator properties, respectively. While McKenzie's autoregressive gamma model lacked full continuity, it advanced understanding of serial dependence, while Al-Saadi and Young highlighted estimator limitations under small samples.

Bentoumi et al. (2021) tackled the issue of negative dependence by proposing models based on the counter-monotonic shock framework, which incorporated the Fréchet lower bound and supported both positive and negative dependencies. Their simulations assessed estimator performance and visualized general dependence structures effectively.

Finally, Grine (2018) introduced a conditional two-parameter bivariate exponential model that assumed joint independence under certain constraints. However, the absence of a correlation parameter like ρ limited its applicability in modeling dependent systems. This restricts its generalizability and applicability in modeling interdependent exponential phenomena.

$$f_{X,Y}(x,y) = \frac{e^{\frac{a}{b}}}{bc} e^{-\left(\frac{1}{b}-\frac{1}{c}\right)x} e^{-\frac{y}{c}} \quad , \; a \leq x \leq y. \; b,c > 0 \tag{2.7}$$

Collectively, the literature reveals a rich and evolving landscape of bivariate exponential distribution, driven by the need for flexible, interpretable and practically applicable models. One of the drawbacks of these models is that they are not absolutely continuous and have restrictions on the range of correlation coefficient.

CHAPTER THREE

METHODOLOGY

3.1 Derivation of the New Correlated Bivariate Exponential Distribution.

The correlation coefficient ρ is a key parameter in describing the linear relationship between two random variables $X \, and \, Y$. For any joint distribution of $X \, and \, Y$, the correlation coefficient satisfies the inequality $-1 \leq \rho \leq 1$ (Robert and Casella, (2004)).

Case 1: Perfect Positive Correlation ($\rho = 1$) In this case, Y is a deterministic linear function of X such that $Y = a + bX, where \; b > 0$. The joint distribution is concentrated along the positively slope line in the $(x, y) - plane$.

Case 2: Perfect Negative Correlation ($\rho = -1$), In this case, Y is again deterministic and linear, but in the opposite direction given that $Y = a - bX, where \; b > 0$. The joint distribution is concentrated along the negatively sloped line.

Case 3: Imperfect Correlation $-1 < \rho < 1$. In some settings, where bivariate distribution involves imperfect correlation. A natural question arises: Does the joint distribution exhibit concentration along some line in the (x, y)-plane. Under the condition that the conditional expectation $E(Y/X)$ is a linear function of X, this question is affirmatively answered. That is there exists a linear regression of the form;

$$E(Y/X) = a + bX \tag{3.1}$$

where a and b are real constants. The coefficient ρ in this context serves as a measure of how close the distribution is clustered around the regression line: the closer ρ is to ± 1, the more concentrated the joint distribution is around this line. Let $f(x, y)$ denote the (pdf) of the continuous random variables X and Y . Let $f_1(x)\ and\ f_2(y)$ denotes the marginal pdfs of $X\ and\ Y$ respectively. Then, the conditional pdf of $Y\ given\ X = x$ is defined as:

$$h(y/x) = \frac{f(x,y)}{f_1(x)},\ \ for\ f_1(x) > 0 \tag{3.2}$$

The conditional expectation of $Y\ given\ X = x, is;$

$$E(Y/X = x) = \emptyset(x) = \int_0^{\infty} y\ \frac{f(x,y)}{f_1(x)}dy \tag{3.3}$$

Suppose $\emptyset(x)$is a linear function of x, say

$$\emptyset(x) = a + bX \tag{3.4}$$

Then Y is said to have a linear conditional mean if

$$E(Y|X = x) = \emptyset(x) = \frac{1}{f_1(x)}\int_{-\infty}^{\infty} y\ f(x,y)\ dy = a + bx \tag{3.5}$$

which implies that;

$$\int_{-\infty}^{\infty} yf(x,y)dy = (a\ + bx)f_1(x) = af_1(x) +\ \ bxf_1(x) \tag{3.6}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dydx = \int_{-\infty}^{\infty} af_1(x)dx\ \ + b\int_{-\infty}^{\infty} xf_1(x)dx$$

$$= a\ + bE(x) \tag{3.7}$$

so that;

$$\int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f(x,y)dx\right]dy = a + b\mu_1\ \ that\ implies$$

$$\int_{-\infty}^{\infty} y\ \ f_2(y)dy = a + b\mu_1 \tag{3.8}$$

$$where\ \int_{-\infty}^{\infty} f(x,y)\ \ dx = f_2(y), so\ that$$

$$\mu_2 = a + b\mu_1\ \ \Longrightarrow\ \ a = \mu_2 - b\mu_1 \tag{3.9}$$

where $\mu_1\ = \int_{-\infty}^{\infty} xf_1(x)dx = E(X)\ and\ \mu_2 = \int_{-\infty}^{\infty} yf_2(y)dy =\ E(Y)$ (3.10)

If we multiply Equation (3.6) by x and integrate to have

$$\int_{-\infty}^{\infty} xy\ f(x,y)dy\ \ \ \ =\ \ \ \ \ \ (a\ + bx)xf_1(x)$$

$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dydx \;=\; \int_{-\infty}^{\infty}(ax\;+bx^2)f_1(x)dx \;=$

$a\int_{-\infty}^{\infty} xf_1(x)dx + b\int_{-\infty}^{\infty} x^2 f_1(x)$

Then, $E(XY) = aE(X) + bE(X^2)$ (3.11)

The correlation coefficient of $X\; and\; Y$ is;

$$\rho = \frac{E(XY) - \mu_1\mu_2}{\sigma_1\sigma_2} \quad (3.12)$$

$$E(XY) \;=\; \rho\sigma_1\sigma_2 + \;\mu_1\mu_2 \quad (3.13)$$

also, recall that $E(X^2) \;=\; \sigma_1^2 \;+\; \mu_1^2$ (3.14)

So, substituting back in equation (3.11) to get

$$E(XY) = \;a\mu_1 + \;b(\sigma_1^2 \;+\; \mu_1^2) \quad (3.15)$$

substituting equations (3.13), (3.14) and (3.9) into equation (3.15) to obtain;

$$\rho\sigma_1\sigma_2 + \;\mu_1\mu_2 = \;\mu_1(\mu_2 - b\;\mu_1) + b\sigma_1^2 + \;b\mu_1^2$$

$$\rho\sigma_1\sigma_2 = \;\mu_1\mu_2 - \;\mu_1\mu_2 - b\mu_1^2 + \;b\;(\sigma_1^2 + \mu_1^2)$$

$$\rho\sigma_1\sigma_2 = b\sigma_1^2 \;\Rightarrow b \;=\; \rho\frac{\sigma_2}{\sigma_1} \quad (3.16)$$

Now substituting b in equation (3.9), we have

$$a \;=\; \mu_2 - \;\rho\frac{\sigma_2}{\sigma_1}\mu_1 \quad (3.17)$$

therefore, equation (3.5) becomes;

$$\emptyset(x) = \;\mu_2 - \;\rho\frac{\sigma_2}{\sigma_1}\mu_1 + \;\rho\frac{\sigma_2}{\sigma_1}x$$

$$\emptyset(x) = \;\mu_2 + \;\rho\frac{\sigma_2}{\sigma_1}(x - \mu_1) \quad (3.18)$$

Hence the linear conditional mean of $Y\; given\; X$ becomes;

$$E(Y|X = x) = \;\emptyset(x) \;\Rightarrow\; \mu_2 + \;\rho\frac{\sigma_2}{\sigma_1}(x - \mu_1) \quad (3.19)$$

Suppose $X\; and\; Y$ are exponentially distributed with parameters $\beta_1 and\; \beta_2$ with means

$\frac{1}{\beta_1}$ and $\frac{1}{\beta_2}$ respectively from equation (3.10);

$$E(X) = \frac{1}{\beta_1}\ ,\ \sigma_1^2 = \frac{1}{\beta_1^2},\ \sigma_1 = \frac{1}{\beta_1} \tag{3.20}$$

and $$E(Y) = \frac{1}{\beta_2}\ ,\ \sigma_2^2 = \frac{1}{\beta_2^2},\ \sigma_2 = \frac{1}{\beta_2} \tag{3.21}$$

Therefore, the exponential distribution of $X\ and\ Y$ are given by;

$$f_1(x) = \beta_1 e^{-\beta_1 x}, \quad x > 0, \beta_1 > 0 \tag{3.22}$$

$$f_2(y) = \beta_2 e^{-\beta_2 y}, \quad y > 0, \beta_2 > 0 \tag{3.23}$$

Substituting for Equation (3.20) and (3.21) in Equation (3.19) to get;

$$\emptyset(x) = \frac{1}{\beta_2} + \rho\frac{\beta_1}{\beta_2}\left(x - \frac{1}{\beta_1}\right) \tag{3.24}$$

Factor out $\frac{1}{\beta_2}$ in equation (3.24)

$$\emptyset(x) = \frac{1}{\beta_2}[1 + \rho(\beta_1 x - 1)] = \frac{1 + \rho(\beta_1 x - 1)}{\beta_2} \tag{3.25}$$

Let $$\lambda = \frac{1 + \rho(\beta_1 x - 1)}{\beta_2} \tag{3.26}$$

Therefore, the conditional density function of $Y\ given\ X$ is defined as;

$$h(y|x) = \frac{1}{\lambda} e^{-\frac{y}{\lambda}} \quad ,\lambda > 0, x > 0 \tag{3.27}$$

Then substituting back for λ, in equation (3.27) to get;

$$h(y|x) = \frac{\beta_2}{1 + \rho(\beta_1 x - 1)} e^{-\left(\frac{\beta_2 y}{1 + \rho(\beta_1 x - 1)}\right)} \tag{3.28}$$

The joint probability density function of X and Y is defined by

$$f(x, y) = f_1(x)\, h(y|x) \tag{3.29}$$

$$f(x, y) = \beta_1 e^{-\beta_1 x} \frac{\beta_2}{1 + \rho(\beta_1 x - 1)} e^{-\left(\frac{\beta_2 y}{1 + \rho(\beta_1 x - 1)}\right)}$$

$$f(x, y) = \frac{\beta_1 \beta_2}{1 - \rho + \rho\beta_1 x} e^{-\left(\frac{\beta_1 x(1 - \rho + \rho\beta_1 x) + \beta_2 y}{1 + \rho(\beta_1 x - 1)}\right)} \tag{3.30}$$

Hence, equation (3.30) is the proposed New Correlated Bivariate Exponential Distribution (NCBED).

Theorem 3.1. Let $X\ and\ Y$ be correlated bivariate exponential random variables with parameters $\beta_1 and\ \beta_2$ respectively, and correlation coefficient $-1 \le \rho \le 1$. The function $f: \mathbb{R}^2 \to \mathbb{R}$ is defined by;

$f(x,y) = \frac{\beta_1\beta_2}{1-\rho+\rho\beta_1 x}\ e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2\,y}{1+\rho(\beta_1 x-1)}\right)}$, $\quad \beta_1 > 0, \beta_2 > 0\ with\ (x,y) \in [0,\infty), and - \rho + \rho\beta_1 x\ \neq -1$. Is a probability density function.

Proof:

it is suffices to show that;

$$\iint_{\mathbb{R}(x,y)}^{\infty} f(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = 1$$

$$\iint_{\mathbb{R}(x,y)}^{\infty} f(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty \frac{\beta_1\beta_2}{1-\rho+\rho\beta_1 x}\ e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2\,y}{1+\rho(\beta_1 x-1)}\right)} dxdy$$

$$= \beta_1\beta_2 \int_0^\infty \int_0^\infty \frac{1}{1-\rho+\rho\beta_1 x}\ e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2\,y}{1+\rho(\beta_1 x-1)}\right)} dxdy \tag{3.31}$$

using substitution and Jacobian method, we

Let $\quad r = 1-\rho+\rho\beta_1 x\ and\ m = \frac{\beta_2\,y}{r}\ , with\ r \in [1-\rho,\infty)\ , then\ x = \frac{r-(1-\rho)}{\rho\beta_1}, y = \frac{r}{\beta_1 m}\ , also\ \ the\ Jacobian\ \ J = \begin{bmatrix} x & y \\ r & m \end{bmatrix} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial m \\ \partial y/\partial r & \partial y/\partial m \end{vmatrix} = \begin{vmatrix} 1/\rho\beta_1 & 0 \\ m/\beta_2 & r/\beta_2 \end{vmatrix} = \frac{r}{\rho\beta_1\beta_2} \neq 0$

Therefore, we substitute back in equation (3.31) to get;

$$= \beta_1\beta_2 \int_{1-\rho}^\infty \int_0^\infty \frac{1}{r}\ e^{-\left(\frac{\beta_1\left(\frac{r-(1-\rho)}{\rho\beta_1}\right)*r+\beta_2\left(\frac{r}{\beta_2}\right)m}{r}\right)} * \frac{r}{\rho\beta_1\beta_2} drdm \tag{3.32}$$

$$\Longrightarrow \frac{\beta_1\beta_2}{\rho\beta_1\beta_2}\int_{1-\rho}^{\infty}\int_{0}^{\infty} e^{-\left(\frac{\frac{1}{\rho}(r-(1-\rho))*r+m}{r}\right)} drdm \tag{3.33}$$

$$= \frac{1}{\rho}\int_{1-\rho}^{\infty} e^{\left(-\frac{r}{\rho}+\frac{1-\rho}{\rho}\right)} dr \int_{0}^{\infty} e^{-m}\, dm$$

Since $\int_{0}^{\infty} e^{-m}\, dm = 1$ (3.34)

we get; $\frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\rho}\int_{1-\rho}^{\infty} e^{-\left(r/\rho\right)}\, dr$ (3.35)

$Let\ \zeta = \frac{r}{\rho} \Longrightarrow r = \rho\zeta\ and\ dr = \rho d\zeta\ , so\ the\ limit\ of\ the\ integral\ becomes\ \zeta \in \left[\frac{1-\rho}{\rho}, \infty\right)$ and substitute back in equation (3.35) to obtain;

$$\frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\rho}\int_{1-\rho/\rho}^{\infty} e^{-\zeta}\, \rho d\zeta = e^{\left(\frac{1-\rho}{\rho}\right)} \int_{1-\rho/\rho}^{\infty} e^{-\zeta}\, d\zeta \tag{3.36}$$

which implies that

$$e^{\left(\frac{1-\rho}{\rho}\right)} * e^{-\left(\frac{1-\rho}{\rho}\right)} = 1 \tag{3.37}$$

Therefore, the proposed New Correlated Bivariate Exponential Distribution (NCBED) is a proper probability density function.

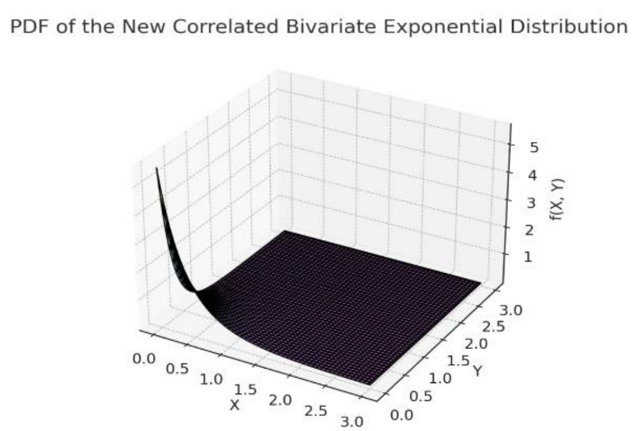


Fig.1.1a the density plot of the NCBED

The 3D surface plot represents the density of the new correlated bivariate exponential distribution (NCBED) for the chosen parameters.

Where,

- ❖ $\beta_1 = 1.5, \beta_2 = 2.0$ (rate parameters)
- ❖ $\rho = 0.5,$ (correlation parameter)
- ❖ $x, y > 0,$ (supports of the distribution)

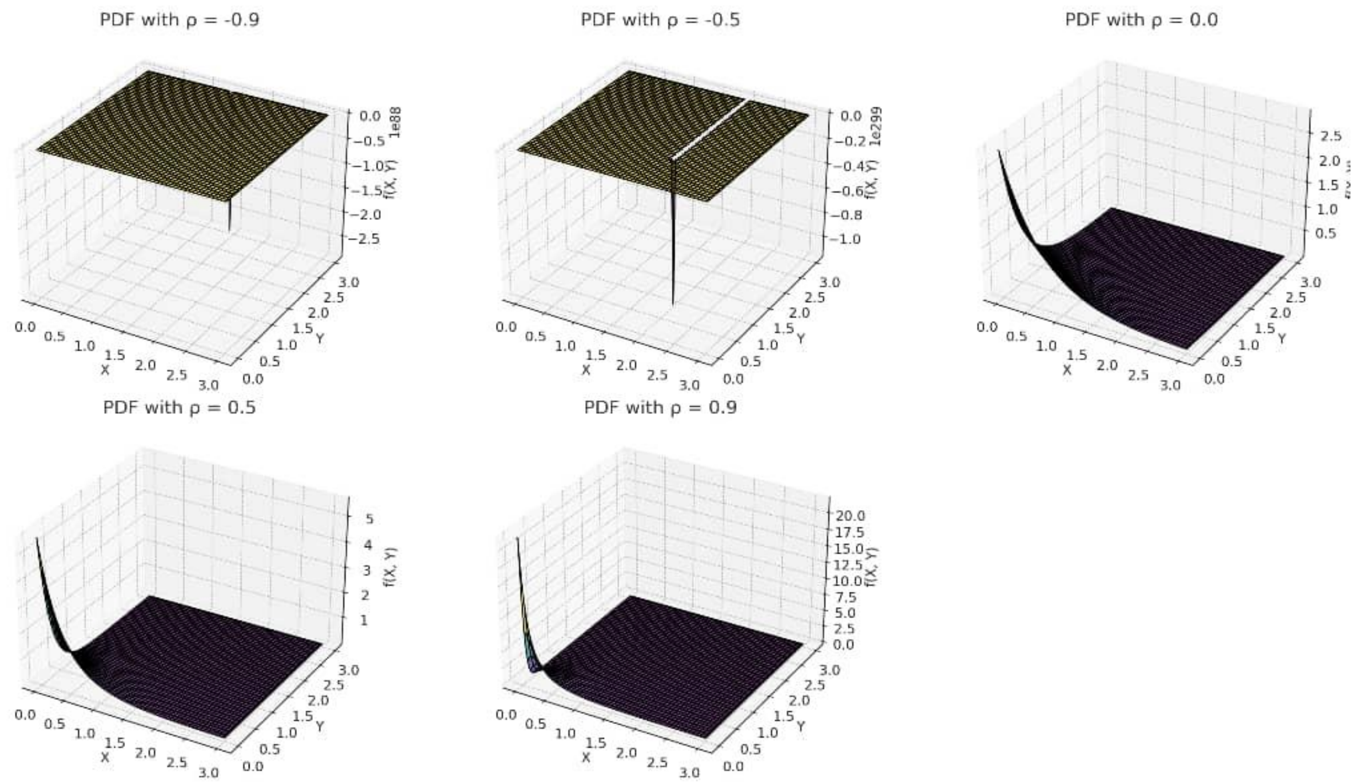


Fig 1.1b The density plots of NCBED at different values of rho

Remark 3.1: Here are some special cases of the proposed new correlated bivariate exponential distribution (NCBED).

(i) When $\beta_1 = 1\ and\ \rho = 1, then$

$$f(x, y) = \frac{\beta_2}{x} e^{-\left(\frac{x^2+\beta_2 y}{x}\right)} = \frac{\beta_2}{x} e^{-x}. e^{-\left(\frac{\beta_2 y}{x}\right)} \tag{3.38}$$

(ii) When $\beta_2 = 1\ and\ \rho = 1, then$

$$f(x,y) = \frac{1}{x}e^{-\beta_1 x}.e^{-\left(\frac{y}{\beta_1 x}\right)} \tag{3.39}$$

Hence, the equations (3.38) & (3.39) are the inverse -type of the proposed NCBED.

(iii) Let $\beta_1 = \frac{1}{b} - \frac{1}{c}, \beta_2 = \frac{1}{c}$ *and suppose* $\rho = 0$. The joint pdf of the New Correlated Bivariate Exponential Distribution (NCBED) in equation (46) simplifies to;

$$f(x,y) = \left(\frac{1}{b} - \frac{1}{c}\right).\frac{1}{c}.exp\left(-\left(\left(\left(\frac{1}{b} - \frac{1}{c}\right)x + \frac{y}{c}\right)\right)\right)$$

$$= \left(\frac{1}{bc} - \frac{1}{c^2}\right).exp\left(-\left(\frac{1}{b} - \frac{1}{c}\right)x\right).exp\left(-\frac{y}{c}\right) \tag{3.40}$$

Now, suppose: $\ln\left(1 - \frac{b}{c}\right) = \frac{a}{b}$ (3.41)

which implies: $1 - \frac{b}{c} = exp\left(\frac{a}{b}\right)$ (3.42)

Therefore, $\left(\frac{1}{bc} - \frac{1}{c^2}\right) = \frac{1}{bc}\left(1 - \frac{b}{c}\right) = \frac{1}{bc}exp\left(\frac{a}{b}\right)$ (3.43)

and the pdf becomes:

$$f(x,y) = \frac{exp\left(\frac{a}{b}\right)}{bc}.exp\left(-\left(\frac{1}{b} - \frac{1}{c}\right)x\right).exp\left(-\frac{y}{c}\right) \tag{3.44}$$

Note, under the assumptions;

$\beta_1 = \frac{1}{b} - \frac{1}{c}, \beta_2 = \frac{1}{c}$ *and* $\ln\left(1 - \frac{b}{c}\right) = \frac{a}{b}$ the probability density function of NCBED reduces to a special form in equation (3.44) which is the probability density function of Grine (2018). Therefore,

Equations (3.38), (3.39) and (3.44) have shown that the proposed NCBED is the generalization of Grine (2018) on Bivariate Distribution with a Two Parameters Exponential Conditional.

Theorem 3.2: Let X and Y be correlated bivariate exponential random variables with parameters $\beta_1 and\ \beta_2$ respectively, and correlation coefficient $-1 \le \rho \le 1$. The function $f: \mathbb{R}^2 \to \mathbb{R}$ is defined by;

$$F(x,y) = \iint_{\mathbb{R}(x,y)}^{\infty} f(x,y)\,dx\,dy$$

$$F(u.v) = \int_{\mathbb{R}_X}^{\infty}\int_{\mathbb{R}_Y}^{\infty} \frac{\beta_1\beta_2}{1-\rho+\rho\beta_1 x}\; e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1+\rho(\beta_1 x-1)}\right)} dxdy \tag{3.45}$$

where, $\beta_1 > 0, \beta_2 > 0\ with\ (x,y) \in [0,\infty) \times [0,\infty)\ for\ u,v \in [0,\infty) and\ 1 - \rho + \rho\beta_1 x \neq -1$

Then, the cumulative density function of the proposed NCBED is given by;

$$F(u,v) = e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right) - e^{\left(\frac{1-\rho}{\rho}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\beta_2^k v^k}{\Gamma(k+1)}$$

$$\left[\frac{1}{\rho}(1-\rho)^{1-k}E_k\left(k,-1+\frac{1}{\rho}\right) - \Gamma\left(1-k,-1+\frac{1}{\rho}+u\beta_1\right)\left(-1+\frac{1}{\rho}+u\beta_1\right)^k(1-\rho+\rho\beta_1)^{-k}\right] \quad (3.46)$$

where $E_k(.)$ is an exponential integral and

$\Gamma(a,x)$is an incomplete gamma function

Proof:

We know that;

$$F(u.v) = \beta_1\beta_2\int_0^u\int_0^v \frac{1}{1-\rho+\rho\beta_1 x}\, e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1+\rho(\beta_1 x-1)}\right)}dxdy \quad (3.47)$$

Using substitution and Jacobian method, we have;

Let $r = 1-\rho+\rho\beta_1 x$ and $m = \frac{\beta_2 y}{r}$ $,with\ r \in [1-\rho, 1-\rho+\rho\beta_1 u]$ $, m \in \left[0, \frac{\beta_2}{r}v\right]$ $,with\ x = \frac{r-(1-\rho)}{\rho\beta_1}, y = \frac{r}{\beta_1}m$ $, also\ the\ Jacobian\ J = \begin{bmatrix} x & y \\ r & m \end{bmatrix} =$

$$\begin{vmatrix} \partial x/\partial r & \partial x/\partial m \\ \partial y/\partial r & \partial y/\partial m \end{vmatrix} = \begin{vmatrix} 1/\rho\beta_1 & 0 \\ m/\beta_2 & r/\beta_2 \end{vmatrix} = \frac{r}{\rho\beta_1\beta_2} \neq 0$$

therefore, we substitute in equation(3.47) to obtain;

$$F(u,v) = \beta_1\beta_2\int_{1-\rho}^{1-\rho+\rho\beta_1 u}\int_0^{\frac{\beta_2}{r}v}\frac{1}{r}\, e^{-\left(\frac{\beta_1\left(\frac{r-(1-\rho)}{\rho\beta_1}\right)*r+\beta_2\left(\frac{r}{\beta_2}\right)m}{r}\right)} * \frac{r}{\rho\beta_1\beta_2}drdm$$

$$= \frac{\beta_1\beta_2}{\rho\beta_1\beta_2}\int_{1-\rho}^{1-\rho+\rho\beta_1 u}\int_0^{\frac{\beta_2}{r}v} e^{-\left(\frac{\frac{1}{\rho}(r-(1-\rho))*r+m}{r}\right)}drdm \quad (3.48)$$

Simplifying further and multiplying out to get;

$$= \frac{e^{\left(1-\rho/\rho\right)}}{\rho}\int_{1-\rho}^{1-\rho+\rho\beta_1 u} e^{\left(-\frac{r}{\rho}+\frac{1-\rho}{\rho}\right)dr}\int_0^{\frac{\beta_2}{r}v} e^{-m}\, dm \quad (3.49)$$

$$\Longrightarrow \frac{e^{\left(1-\rho/\rho\right)}}{\rho}\int_{1-\rho}^{1-\rho+\rho\beta_1 u} e^{\left(-\frac{r}{\rho}\right)}\left(1-e^{-\frac{\beta_2}{r}v}\right)dr \tag{3.50}$$

so, further simplification and decomposition of equation (3.50) we obtain

$$= \frac{e^{\left(1-\rho/\rho\right)}}{\rho}\left[\int_{1-\rho}^{1-\rho+\rho\beta_1 u} e^{\left(-\frac{r}{\rho}\right)}dr - \int_{1-\rho}^{1-\rho+\rho\beta_1 u} e^{-\frac{r}{\rho}}\left(e^{-\frac{\beta_2}{r}v}\right)dr\right] \tag{3.51}$$

$$\Longrightarrow e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right)-\frac{1}{\rho}e^{\left(\frac{1-\rho}{\rho}\right)}\int_{1-\rho}^{1-\rho+\rho\beta_1 u} e^{-r/\rho} * e^{-\frac{\beta_2}{r}v}\,dr$$

also, we decompose further the integral into;

$$= e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right)-\frac{1}{\rho}e^{\left(\frac{1-\rho}{\rho}\right)}\int_{1-\rho}^{1-\rho+\rho\beta_1 u} e^{-r/\rho}\sum_{k=0}^{\infty}\frac{(-1)^k\beta_2^k v^k r^{-k}}{\Gamma(k+1)}dr \tag{3.52}$$

we can say that the integrand is uniformly convergent, which makes it easier to pull summation outside the integral.

$$= e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right)-\frac{1}{\rho}e^{\left(\frac{1-\rho}{\rho}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\beta_2^k v^k r^{-k}}{\Gamma(k+1)}\int_{1-\rho}^{1-\rho+\rho\beta_1 u} e^{-r/\rho}.r^{-k}dr \tag{3.53}$$

$Let\ \zeta = \frac{r}{\rho} \Longrightarrow r = \rho\zeta\ and\ dr = \rho d\zeta\ , so\ the\ limit\ of\ the\ integral\ becomes\ \zeta \in \left[\frac{1-\rho}{\rho}, \frac{1-\rho+\rho\beta_1 u}{\rho}\right]$

Substitute back in equation (3.53) to get

$$= e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right)-\frac{1}{\rho}e^{\left(\frac{1-\rho}{\rho}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\beta_2^k v^k r^{-k}}{\Gamma(k+1)}\int_{1-\rho}^{1-\rho+\rho\beta_1 u} e^{-\zeta} * \rho^{-k}r^{-k}\,\rho dr \tag{3.54}$$

By decomposition of the integral and inverse limits of the integral in equation (3.54) we get

$$= e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right)-e^{\left(\frac{1-\rho}{\rho}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\beta_2^k v^k}{\Gamma(k+1)} *$$

$$\rho^{-k}\left[\int_{\frac{1-\rho}{\rho}}^{\infty} e^{-\zeta}\,r^{-k}dr - \int_{1-\rho+\rho\beta_1 u}^{\infty} r^{-k}e^{-\zeta}dr\right] \tag{3.55}$$

$$= e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right) - e^{\left(\frac{1-\rho}{\rho}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \beta_2^k v^k}{\Gamma(k+1)} \left[\rho^{-k} \int_{\frac{1-\rho}{\rho}}^{\infty} \frac{e^{-m}}{m^k} dr \ - \Gamma\left(1-k, -1+\frac{1}{\rho}+ u\beta_1\right)\left(-1+\frac{1}{\rho}+u\beta_1\right)^k (1-\rho+\rho\beta_1)^{-k}\right] \quad (3.56)$$

Taking $\varsigma = \frac{1-\rho}{\rho}$, we have;

$$F(u,v) \ = e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right) - e^{\left(\frac{1-\rho}{\rho}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \beta_2^k v^k}{\Gamma(k+1)} \times \left[\frac{1}{\rho}(1-\rho)^{1-k} Integral\ E\left(k, -1+\frac{1}{\rho}\right) - \Gamma\left(1-k, -1+\frac{1}{\rho}+u\beta_1\right)\left(-1+\frac{1}{\rho}+u\beta_1\right)^k (1-\rho+\rho\beta_1)^{-k}\right] \quad (3.57)$$

Therefore, the cumulative density function (cdf) of the New Bivariate Exponential Distribution is given as;

$$F(u,v) \ = e^{-u\beta_1}\left(-1+e^{-u\beta_1}\right) - e^{\left(\frac{1-\rho}{\rho}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \beta_2^k v^k}{\Gamma(k+1)}$$

$$\left[\frac{1}{\rho}(1-\rho)^{1-k} E_k\left(k, -1+\frac{1}{\rho}\right) \ - \Gamma\left(1-k, -1+\frac{1}{\rho}+u\beta_1\right)\left(-1+\frac{1}{\rho}+u\beta_1\right)^k (1-\rho+\rho\beta_1)^{-k}\right]$$

(3.58)

where $E_k(.)$ is an exponential integral and

$\Gamma(a,x)$is an incomplete gamma function

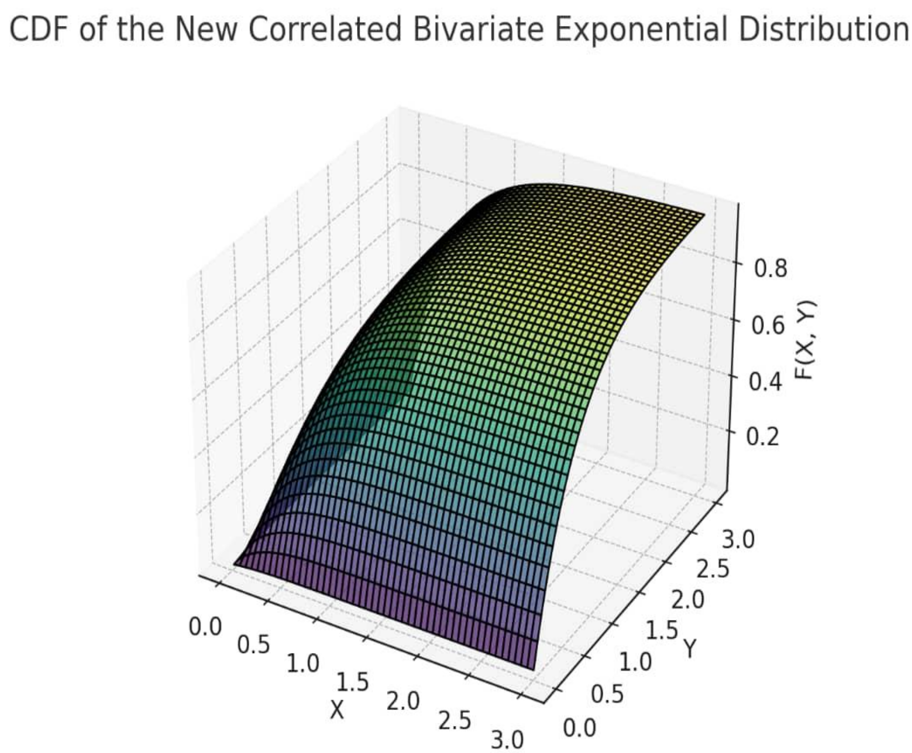


Fig.1.2 The cumulative density plot of NCBED

The 3D surface plot represents the cumulative density function (cdf) of the proposed new correlated bivariate exponential distribution (NCBED) for the chosen parameters.

3.2 The Statistical Properties of the NCBED Distribution.

In this section, the statistical properties of the NCBED are presented. The properties include: Moment Generating Function, the Mean and Variance, Characteristic function, Maximum Likelihood Estimation, Survival function and Hazard function.

Theorem 3.3: Let $X \; and \; Y$ be random variables that follows new correlated bivariate exponential distribution. Then the moment generating function

$M_{X,Y}(e^{t_{1x}+t_{2y}})$ is given by;

$$= \frac{1}{\rho} e^{\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \sum_{k=0}^{\infty} \left(\frac{t_2}{\beta_2}\right)^k \left(\frac{1}{\rho}\right)^k \left(1 - \frac{t_1}{\beta_1}\right)^{-k-1} \left[\Gamma(k+1) - \gamma(k+1, \frac{1-\rho}{\rho}\left(1 - \frac{t_1}{\beta_1}\right))\right] \quad (3.59)$$

where $\gamma(.,.)$ is an incomplete gamma function.

Proof:

Moment generating function (mgf) is defined as;

$$M_{X,Y}\left(e^{t_{1x}+t_{2y}}\right) = \int_{\mathbb{R}_x} \int_{\mathbb{R}_y} e^{t_{1x}+t_{2y}} f(x,y)dxdy \quad (3.60)$$

where $t_1 > 0$, $t_2 > 0$.

$$M_{X,Y}(e^{t_{1x}+t_{2y}}) = \beta_1\beta_2 \int_0^{\infty} \int_0^{\infty} e^{-\beta_1 x} e^{t_1 x} e^{t_2 y} \frac{1}{1-\rho+\rho\beta_1 x} \, e^{-\left(\frac{\beta_2 y}{1-\rho+\rho\beta_1 x}\right)} dxdy \quad (3.61)$$

$$Let\ r = 1-\rho + \rho\beta_1 x, dr = \rho\beta_1 dx \implies dx = \frac{dr}{\rho\beta_1}, and\ x = \frac{r-(1-\rho)}{\rho\beta_1}$$

$Also, -\beta_1 x = \frac{r+(1-\rho)}{\rho}$, $r \in [0,\infty]$

using substitution method, equation (3.61) becomes

$$M_{X,Y}(e^{t_{1x}+t_{2y}}) = \beta_1\beta_2 \int_{1-\rho}^{\infty} \int_0^{\infty} e^{-\left(\frac{r+(1-\rho)}{\rho}\right)} e^{t_1\left(\frac{r-(1-\rho)}{\rho\beta_1}\right)} e^{t_2 y} \frac{1}{r} \, e^{-\left(\frac{\beta_2 y}{r}\right)} \frac{dr}{\rho\beta_1} dy \quad (3.62)$$

$$= \frac{\beta_1\beta_2}{\rho\beta_1} e^{\left(\frac{1-\rho}{\rho}\right)} e^{-\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \int_{1-\rho}^{\infty} \int_0^{\infty} e^{-\left(\frac{r}{\rho}\right)} e^{t_1\left(\frac{r}{\rho\beta_1}\right)} \frac{dr}{r} \, e^{t_2 y - \left(\frac{\beta_2 y}{r}\right)} dy \quad (3.63)$$

$Let\ m = \frac{\beta_2 y}{r} \implies y = \frac{r}{\beta_2} m\ and\ dy = \frac{r}{\beta_2} dm$, $m \in [0,\infty]$

we substitute back into equation (3.63) to obtain;

$$= \frac{\beta_1\beta_2}{\rho\beta_1} e^{\left(\frac{1-\rho}{\rho}\right)} e^{-\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \int_{1-\rho}^{\infty} \int_0^{\infty} e^{-\left(\frac{r}{\rho}\right)} e^{\left(\frac{t_1 r}{\rho\beta_1}\right)} \, e^{t_2\left(\frac{r}{\beta_2} m\right) - m} drdm \quad (3.64)$$

$$= \frac{\beta_1\beta_2}{\rho\beta_1\beta_2} e^{\left(\frac{1-\rho}{\rho}\right)} e^{-\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \int_{1-\rho}^{\infty} \frac{1}{1-\frac{rt_2}{\beta_2}} e^{-r\left(\frac{1}{\rho}-\frac{t_1}{\rho\beta_1}\right)} dr \tag{3.65}$$

$Set\ a = \frac{1}{\rho} - \frac{t_1}{\rho\beta_1}\ , and\ b = \frac{t_2}{\beta_2}$, then we substitute and apply binomial expansion to obtain;

$$= \frac{1}{\rho} e^{\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \int_{1-\rho}^{\infty} \sum_{k=0}^{\infty} r^k b^k * e^{-ra} dr \tag{3.66}$$

$$= \frac{1}{\rho} e^{\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \sum_{k=0}^{\infty} b^k \int_{1-\rho}^{\infty} r^k * e^{-ra} dr \tag{3.67}$$

Next, we substitute for b in above equation to get;

$$= \frac{1}{\rho} e^{\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \sum_{k=0}^{\infty} \left(\frac{t_2}{\beta_2}\right)^k \int_{1-\rho}^{\infty} r^k * e^{-ra} dr \tag{3.68}$$

$Let\ ra = s\ \ which\ implies\ r = \frac{s}{a}, dr = \frac{ds}{a}, s\ \in [a(1-\rho), \infty]$, substitute and decompose the integral to obtain;

$$= \frac{1}{\rho} e^{\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \sum_{k=0}^{\infty} \left(\frac{t_2}{\beta_2}\right)^k a^{-k-1} \left[\int_0^{\infty} s^k e^{-s} ds - \int_0^{a(1-\rho)} s^k e^{-s} ds\right]$$

$$= \frac{1}{\rho} e^{\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \sum_{k=0}^{\infty} \left(\frac{t_2}{\beta_2}\right)^k a^{-k-1} [\Gamma(k+1) - \gamma(k+1, a(1-\rho))] \tag{3.69}$$

we substitute back for a in equation above to get;

$$= \frac{1}{\rho} e^{\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \sum_{k=0}^{\infty} \left(\frac{t_2}{\beta_2}\right)^k \left(\frac{1}{\rho}\right)^k \left(1 - \frac{t_1}{\beta_1}\right)^{-k-1} \left[\Gamma(k+1) - \gamma(k+1, \frac{1-\rho}{\rho}\left(1 - \frac{t_1}{\beta_1}\right))\right] \tag{3.70}$$

Hence, the moment generating function of the proposed new correlated bivariate distribution(NCBED) is given as;

$$M_{X,Y}(e^{t_1x+t_2y}) = \frac{1}{\rho} e^{\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{t_1(1-\rho)}{\rho\beta_1}\right)} \sum_{k=0}^{\infty} \left(\frac{t_2}{\beta_2}\right)^k \left(\frac{1}{\rho}\right)^k \left(1 - \frac{t_1}{\beta_1}\right)^{-k-1} \left[\Gamma(k+1) - \gamma(k+ 1, \frac{1-\rho}{\rho}\left(1 - \frac{t_1}{\beta_1}\right))\right] \tag{3.71}$$

where $\gamma(.,.)$ is an incomplete gamma function.

Theorem 3.4: Let $X\ and\ Y$ be random variables that follow correlated bivariate exponential distribution. Then the moment of the propose new correlated bivariate exponential distribution $E(x^r y^m)$ is given by;

$= \rho^m(\beta_1)^{-r}(\beta_2)^{-m}\Gamma(m+1)\Gamma(r+1)\mathbb{U}\left(-m,-m-r,\frac{1-\rho}{\rho}\right)$, where $\mathbb{U}(.,.,.)$is confluent hypergeometric function.

Proof:

We know that;

$$E\left(x^r y^m\right) = \int_{\mathbb{R}_x}\int_{\mathbb{R}_y} x^r y^m f(x,y)dydx \qquad (3.72)$$

$$E(x^r y^m) = \beta_1\beta_2\int_0^\infty\int_0^\infty \frac{x^r y^m}{1-\rho+\rho\beta_1 x}\ e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1+\rho(\beta_1 x-1)}\right)}dxdy \qquad (3.73)$$

$$Let\ \zeta = 1-\rho+\rho\beta_1 x,\ \ dx = \frac{d\zeta}{\rho\beta_1}, and\ x = \frac{\zeta-(1-\rho)}{\rho\beta_1}, with -\beta_1 x = \frac{-\zeta+(1-\rho)}{\rho\beta_1},$$

$and\ the\ limit\ of\ integral\ becomes\ \zeta \in [1-\rho,\infty]$

Again, let $\eta = \frac{\beta_2 y}{\zeta} \Longrightarrow y = \frac{\zeta}{\beta_2}\eta; with\ \eta \in [0,\infty]$

Substituting in equation (3.73) we have;

$$= \beta_1\beta_2\int_{1-\rho}^\infty\int_0^\infty \frac{\left(\frac{\zeta-(1-\rho)}{\rho\beta_1}\right)^r,\left(\frac{\zeta}{\beta_2}\eta\right)^m}{\zeta}\ e^{-\left\{\beta_1\left(\frac{\zeta-(1-\rho)}{\rho\beta_1}\right)*\zeta+\beta_2\left(\frac{\zeta}{\beta_2}\eta\right)\right\}}\frac{d\zeta}{\rho\beta_1}*\frac{\zeta d\eta}{\beta_2} \qquad (3.74)$$

$$Factor\ out\ \left(\frac{1}{\rho\beta_1}\right)^r\ and\ \left(\frac{\zeta}{\beta_2}\eta\right)^m\ in\ equation\ (3.73)\ to\ get;$$

$$\Longrightarrow \frac{e^{\frac{1-\rho}{\rho}}\left(\frac{1}{\rho\beta_1}\right)^r\left(\frac{1}{\beta_2}\right)^m}{\rho}\int_{1-\rho}^\infty\int_0^\infty(\zeta-(1-\rho))^r\zeta^m\eta^m e^{-\frac{\zeta}{\rho}}e^{-\eta}\ d\zeta d\eta \qquad (3.75)$$

$$= \frac{e^{\frac{1-\rho}{\rho}}\left(\frac{1}{\rho\beta_1}\right)^r\left(\frac{1}{\beta_2}\right)^m}{\rho}\int_{1-\rho}^{\infty}\zeta^m(\zeta-(1-\rho))^r\, e^{-\frac{\zeta}{\rho}}d\zeta\int_0^{\infty}\eta^m e^{-\eta}d\eta \tag{3.76}$$

Since $\int_0^{\infty}\eta^m e^{-\eta}d\eta = \Gamma(m+1)\ and\ we\ set\ w = \frac{\zeta}{\rho} \Longrightarrow \zeta = \rho w\ with\ d\zeta = \rho dw\ and\ w \in \left[\frac{1-\rho}{\rho},\infty\right)$ then equation (3.76) becomes;

$$= \rho^m e^{\frac{1-\rho}{\rho}}\left(\frac{1}{\rho\beta_1}\right)^r\left(\frac{1}{\beta_2}\right)^m\Gamma(m+1)\int_{\frac{1-\rho}{\rho}}^{\infty}w^m\,(\rho w-(1-\rho))^r e^{-w}dw \tag{3.77}$$

Expressing further as we obtain;

$$= \rho^{m+r} e^{\frac{1-\rho}{\rho}}\left(\frac{1}{\rho\beta_1}\right)^r\left(\frac{1}{\beta_2}\right)^m\Gamma(m+1)\int_{\frac{1-\rho}{\rho}}^{\infty}w^m\,(w-k)^r e^{-w}dw \tag{3.78}$$

Substituting further as we let $u = w-k, \Longrightarrow du = dw\ and\ u \in [0,\infty) with\ w = u+k$ we get;

$$= \rho^{m+r}\left(\frac{1}{\rho\beta_1}\right)^r\left(\frac{1}{\beta_2}\right)^m\Gamma(m+1)\int_{\frac{1-\rho}{\rho}}^{\infty}(u+k)^m\,u^r e^{-u}du \tag{3.79}$$

Now, adopting Binomial expansion $(u+k)^m = \sum_{i=0}^{m}\binom{m}{i}u^i k^{m-i}$

Hence,

$$= \rho^{m+r}\left(\frac{1}{\rho\beta_1}\right)^r\left(\frac{1}{\beta_2}\right)^m\Gamma(m+1)\sum_{i=0}^{m}\binom{m}{i}k^{m-i}\int_{\frac{1-\rho}{\rho}}^{\infty}u^{r+i}\,e^{-u}du \tag{3.80}$$

Therefore, the raw moment of the NCBED is given as;

$$= \rho^{m+r}\left(\frac{1}{\rho\beta_1}\right)^r\left(\frac{1}{\beta_2}\right)^m\Gamma(m+1)\Gamma(r+1)\mathbb{U}(-m,-m-r,k)$$

$$E(x^r,y^m) = \rho^m(\beta_1)^{-r}(\beta_2)^{-m}\Gamma(m+1)\Gamma(r+1)\mathbb{U}\left(-m,-m-r,\frac{1-\rho}{\rho}\right) \tag{3.81}$$

Where $\mathbb{U}(-m,-m-r,k)\ is\ the\ confluent\ hypergeometric\ function$

3.3 Special Cases of Moment

i. The first moment of the NCBED

$Take\ r = 1\ , m = 1, we\ get$

$$E(x^1, y^1) = \rho^1(\beta_1)^{-1}(\beta_2)^{-1}\Gamma(2)\Gamma(2)\mathbb{U}\left(-1,-2,\frac{1-\rho}{\rho}\right)$$

$$\frac{1}{\beta_1\beta_2}\left[(1-\rho)\left(\frac{2\rho}{1-\rho}+1\right)\right] = \frac{1+\rho}{\beta_1\beta_2} \tag{3.82}$$

ii. The second moment, of the NCBED

for $r = 2,\ m = 2$ we get

$$E(x^2, y^2) = \frac{\rho^2}{\beta_1^2\beta_2^2}\ \Gamma(3)\Gamma(3)\mathbb{U}\left(-2,-4,\frac{1-\rho}{\rho}\right)$$

$$= \frac{\rho^2}{\beta_1^2\beta_2^2}\ 4\mathbb{U}\left(-2,-4,\frac{1-\rho}{\rho}\right)$$

$further\ \ simplification\ yields$;

$$= \frac{4}{\beta_1^2\beta_2^2}\left[(1-\rho)^2\left(\frac{12\rho^2}{(1-\rho)^2}+\frac{6\rho}{1-\rho}+1\right)\right] = \frac{4(7\rho^2+4\rho+1)}{\beta_1^2\beta_2^2} \tag{3.83}$$

iii. The third moment about the origin; for $r = 3,\ m = 3$ we get

$$E(x^3, y^3) = \frac{\rho^3}{\beta_1^3\beta_2^3}\ 3!\,3!\ \mathbb{U}\left(-4,-8,\frac{1-\rho}{\rho}\right)$$

$$= \frac{36}{\beta_1^3\beta_2^3}\left(\frac{(1-\rho)^2\left(\frac{1680\rho^4}{(1-\rho)^4}+\frac{840\rho^3}{(1-\rho)^3}+\frac{180\rho^2}{(1-\rho)^2}+\frac{20\rho^4}{1-\rho}+1\right)}{\boldsymbol{\rho}}\right)$$

$$= \frac{36}{\beta_1^3\beta_2^3}\left(\frac{1680\rho^4+840\rho^3(1-\rho)+180\rho^2(1-\rho)^2+20\rho(1-\rho)^3+(1-\rho)^4}{\rho}\right) \tag{3.84}$$

Theorem 3.5: The mean of the NCBED random variable is given by;

$E(X) = \frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\left[\rho^2\Gamma(2) - \gamma\left(2, \frac{1-\rho}{\rho}\right) - (1-\rho)\rho\, e^{\left(1-\rho/\rho\right)}\right]$ where $\gamma(.,.)$is an incomplete gamma function.

Proof:

We know that;

$E(X) = \int\limits_{\mathbb{R}x}\int\limits_{\mathbb{R}y} xf(x,y)dxdy$, supports $0 \le x \le \infty, 0 \le y \le \infty, -\rho + \rho\beta_1 x \neq -1$ $with\ \mathbb{R}(\beta_1) > 0\ and\ \beta_2 > 0$.

$$E(X) = \beta_1\beta_2 \int_0^\infty \int_0^\infty \frac{1}{1-\rho+\rho\beta_1 x} * xe^{-\beta_1 x} * e^{-\left(\frac{\beta_2 y}{1+\rho(\beta_1 x-1)}\right)}dxdy \tag{3.85}$$

$Let\ k = 1-\rho + \rho\beta_1 x\ and\ Let\ u = \frac{\beta_2 y}{k} \Rightarrow y = \frac{k}{\beta_1}u\ and\ dy = \frac{k}{\beta_2}\ with\ k \in [1-\rho, \infty)\ , then\ x = \frac{k-(1-\rho)}{\rho\beta_1},$

using substitution method, in equation (3.85) to get;

$$E(X) = \frac{\beta_1\beta_2 e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1^2\beta_2}\int_{1-\rho}^\infty (k-(1-\rho))\, e^{-\left(\frac{k}{\rho}\right)}dk \int_0^\infty e^{-u}\, du \tag{3.86}$$

$$E(X) = \frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\int_{1-\rho}^\infty (k-(1-\rho))\, e^{-\left(\frac{k}{\rho}\right)}dk \tag{3.87}$$

Now, we multiply out in equation (3.87) to get;

$$= \frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\int_{1-\rho}^\infty k\, e^{-\left(\frac{k}{\rho}\right)}dk - (1-\rho)\int_{1-\rho}^\infty e^{-\left(\frac{k}{\rho}\right)}dk \tag{3.88}$$

Also, we set $s = \frac{k}{\rho} \Rightarrow k = \rho s\ and\ ds = \frac{1}{\rho}dk, so\ dk = \rho ds. with\ s \in \left[\frac{1-\rho}{\rho}, \infty\right)$

$$\Rightarrow \frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\left[\rho^2\int_{1-\rho}^\infty s\, e^{-(s)}ds - (1-\rho)\,\rho\int_{1-\rho}^\infty e^{-s}ds\right] \tag{3.89}$$

Next, we decompose the integral

$$\Rightarrow \frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\left[\rho^2\int_{1-\rho}^{\infty} s^{2-1}e^{-s}\,ds - (1-\rho)\rho\left(-e^{-s}\Big|_{\frac{1-\rho}{\rho}}^{\infty}\right]\right. \tag{3.90}$$

$$= \frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\left[\rho^2\int_{1-\rho}^{\infty} s^{2-1}e^{-s}\,ds - \int_0^{1-\rho} s^{2-1}e^{-s}\,ds - (1-\rho)\rho\left(-e^{-s}\Big|_{\frac{1-\rho}{\rho}}^{\infty}\right]\right. \tag{3.91}$$

$$= \frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\left[\rho^2\Gamma(2) - \gamma\left(2,\frac{1-\rho}{\rho}\right) - (1-\rho)\rho\left(0 + e^{\left(1-\rho/\rho\right)}\right)\right] \tag{3.92}$$

Therefore, the expected mean of the NCBED random variable X is given by;

$$E(X) = \frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\left[\rho^2\Gamma(2) - \gamma\left(2,\frac{1-\rho}{\rho}\right) - (1-\rho)\rho\, e^{\left(1-\rho/\rho\right)}\right] \tag{3.93}$$

Theorem 3.6:. Then the Variance of the NCBED random variable X is given by;

$$Var(X) = \frac{e^{\frac{1-\rho}{\rho}}}{\beta_1^2}\left[\left(\Gamma(3) - \gamma(3,\frac{1-\rho}{\rho})\right) - 2\rho^2(1-\rho)\left(\Gamma(2) - \gamma\left(2\,,\frac{1-\rho}{\rho}\right)\right) - \rho(1-\rho)^2\left(-e^{-\frac{1-\rho}{\rho}}\right)\right] - \left(\frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\left[\rho^2\Gamma(2) - \gamma\left(2,\frac{1-\rho}{\rho}\right) - (1-\rho)\rho\, e^{\left(1-\rho/\rho\right)}\right]\right)^2$$

where $\gamma(.,.,.)$ is an incomplete gamma function.

Proof:

The variance of the NCBED random variable X is written as; $Var(X) = E(X^2) - (E(X))^2$

since $E(X^2) = \int_{\mathbb{R}x}\int_{\mathbb{R}y} x^2 f(x,y)\,\mathrm{dxdy}$ $, 0 \le \mathrm{x} \le \infty, 0 \le \mathrm{y} \le \infty, -\rho + \rho\beta_1 x \neq -1\ with\ \beta_1 > 0\ and\ \beta_2 > 0$.

$$E(X^2) = \beta_1\beta_2 \int_0^\infty \int_0^\infty \frac{1}{1-\rho+\rho\beta_1 x} * x^2 e^{-\beta_1 x} * e^{-\left(\frac{\beta_2 y}{1+\rho(\beta_1 x-1)}\right)} dydx \tag{3.94}$$

$Let\ t = 1-\rho+\rho\beta_1 x \Longrightarrow dt = \rho\beta_1, and\ Let\ z = \frac{\beta_2 y}{t} \Longrightarrow y = \frac{t}{\beta_1} z\ and\ dy = \frac{t}{\beta_2} dz,$ $with\ t \in [1-\rho, \infty]\ and\ z \in [0,\infty], then\ x = \frac{t-(1-\rho)}{\rho\beta_1} \Longrightarrow dx = \frac{dt}{\rho\beta_1}\ \beta_1 x = \frac{-(t-(1-\rho))}{\rho}$

Now,

$$E(X^2) = \frac{\beta_1\beta_2}{\rho^3\beta_1^2} \int_{1-\rho}^\infty (t-(1-\rho))^2 * e^{-t/\rho} dt \int_0^\infty e^{-z}\, dz \tag{3.95}$$

$$E(X^2) = \frac{\beta_1\beta_2}{\rho^3\beta_1^2} \int_{1-\rho}^\infty (t-(1-\rho))^2 * e^{-t/\rho} dt \tag{3.96}$$

simplifying further we have;

$$E(X^2) = \frac{\beta_1\beta_2}{\rho^3\beta_1^2} \int_{1-\rho}^\infty (t)^2\, e^{-t/\rho} dt - 2(1-\rho)\int_{1-\rho}^\infty t e^{-t/\rho}\, dt + \int_{1-\rho}^\infty (1-\rho)^2 e^{-t/\rho} dt \tag{3.97}$$

Let $h = \frac{t}{\rho} \Longrightarrow dh = \frac{1}{\rho} dt, and\ dt = \rho dh, with\ t = \rho h\ and\ limit\ of\ the\ integral\ changes\ when\ t = (1-\rho) \Longrightarrow h = \frac{1-\rho}{\rho}\ and\ when\ t = \infty \Longrightarrow h = \infty; supports\ h \in \left[\frac{1-\rho}{\rho}, \infty\right)$

Further simplification gives;

$$= \frac{e^{\frac{1-\rho}{\rho}}}{\rho^3\beta_1^2}\left[\rho^3 \int_{\frac{1-\rho}{\rho}}^\infty h^2 e^{-h} dh - 2\rho^2(1-\rho)\int_{\frac{1-\rho}{\rho}}^\infty h e^{-h} dh + \rho(1-\rho)^2 \int_\infty^{\frac{1-\rho}{\rho}} e^{-h} dh\right] \tag{3.98}$$

The first integral from equation (3.98) gives;

$$\rho^3 \int_{\frac{1-\rho}{\rho}}^\infty h^2\ e^{-h} dh = \rho^3 \left(\Gamma(3) - \gamma(3, \tfrac{1-\rho}{\rho})\right) \tag{3.99a}$$

The second integral from equation (3.98) gives;

$$2\rho^2(1-\rho)\int_{\frac{1-\rho}{\rho}}^{\infty} h\, e^{-h} dh \;=\; 2\rho^2(1-\rho)\left[\Gamma(2)-\gamma\left(2\,,\frac{1-\rho}{\rho}\right)\right] \tag{3.99b}$$

The last integral from equation (3.98) decompose to;

$$\rho(1-\rho)^2\int_{\infty}^{\frac{1-\rho}{\rho}} e^{-h} dh \;=\; \rho(1-\rho)^2\left(-e^{-\frac{1-\rho}{\rho}}\right) \tag{3.99c}$$

Therefore,

$$E(X^2)=\frac{e^{\frac{1-\rho}{\rho}}}{\beta_1^2}\left[\left(\Gamma(3)-\gamma(3,\frac{1-\rho}{\rho})\right)-2\rho^2(1-\rho)\left(\Gamma(2)-\gamma\left(2\,,\frac{1-\rho}{\rho}\right)\right)-\rho(1-\rho)^2\left(-e^{-\frac{1-\rho}{\rho}}\right)\right] \tag{3.100}$$

Hence, the Variance of the NCBED random variable X is given as;

$$Var(X)=\frac{e^{\frac{1-\rho}{\rho}}}{\beta_1^2}\left[\left(\Gamma(3)-\gamma(3,\frac{1-\rho}{\rho})\right)-2\rho^2(1-\rho)\left(\Gamma(2)-\gamma\left(2\,,\frac{1-\rho}{\rho}\right)\right)-\rho(1-\rho)^2\left(-e^{-\frac{1-\rho}{\rho}}\right)\right]-\left(\frac{e^{\left(1-\rho/\rho\right)}}{\rho^2\beta_1}\left[\rho^2\Gamma(2)-\gamma\left(2,\frac{1-\rho}{\rho}\right)-(1-\rho)\rho\, e^{\left(1-\rho/\rho\right)}\right]\right)^2$$

3.4 The Characteristic Function of proposed New Correlated Bivariate Exponential Distribution (NCBED).

A mathematical tool used to characterize a random variable's distribution is called a characteristic function. By transforming a random variable into the frequency domain, this complex-valued function enhances the analysis of the variable's characteristics and connections to other random variables.

Theorem 3.5: Let $X\ and\ Y$ be random variables that follows correlated bivariate exponential distribution. Then the characteristic function $\phi_{X,Y}(t_1,t_2)=E\left[e^{i(t_1X+t_2Y)}\right]$ is defined as;

$$\phi_{X,Y}(t_1,t_2)=E\left[e^{i(t_1X+t_2Y)}\right] \tag{3.101}$$

Where; $E[\cdot]$ denotes the expectation operator.

$t_1,t_2 \in \mathbb{R}\ are\ real\ numbers$

$i=\sqrt{-1}\ is\ the\ imarginary\ unit.$

Proof:

$$\phi_{X,Y}(t_1,t_2) = \beta_1\beta_2 \int_0^\infty \int_0^\infty \frac{e^{i(t_1x+t_2y)}}{1-\rho+\rho\beta_1 x} e^{-\left(\frac{\beta_2 y}{1+\rho(\beta_1 x-1)}\right)} dxdy \tag{3.102}$$

$Let\ r = 1-\rho+\rho\beta_1 x \Rightarrow dr = \rho\beta_1, and\ Let\ m = \frac{\beta_2 y}{t} \Rightarrow y = \frac{t}{\beta_1} m\ and\ dy = \frac{t}{\beta_2} dm,$ $with\ r \in [1-\rho,\infty]\ and\ m \in [0,\infty], then\ x = \frac{r-(1-\rho)}{\rho\beta_1} \Rightarrow dx = \frac{dr}{\rho\beta_1}\ \beta_1 x = \frac{-(r-(1-\rho))}{\rho}$

Substituting in equation (3.102) we have;

$$= \frac{e^{-it_1\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{1-\rho}{\rho}\right)}}{\rho} \int_{1-\rho}^\infty e^{\left(\frac{it_1 r}{\rho\beta_1}\right)} e^{-\left(\frac{r}{\rho}\right)} dr \int_0^\infty e^{-m\left(\frac{-it_2 r}{\beta_2}+1\right)} dm \tag{3.103}$$

$Let\ k = \left(\frac{it_2 r}{\beta_2}+1\right)\ and\ we\ set\ u = km \Rightarrow du = kdm\ and\ dm = \frac{du}{k}$, so, it becomes;

$$= \frac{e^{-it_1\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{1-\rho}{\rho}\right)}}{\rho} \int_{1-\rho}^\infty e^{\left(\frac{it_1 r}{\rho\beta_1}\right)} e^{-\left(\frac{r}{\rho}\right)} * \frac{1}{\left(1-\frac{it_2 r}{\beta_2}\right)} dr \tag{3.104}$$

Now, we $take\ a = \frac{it_1}{\rho\beta_1}\ and\ b = \frac{it_2}{\beta_2}$ we have;

$$= \frac{e^{-it_1\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{1-\rho}{\rho}\right)}}{\rho} \int_{1-\rho}^\infty \frac{e^{r\left(\frac{1}{\rho}-a\right)}}{1-br} dr \tag{3.104}$$

Also, we $take\ c = \frac{1}{\rho} - a, and\ set\ h = 1-br \Rightarrow dh = bdr\ and\ dr = \frac{dh}{b},\ r = \frac{1-h}{b}\ with\ h \in [1-b(1-\rho),\infty], then\ equation\ (3.104)\ becomes;$ S

$$\Rightarrow \frac{e^{-it_1\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{1-\rho}{\rho}\right)} e^{-\frac{c}{b}}}{b\rho} \int_{1-b(1-\rho)}^\infty \frac{e^{ch}}{h} dh \tag{3.105}$$

Hence, the characteristic function of the new correlated bivariate exponential distribution (NCBED) is given as;

$$\phi_{X,Y}(t_1,t_2) = \frac{e^{-it_1\left(\frac{1-\rho}{\rho}\right)} e^{\left(\frac{1-\rho}{\rho}\right)} e^{-\frac{c}{b}}}{b\rho} \Gamma\big(0,-c(1+b(-1+\rho))\big) \tag{3.106}$$

Where;

$a = \frac{it_1}{\rho\beta_1}\ , b = \frac{it_2}{\beta_2}\ \ c = \frac{1}{\rho} - a \Rightarrow \frac{1}{\rho} - \frac{it_1}{\rho\beta_1}$

3.5 The Survival Function of the New Correlated Bivariate Exponential Distribution(NCBED).

In a bivariate exponential distribution, the survival function is an important component in describing the join behavior of two random variables. It provides information on lifespan modeling, risk and dependability for range of application.

Let us assume that our data has a bivariate marginal consist of independent identically distributed random variables $X, Y \sim F$.

Theorem 3.6: Let $X \; and \; Y$ be random variables that follows correlated bivariate exponential distribution. Then the survival function of the propose new correlated bivariate exponential distribution $S(.,.)$ is given by;

$$\begin{aligned} S(x,y) = 1 - e^{-u\beta_1}\left(-1 + e^{-u\beta_1}\right) \\ - e^{\left(\frac{1-\rho}{\rho}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \beta_2^k v^k}{\Gamma(k+1)} \Bigg[\frac{1}{\rho}(1-\rho)^{1-k} E_k\left(k, -1 + \frac{1}{\rho}\right) \\ - \Gamma\left(1 - k, -1 + \frac{1}{\rho} + u\beta_1\right)\left(-1 + \frac{1}{\rho} + u\beta_1\right)^k (1 - \rho + \rho\beta_1)^{-k}\Bigg] \end{aligned}$$

Proof:

Since, we know that;

$S(x,y) = \int_x \int_y f(x,y)dxdy = 1 - F(u,v)$, which supports $x \in [0,\infty], y \in [0,\infty] \; and \; u, v \in [0,\infty]$.

Then, the survival function of the NCBED is defined as;

$$S(x,y) = 1 - e^{-u\beta_1}\left(-1 + e^{-u\beta_1}\right) - e^{\left(\frac{1-\rho}{\rho}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \beta_2^k v^k}{\Gamma(k+1)}$$

$$\left[\frac{1}{\rho}(1-\rho)^{1-k} E_k\left(k, -1 + \frac{1}{\rho}\right) - \Gamma\left(1 - k, -1 + \frac{1}{\rho} + u\beta_1\right)\left(-1 + \frac{1}{\rho} + u\beta_1\right)^k (1 - \rho + \rho\beta_1)^{-k}\right] \quad (3.107)$$

3.6 The Hazard Function

Hazard function describes the instantaneous failure rate at a given time t, given that an individual already survived past time t. The hazard function of the proposed NCBED is defined as

Theorem 3.7: Let $X\ and\ Y$ be random variables that follows correlated bivariate exponential distribution. Then the hazard function of the new correlated bivariate exponential distribution $h(.,.)$ is given by;

$$h(x,y) = \frac{f(x,y)}{S(x,y)} \tag{3.108}$$

$$= \frac{\frac{\beta_1\beta_2}{1-\rho+\rho\beta_1 x} e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1+\rho(\beta_1 x-1)}\right)}}{1-e^{-u\beta_1}(-1+e^{-u\beta_1})-e^{\left(\frac{1-\rho}{\rho}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\beta_2^k v^k}{\Gamma(k+1)}\left[\frac{1}{\rho}(1-\rho)^{1-k}E_k\left(k,-1+\frac{1}{\rho}\right)-\Gamma\left(1-k,-1+\frac{1}{\rho}+u\beta_1\right)\left(-1+\frac{1}{\rho}+u\beta_1\right)^k(1-\rho+\rho\beta_1)^{-k}\right]} \tag{3.109}$$

3.7 The Shannon Entropy of the NCBED

The father of information theory, Claude Shannon, is honored by the term "Shannon entropy." This idea was first presented by Shannon in his (1948) paper, which established the groundwork for contemporary information theory. The beauty of Shannon Entropy is that, it measures the uncertainty of information in the system based on the probabilities of different situations within the system.

Since Shannon Entropy gives insight on average rate at which information is produced by a stochastic source of data. The higher the Shannon entropy the bigger the information is produced by a new value in process.

Theorem 3.8: Let X, Y be two independent identically distributed random variates such that $(x_{1,}y_1),(x_{2,}y_2) \dots \dots (x_{m,}y_m)$. The Shannon Entropy of the NCBED is given as;

$$H(x,y) = \log_e(\beta_1\beta_2) - \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2}\left(\log_e\left(\frac{1-\rho}{\rho}\right) + \log_e\rho\right) - \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2}\left[\Gamma(2) - \gamma\left(2,\frac{1-\rho}{\rho}\right) - \rho(1-\rho)e^{\left(\frac{1-\rho}{\rho}\right)} + \rho\Gamma(2)\right]. \quad (3.110)$$

Proof:

We have that;

$$H(x,y) = \iint_{\mathbb{R}_x\mathbb{R}_y} -\log_e\big(f(x,y)\big)f(x,y)dxdy \quad (3.111)$$

$$H(x,y) = \int_0^\infty\int_0^\infty -\log_e\left(\frac{\beta_1\beta_2}{1-\rho+\rho\beta_1 x}\, e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1+\rho(\beta_1 x-1)}\right)}\right) * \frac{\beta_1\beta_2}{1-\rho+\rho\beta_1 x}\, e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1+\rho(\beta_1 x-1)}\right)}dxdy \quad (3.112)$$

Expanding equation (3.112) to get;

$$H(x,y) = -\beta_1\beta_2\int_0^\infty\int_0^\infty\left[\log_e\beta_1\beta_2 - \log_e(1-\rho+\rho\beta_1 x) - \frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1-\rho+\rho\beta_1 x}\right]\times \frac{1}{1-\rho+\rho\beta_1 x}e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1-\rho+\rho\beta_1 x}\right)}dxdy \quad (3.113)$$

So, we have a break-down of equation (3.113) below as;

$$-\beta_1\beta_2\int_0^\infty\int_0^\infty \log_e\beta_1\beta_2 \times \frac{1}{(1-\rho+\rho\beta_1 x)}\, e^{-\left\{\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1-\rho+\rho\beta_1 x}\right\}}dxdy \quad (3.114a)$$

$$-\int_0^\infty\int_0^\infty \frac{\log_e(1-\rho+\rho\beta_1 x)}{(1-\rho+\rho\beta_1 x)} \times e^{-\left\{\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1-\rho+\rho\beta_1 x}\right\}}dxdy \quad (3.114b)$$

$$\int_0^\infty\int_0^\infty \frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{(1-\rho+\rho\beta_1 x)^2} \times e^{-\left\{\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1-\rho+\rho\beta_1 x}\right\}}dxdy \quad (3.114c)$$

From equations (3.114a) to get;

Let $k = 1-\rho+\rho\beta_1 x \Rightarrow x = \frac{k-(1-\rho)}{\rho\beta_1}, dk = \rho\beta_1 dx \Rightarrow dx = \frac{dk}{\rho\beta_1}$ $with\ k \in [1-\rho,\infty]$

$and\ let\ t = \frac{\beta_2 y}{k} \Longrightarrow y = \left(\frac{k}{\beta_2}\right)t, and\ dy = \frac{k}{\beta_2}dt\ , with\ t \in [0, \infty)$

substituting to get;

$$-\frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\rho} log_e(\beta_1\beta_2) \int_{1-\rho}^{\infty} e^{-\left(\frac{k}{\rho}\right)} dk \int_0^{\infty} e^{-t}\, dt \tag{3.115}$$

Again, $Let\ h = \frac{k}{\rho} \Longrightarrow k = \rho dh\ with\ k \in \left[\frac{1-\rho}{\rho}, \infty\right]$

we substitute further to obtain equation;

$$-e^{\left(\frac{1-\rho}{\rho}\right)} log_e(\beta_1\beta_2)\ [-e^{-h} \Big|_{1-\rho/\rho}^{\infty} = log_e(\beta_1\beta_2) \tag{3.116}$$

Next, from equations (3.114b) ;

Let $u = 1-\rho+\rho\beta_1 x \Longrightarrow x = \frac{u-(1-\rho)}{\rho\beta_1}, du = \rho\beta_1 dx \Longrightarrow dx = \frac{du}{\rho\beta_1}\ with\ u \in [1-\rho, \infty)$

$and\ let\ r = \frac{\beta_2 y}{u} \Longrightarrow y = \left(\frac{u}{\beta_2}\right)r, and\ dy = \frac{u}{\beta_2}dr\ , with\ r \in [0, \infty)$

So, we substitute into equation (3.114b) to obtain;

$$= \int_{1-\rho}^{\infty}\int_0^{\infty} \frac{log_e\, u}{u} \times e^{-\left\{\frac{\beta_1\left(\frac{u-(1-\rho)}{\rho\beta_1}\right)u + \beta_2\left(\frac{u}{\beta_2}\right)}{u}\right\}} \frac{du}{\rho\beta_1}\frac{udr}{\beta_2} \tag{3.117}$$

$$= \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\rho\beta_1\beta_2} \int_{1-\rho}^{\infty} \log_e u\ e^{-\left(\frac{u}{\rho}\right)} du \times \int_0^{\infty} e^{-r}\, dr \tag{3.118}$$

Also, we $Let\ z = \frac{u}{\rho} \Longrightarrow u = \rho z\ , du = \rho dz, with\ z \in \left[\frac{1-\rho}{\rho}, \infty\right)$

substituting into equation (3.118) to get;

$$= \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2} \int_{1-\rho}^{\infty} \log_e(\rho z)\ e^{-(z)} \rho dz \tag{3.119}$$

$$= \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2} \left[\int_{\frac{1-\rho}{\rho}}^{\infty} log_e(z) e^{-z} dz + log_e \rho\ [-e^{-z} \Big|_{1-\rho/\rho}^{\infty}\right] \tag{3.120}$$

$$= \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2}\left(log_e\left(\frac{1-\rho}{\rho}\right) + log_e\rho\right) \tag{3.121}$$

Again, from equation (3.114c)

$$Let\ k = 1-\rho+\rho\beta_1 x \Rightarrow x = \frac{k-(1-\rho)}{\rho\beta_1}, dk = \rho\beta_1 dx \Rightarrow dx = \frac{dk}{\rho\beta_1}\ with\ k \in [1-\rho, \infty]$$

$$and\ let\ s = \frac{\beta_2 y}{k} \Rightarrow y = \left(\frac{k}{\beta_2}\right)s, and\ dy = \frac{k}{\beta_2}ds\ , with\ s \in [0,\infty)$$

Now, we substitute in equation (3.114c) to get;

$$= \frac{1}{\rho\beta_1\beta_2}\int_{1-\rho}^{\infty}\int_0^{\infty}\left(\frac{(k-(1-\rho))}{\rho} + s\right) \times e^{-\left(\frac{k}{\rho}\right)} e^{\left(\frac{1-\rho}{\rho}\right)} e^{-s}\ kdkds \tag{3.122}$$

Also, we $Let\ \xi = \frac{k}{\rho} \Rightarrow k = \rho\xi\ , dk = \rho d\xi, with\ \xi \in \left[\frac{1-\rho}{\rho}, \infty\right)$ and substituting to obtain;

$$= \frac{\rho^2 e^{\left(\frac{1-\rho}{\rho}\right)}}{\rho^2\beta_1\beta_2}\left[\int_{\frac{1-\rho}{\rho}}^{\infty}\xi e^{-\xi}d\xi - \rho(1-\rho)\int_{\frac{1-\rho}{\rho}}^{\infty}e^{-\xi}d\xi + \rho\int_0^{\infty}se^{-s}ds\right] \tag{3.123}$$

$$= \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2}\left[\int_0^{\infty}\xi^{2-1}e^{-\xi}d\xi - \int_0^{1-\rho}\xi^{2-1}e^{-\xi}d\xi - \rho(1-\rho)\left(-e^{-\infty} + e^{-\left(\frac{1-\rho}{\rho}\right)}\right) + \rho\int_0^{\infty}s^{2-1}e^{-s}ds\right] \tag{3.124}$$

Hence, equation (3.114c) becomes;

$$= \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2}\left[\Gamma(2) - \gamma\left(2, \frac{1-\rho}{\rho}\right) - \rho(1-\rho)e^{\left(\frac{1-\rho}{\rho}\right)} + \rho\Gamma(2)\right] \tag{3.125}$$

Hence, the Shannon Entropy of the NCBED is obtained as;

$$H(x,y) = log_e(\beta_1\beta_2) - \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2}\left(log_e\left(\frac{1-\rho}{\rho}\right) + log_e\rho\right) - \frac{e^{\left(\frac{1-\rho}{\rho}\right)}}{\beta_1\beta_2}\left[\Gamma(2) - \gamma\left(2, \frac{1-\rho}{\rho}\right) - \rho(1-\rho)e^{\left(\frac{1-\rho}{\rho}\right)} + \rho\Gamma(2)\right] \tag{3.126}$$

3.9 The Maximum Likelihood Estimation of the New Correlated Bivariate Exponential Distribution (NCBED).

Theorem 3.9: Let $(X_1, Y_1), (X_2, Y_2), .., (X_n, Y_n)$, be a random variable that follows a bivariate exponential distribution of NCBED with the probability density function

$f(x, y) = \frac{\beta_1\beta_2}{1-\rho+\rho\beta_1 x} e^{-\left(\frac{\beta_1 x(1-\rho+\rho\beta_1 x)+\beta_2 y}{1+\rho(\beta_1 x-1)}\right)}$, then there parameter estimations are given as;

$$\hat{\beta}_1 = \frac{2B^2 + C\left(\sqrt[3]{2}C\left(-\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}\right)^{\frac{2}{3}} - 6A\right)}{3 \times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}} - \frac{B}{3C}$$

$$\hat{\beta}_2 = \frac{n}{\sum_{i=1}^{n}\frac{y_i}{(1-\rho+\rho\hat{\beta}_1 x)}}.$$

were,

$A = (1-\rho)(2\rho-1)n\bar{x} + \rho\beta_2\sum XY$, $B = 2\rho(1-\rho)\sum X^2$, $C = \rho^2\sum X^3$, $D = n(1-\rho)^2$

Proof:

We use the Joe (1997) estimation procedures to estimate the parameters β_1, β_2 *and* ρ. The likelihood function is given as;

$$L(\beta_1, \beta_2 ; \rho) = \prod_{i=1}^{n} f(x_i y_i) \tag{3.127}$$

$$= \frac{(\beta_1\beta_2)^n}{\prod_{i=1}^{n}(1-\rho+\rho\beta_1 x_i)} e^{-\sum_{i=1}^{n}\frac{1}{(1-\rho+\rho\beta_1 x_i)} * \{\beta_1 x_i(1-\rho+\rho\beta_1 x_i)+\beta_2 y_i\}} \tag{3.128}$$

so, we take the logarithm of both sides as in equation (3.128) which yields;

$$\ln L(\beta_1, \beta_2 ; \rho) = n(\log_e \beta_1 + \log_e \beta_2) - \sum_{i=1}^{n}\log(1-\rho+\rho\beta_1 x_i) - \sum_{i=1}^{n}\frac{\beta_1 x_i(1-\rho)+\rho\beta_1^2 x_i^2+\beta_2 y_i}{(1-\rho+\rho\beta_1 x)} \tag{3.129}$$

Now, we take partial derivatives with respect to parameters $\beta_1, \beta_2\ and\ \rho$ respectively and equate to zero with $i = 1(n)$and set $L = \partial L(\beta_1, \beta_2\ ;\ \rho)$.

$$\frac{\partial L(\beta_1,\beta_2\,;\rho)}{\partial\beta_1} =$$

$$\frac{n}{\beta_1} - \sum_{i=1}^{n}\frac{\rho x_i}{(1-\rho+\rho\beta_1 x)} - \sum_{i=1}^{n}\left(\frac{(1-\rho)x_i+2\rho x_i^2\beta_1}{(1-\rho+\rho\beta_1 x)} - \frac{\rho x_i\big((1-\rho)x_i\beta_1-\rho x_i^2\beta_1^2+y_i\beta_2\big)}{(1-\rho+\rho\beta_1 x)^2}\right) = 0 \qquad (3.130)$$

we expand the numerator in equation (3.130) as;

$$\sum_{i=1}^{n}\left\{\frac{n}{\beta_1} - \frac{\rho x_i}{(1-\rho+\rho\beta_1 x_i)} - \left(\frac{(1-\rho)x_i+2\rho x_i^2\beta_1}{(1-\rho+\rho\beta_1 x_i)} - \frac{\rho x_i\big((1-\rho)x_i\beta_1-\rho x_i^2\beta_1^2+y_i\beta_2\big)}{(1-\rho+\rho\beta_1 x_i)^2}\right)\right\} \qquad (3.131)$$

by algebraic expression, take L.C.M and equate to zero, we have;

$$\sum_{i=1}^{n}\left\{\frac{(1-\rho+\rho\beta_1 x_i)^2-\beta_1\rho x_i(1-\rho+\rho\beta_1 x_i)-\beta_1\big((1-\rho)^2x_i-\rho x_i y_i\beta_2+2\rho(1-\rho)x_i^2\beta_1+\rho^2x_i^3\beta_1^2\big)}{\beta_1(1-\rho+\rho\beta_1 x_i)^2}\right\} = 0 \qquad (3.132)$$

so, expressing only the numerator in equation (3.132) as;

$$= (1-\rho)^2 + 2\rho(1-\rho)\beta_1 x_i + \rho^2 {x_i}^2\beta_1^2 - \rho(1-\rho)x_i\beta_1 - \rho^2{x_i}^2\beta_1^2 - [(1-\rho)^2x_i - \rho x_i y_i\beta_2]\beta_1 - 2\rho(1-\rho){x_i}^2\beta_1^2 - \rho^2{x_i}^3\beta_1^3 \qquad (3.133)$$

which implies

$$= (1-\rho)^2 + [\rho(1-\rho)x - (1-\rho)^2x + \rho x y\beta_2]\beta_1 - 2\rho(1-\rho)x^2\beta_1^2 - \rho^2x^3\beta_1^3 \qquad (3.134)$$

Now we can write equation (3.134) as;

$$\sum_{i=1}^{n}\frac{(1-\rho)^2+[\rho(1-\rho)x_i-(1-\rho)^2x_i+\rho x_i y_i\beta_2]\beta_1-2\rho(1-\rho){x_i}^2\beta_1^2-\rho^2{x_i}^3\beta_1^3}{\beta_1(1-\rho+\rho\beta_1 x_i)^2} = 0 \qquad (3.135)$$

So, equation (3.135) becomes;

$$\sum_{i=1}^{n}\frac{[(1-\rho)(2\rho-1)x_i+\rho x_i y_i\beta_2]\beta_1-2\rho(1-\rho){x_i}^2\beta_1^2-\rho^2{x_i}^3\beta_1^3}{\beta_1(1-\rho+\rho\beta_1 x_i)^2} = -\sum_{i=1}^{n}\frac{(1-\rho)^2}{\beta_1(1-\rho+\rho\beta_1 x_i)^2} \qquad (3.136)$$

we can equate the numerators since it has same base as;

$$\sum_{i=1}^{n}[(1-\rho)(2\rho-1)x_i + \rho x_i y_i\beta_2]\beta_1 - 2\rho(1-\rho){x_i}^2\beta_1^2 - \rho^2{x_i}^3\beta_1^3 = -\sum_{i=1}^{n}(1-\rho)^2 \qquad (3.137)$$

Now, summing through equation we obtain;

$$[(1-\rho)(2\rho-1)\sum_{i=1}^{n}x_i + \rho\beta_2\sum_{i=1}^{n}x_i y_i]\beta_1 - 2\rho(1-\rho)\sum_{i=1}^{n}{x_i}^2\beta_1^2 - \rho^2\sum_{i=1}^{n}{x_i}^3\beta_1^3 = -n(1-\rho)^2 \qquad (3.138)$$

Which forms a cubic polynomial of degree three by Cardano Tartaglia (1545).

$$A\beta_1 + B\beta_1^2 + C\beta_1^3 + D = 0$$

Where;

$A = (1-\rho)(2\rho-1)n\bar{X} + \rho\beta_2 \sum_{i=1}^{n} x_i y_i$

$B = -2\rho(1-\rho)\sum_{i=1}^{n} x_i^2$

$C = -\rho^2 \sum_{i=1}^{n} x_i^3$

$D = -n(1-\rho)^2$

After some mathematical simplifications, we obtain these results of a real root and imaginary roots of the cubic polynomial function; $C\beta_1^3 + B\beta_1^2 + A\beta_1 + D = 0$

$$\hat{\beta}_1 = \frac{2B^2 + C\left(\sqrt[3]{2}C\left(-\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}\right)^{\frac{2}{3}} - 6A\right)}{3 \times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}} - \frac{B}{3C}$$

$$\hat{\beta}_1 = \frac{2(-1)^{\frac{2}{3}}B^2 - 6(-1)^{\frac{2}{3}}AC - \sqrt[3]{-2}C^2\left(-\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}\right)^{\frac{2}{3}}}{3 \times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}} - \frac{B}{3C}$$

$$\hat{\beta}_1 = \frac{\sqrt{-1}\left(C\left(6A + \sqrt[3]{-2}C\left(-\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}\right)^{\frac{2}{3}}\right) - 2B^2\right)}{3 \times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}} - \frac{B}{3C}$$

See Appendix A

REMARKS ON CUBIC POLYNOMIAL

In solving the cubic polynomial involving the following parameters A,B,C &D of which non of these parameters are zeros, we invoke the Cardano (1545) methods on $A\beta + B\beta^2 + C\beta^3 + D = 0$ which entails the elimination of variables with the highest factor in that polynomial and reconvert the resulting equation by substituting $x = \frac{B}{3C} + \beta$

If $C = 0$, $A \neq 0, \ \beta \neq 0$. The resulting equation becomes a quadratic which is easy to solve; $\beta = \frac{-\sqrt{A^2} - 4BD}{2B} - A$ and $\beta = \frac{\sqrt{A^2} - 4BD}{2B} - A$

If $C = 0$, $B = 0$, the polynomial becomes a Linear equation upon which $\beta = \frac{-B}{A}$

If $D = 0$, $C = 0$, The polynomial becomes a quadratic equation that is

$$\beta = A\beta + B\beta^2 = 0$$
$$\Rightarrow \beta(A + B\beta) = 0$$
$$therefore,$$
$$\beta = 0 \text{ or } -\frac{A}{B}$$

Next, we differentiate equation (3.129) partially with respect to β_2 and ρ we obtain;

$$\frac{\partial L}{\partial \beta_2} = \frac{n}{\hat{\beta}_2} - \sum_{i=1}^{n} \frac{y_i}{(1-\rho+\rho\hat{\beta}_1 x_i)} \tag{3.139}$$

Therefore $\hat{\beta}_2 = \frac{y_i}{(1-\rho+\rho\hat{\beta}_1 x)}$ (3.140)

Also,

$$\frac{\partial L}{\partial \rho} = \frac{(-1+x_i\hat{\beta}_1)}{(1-\rho+\rho\hat{\beta}_1 x_i)} \tag{3.141}$$

Again, we estimate $\frac{\partial^2 L}{\partial \beta_1^2}$ from equation (3.135) to obtain;

$$\frac{\partial^2 L}{\partial \beta_1^2} = \frac{2\rho x_i^2}{1-\rho+\rho\beta_1 x_i} - \frac{2\rho x_i\left((1-\rho)x_i+2\rho\beta_1 x_i^2\right)}{(1-\rho+\rho\beta_1 x_i)^2} + \frac{2\rho^2 x_i^2\left((1-\rho)x_i\beta_1+\rho x_i^2\beta_1^2+y_i\beta_2\right)}{(1-\rho+\rho\beta_1 x_i)^3} \tag{3.142}$$

Also, we estimate $\frac{\partial^2 L}{\partial \beta_2^2}$ from equation (3.135) to obtain;

$$\frac{\partial^2 L}{\partial \beta_2^2} = -\frac{y_i}{(1-\rho+\rho\widehat{\beta}_1 x)} \tag{3.143}$$

Also, we estimate $\frac{\partial^2 L}{\partial \rho^2}$ from equation (3.135) to obtain;

$$= \frac{-1+\beta_1 x_i}{1-\rho+\rho\beta_1 x_i} \tag{3.144}$$

Next, we estimate $\frac{\partial^2 L}{\partial \beta_1 \beta_2}$ from equation (3.135) to obtain;

$$\frac{\partial^2 L}{\partial \beta_1 \beta_2} = -\sum_{i=1}^{n} \frac{\rho x_i y_i}{(1-\rho+\rho\beta_1 x)} \tag{3.145}$$

Next, we estimate $\frac{\partial^2 L}{\partial \beta_1 \partial \rho}$ from equation (3.135) to obtain;

$$= -\frac{\rho x_i(-1+\beta_1 x_i)}{(1-\rho+\rho\beta_1 x_i)^2} \tag{3.146}$$

The Fishers information measures the amount of information that an observable random variable X carries about an unknown parameter θ of a distribution that models X. Also, it is the matrix of the second cross moment of the score vector. The Fishers information Matrix is given as;

$$\mathrm{I}_{ij}(\theta) = E_\theta \begin{bmatrix} \mathrm{I}_{\beta_1\beta_2} & \mathrm{I}_{\beta_1\beta_2} & \mathrm{I}_{\beta_1\rho} \\ \mathrm{I}_{\beta_2\beta_1} & \mathrm{I}_{\beta_2\beta_1} & \mathrm{I}_{\beta_2\rho} \\ \mathrm{I}_{\beta_1\rho} & \mathrm{I}_{\beta_2\rho} & \mathrm{I}_{\rho\rho} \end{bmatrix}$$

Where $\mathrm{I}_{ij}(\theta) = \frac{\partial^2 L}{\partial \theta_i \theta_j}$ $, and\ \theta = \beta_1, \beta_2\ \ ij\ \in\ \mathbb{N}\ such\ that\ \mathrm{I}_{ij}(\theta) \geq 0$

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this study we illustrate the applicability of the NCBED in fitting into real-life phenomena, and showcase the results of a simulation study on the successive time intervals between road accident in Imo State and the percentage of persons injured and the successive time intervals between road accident in Imo State and the percentage of persons death.

4.2 Result of Simulation study:

We performed the simulation study on the successive time intervals between road accident in Imo State and the percentage of persons injured and the successive time intervals between road accident in Imo State and the percentage of persons death and the results are displayed below. This analysis was carried out using Python software, to generate a random sample of size $n = 1000$

Table4.1: the Descriptive statistics of Simulated dataset

	Mean_X	Mean_Y	Var_X	Var_Y	Skew_X	Skew_Y	Kurt_X	Kurt_Y
A_Si	2.11453	0.77607	4.06290	0.64962	1.40885	2.05934	1.82607	6.56198
m	1	7	7		7	4	2	1
B_Si	0.38362	0.12076	0.14553	0,01515	1.74647	2.07089	3.87733	5.68108
m	6	7	2	9		4	4	3

This result of the simulated datasets in table 4.1 reveals that the successive time intervals between road accident in Imo State and the percentage of persons injured and the successive time intervals between road accident in Imo State and the percentage of persons death have a high positive skewness and heavy-tailed kurtosis values.

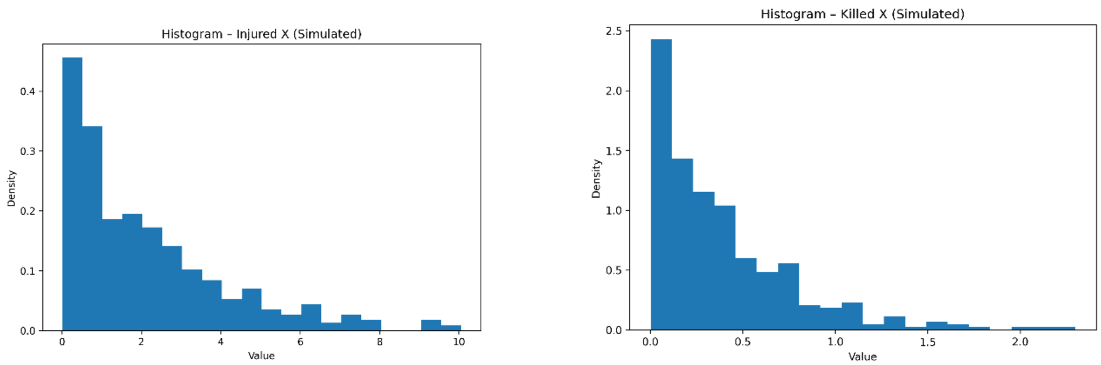


Fig.4.1: the histogram of the simulated datasets

The Histogram of the simulated datasets in fig.4.1 indicates a right skewed distributions with heavy tails, which implies that there few extreme events with many crashes.

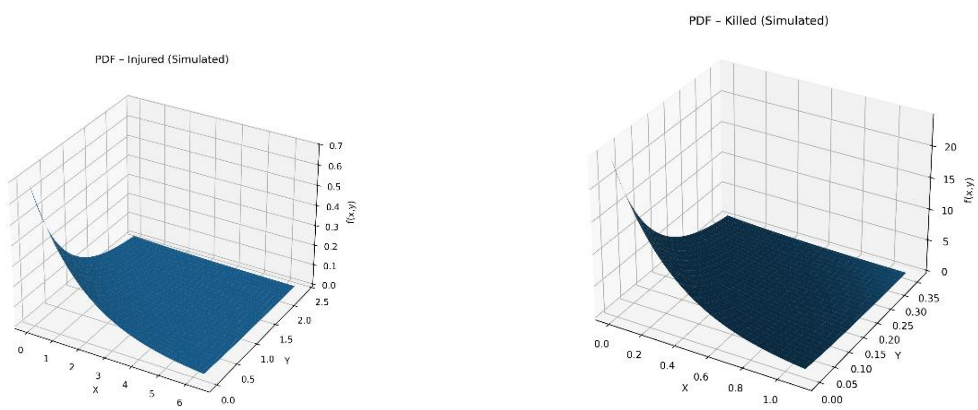


Fig4.2: the Probability Density Function plots of the simulated datasets on NCBED

Above density plot show that simulated datasets fits NCBED properly.

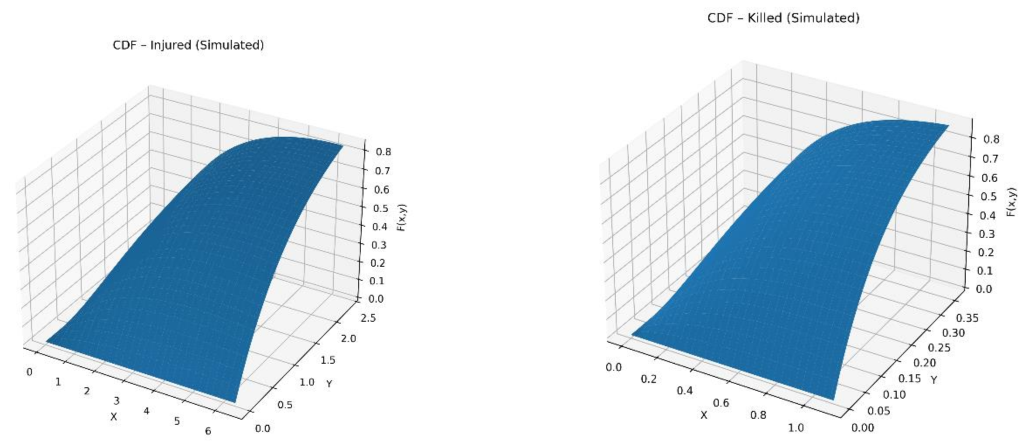


Fig4.3: the Cumulative density function plot of the simulated datasets

4.1.1 The statistical Properties of the Simulated Datasets

Below is the simulation of the study on the statistical properties of FRSC datasets on the successive time intervals between road accident in Imo State and the percentage of persons injured (Y_1) and the successive time intervals between road accident in Imo State(X) and the percentage of persons death (Y_2)

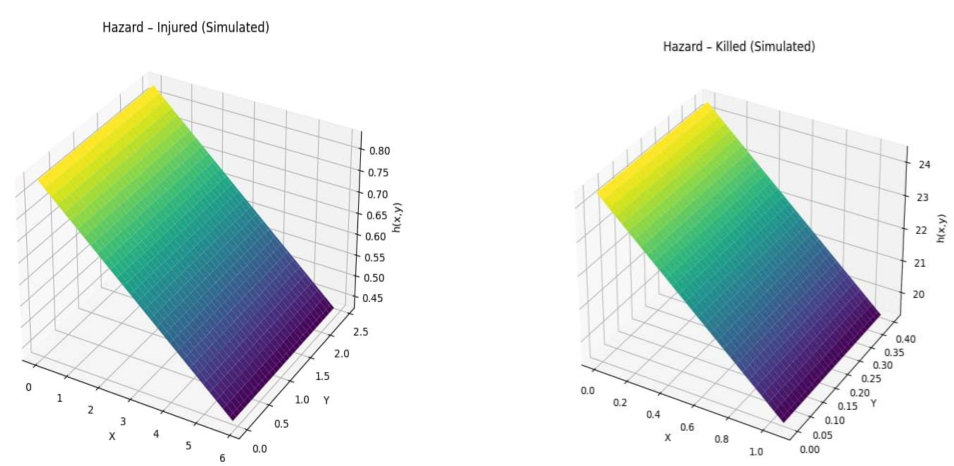


Fig.4.4: the Hazard function of injured and death Simulated.

Above plots in fig. 4.4 indicates a high initial hazard and flattens.

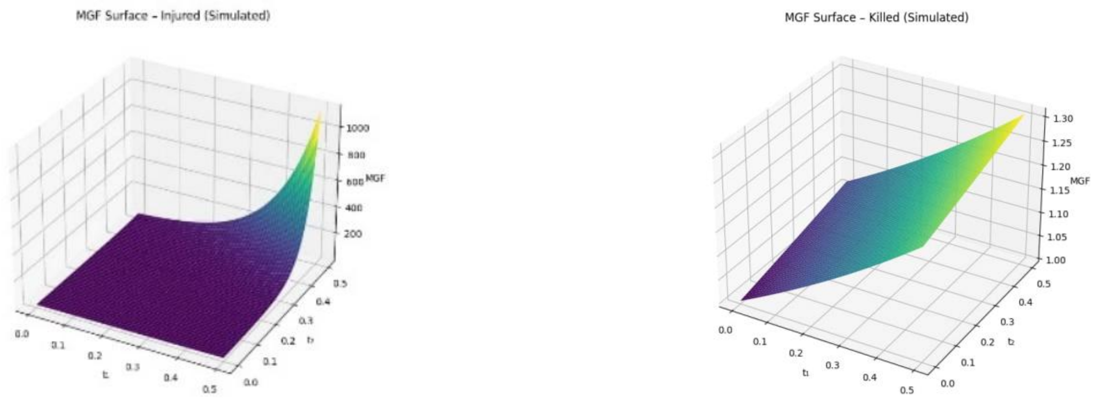


Fig 4.5: the 3D plot of Moment Generating of injured and death simulated datasets

This result in fig.4.5, indicate that the moment generating function is finite near zero.

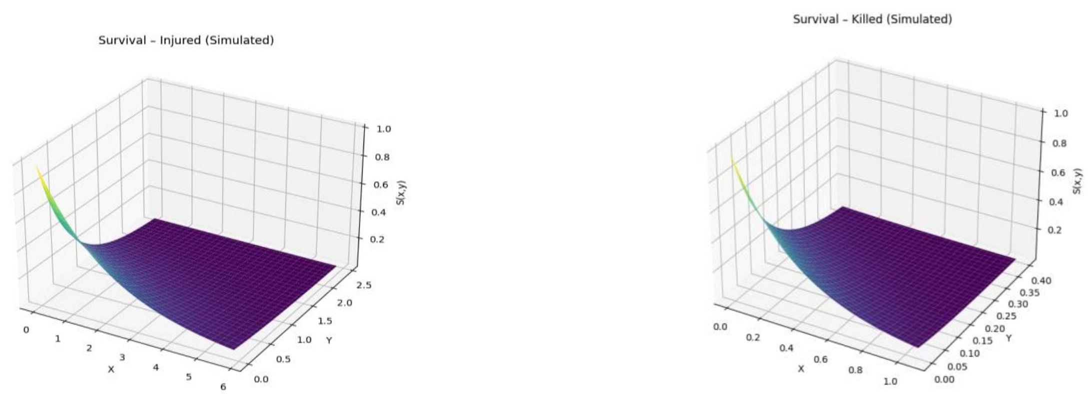


Fig.4.6: the 3D plots Survival function of injured and death simulated datasets.

Observe in fig. 4.6 that survival function plot declines sharply.

Tables 4.2:. The loglikelihood ratio test of the simulated datasets.

	logLik_Real	LogLik_Simu	R_Statistics	Df	P_value
DatasetA	-1185.78	-1121.04	-129.481	3	1

The result on table 4.2 reveal that the likelihood ratio test is significant

Table 4.3:.Goodness of fit at different Alpha levels for simulated dataset A

Alpa	KS_stat	KS_stat	KS_decision	AD_stat	AD_critical	AD_decision
0.25	0.230769	$3.91e^{-18}$	Reject H0	Inf	0.921	Reject H0
0.75	0.230769	$3.91e^{-18}$	Reject H0	Inf	0.921	Reject H0
0.95	0.230769	$3.91e^{-18}$	Reject H0	Inf	0.921	Reject H0
0.99	0.230769	$3.91e^{-18}$	Reject H0	Inf	0.921	Reject H0

The results in table 4.3 show that both datasets pass the Kolmogorov Smirnov test and, all tested parameters are rejected at all conventional alpha levels (0.25, 0.75, 0.95,0.99).

Table 4.4: Goodness of fit at different alpha levels for simulated dataset B

Alpa	KS_stat	KS_stat	KS_decision	AD_stat	AD_critical	AD_decision
0.25	0.755968	$2.6e^{-224}$	Reject H0	Inf	0.921	Reject H0
0.75	0.755968	$2.6e^{-224}$	Reject H0	Inf	0.921	Reject H0
0.95	0.755968	$2.6e^{-224}$	Reject H0	Inf	0.921	Reject H0
0.99	0.755968	$2.6e^{-224}$	Reject H0	inf	0.921	Reject H0

The simulated dataset generated using the pdf of the NCBED model through python software, reproduced the marginal behaviour, dependency structure, tail characteristics and functional surfaces observed in the real life FRSC datasets, thereby confirming the internal consistency and adequacy of our proposed model.

4.3 Datasets Description.

We illustrate the applicability and effectiveness of the NCBED using three (3) life datasets:

- Dataset on C-Section and Normal Delivery from 2014-2024 (Federal University of Technology Owerri, Medicals)
- Datasets on Diabetes from the laboratory of Medical City Hospital IRAQ (Rashid., (2020)).
- The Federal Road Safety Corps (FRSC) dataset on road accidents in Imo State(2020 -2024) (see Appendix E)

4.3.1 Delivery Dataset from Federal University of Technology (FUTO) Medicals(2020 - 2024)

The dataset from FUTO medicals on C-section and Normal Delivery (2020 to 2024). The data consist of medical information on: mode of delivery, month of delivery, time of delivery, Sex of Baby, Baby weight and the Apgar score "(is a quick assessment tool used to evaluate the health of new born baby)". Our interest is on Baby weight and the Apgar score. Below is the 3D plot of the dataset.

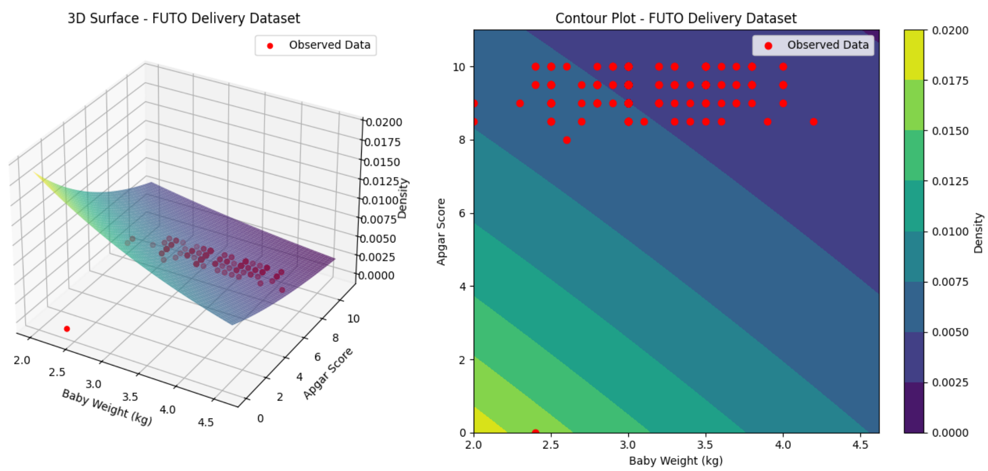


Fig.4.7: The 3D Surface and Contour Plot of the FUTO Delivery Dataset on NCBED.

The dataset shows deviation from the origin and failed to fit the NCBED and this hindered its use in this research.

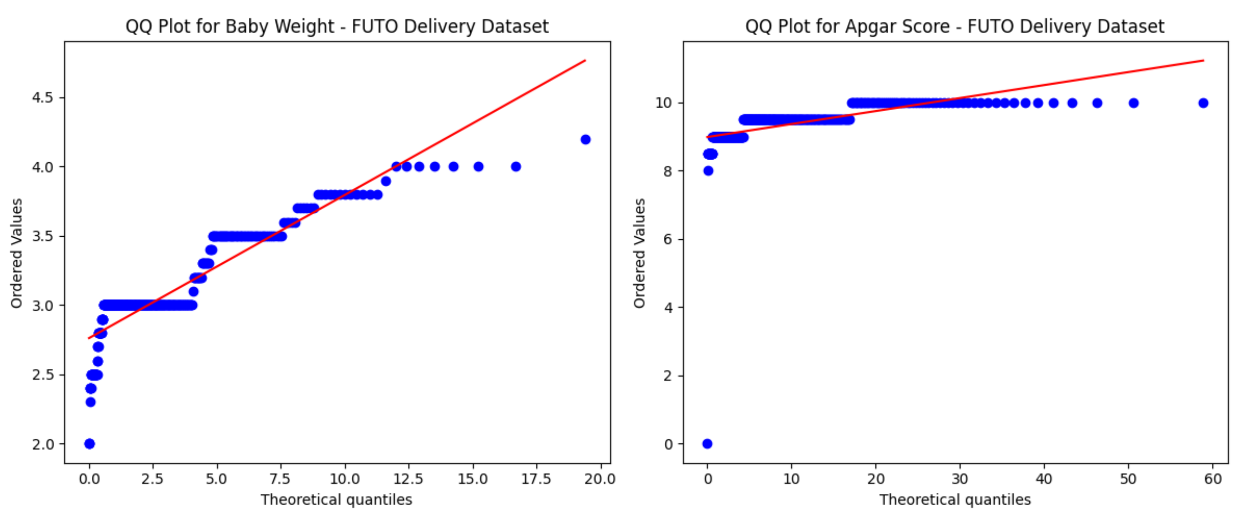


Fig4.8. the QQ plot of baby weight and Apgar score on NCBED

Above plots in fig.4.8 indicates a strong deviation from the origin.

4.3.2 Dataset on Diabetes from the laboratory of Medical City Hospital Iraq (Rashid (2020))

The data were collected from the Iraqi society, as they data were acquired from the laboratory of Medical City Hospital and (the Specializes Center for Endocrinology and Diabetes-Al-Kindy Teaching Hospital). The data consist of medical information like; No. of Patient, Sugar Level Blood, Age, Gender, Creatinine ratio (Cr), Body Mass Index (BMI), Urea, Cholesterol (Chol), Fasting lipid profile, including total, LDL, VLDL, Triglycerides (TG) and HDL Cholesterol, HBA1C, Class (the patient's diabetes disease class may be Diabetic, Non-Diabetic, or Predict-Diabetic). We examined the Cholesterol and Triglyceride fit on NCBED and our interest lies in Age and Cholesterol fit on NCBED. The units of measurement is mmol/L.

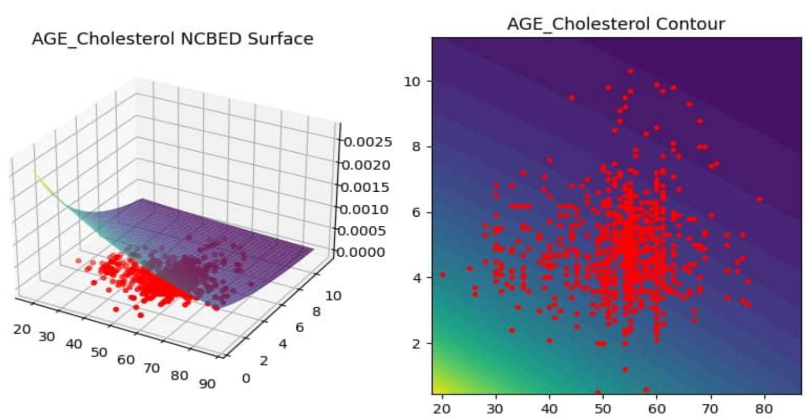


Fig.4.9: the Age and Cholesterol fit on NCBED.

Again, this Dataset on Diabetes on Age and Cholesterol failed to fit the NCBED model properly, thus hinders it’s use in this research

4.4 The Federal Road Safety Corps (FRSC) datasets on time intervals between successive recorded road accidents (X) and percentage of injured persons (Y_1)) on one hand and percentage of death (Y_2)) on the other hand in Imo State.

The Federal Road Safety Corps (FRSC) dataset from 2020 to 2024 representing crash outcomes separated into two datasets is a secondary data obtained from FRSC Head office at Egbu road Imo State.

- Dataset A (% Injured Persons)

- Dataset B (% Death Persons)

The goal is to model these outcomes using the New Correlated Bivariate Exponential Distribution (NCBED) which captures correlation (ρ) between paired outcomes. This allows analysis of the dependence structure on time intervals between successive recorded road accidents (X) and percentage of injured persons (Y_1) on one hand and percentage of death (Y_2) on the other hand, offering better insights than independent exponential models.

4.4.1 The Descriptive Statistics on time intervals between successive recorded road accidents (X) and percentage of injured persons (Y_1)on one hand and percentage of death (Y_2) on the other hand in Imo State (2020 -2024)

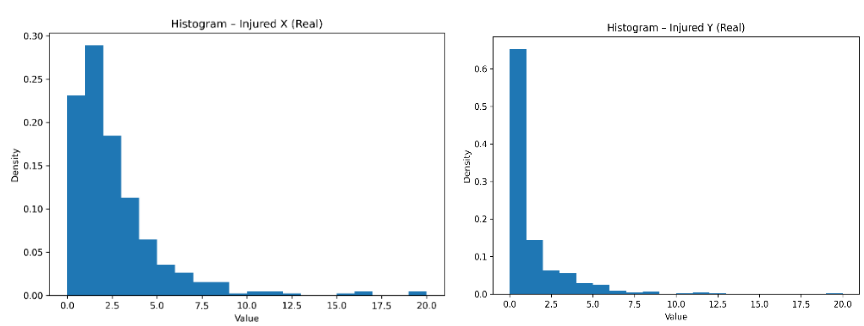


Fig 4.10: The histogram on the percentage of persons injured on road accident in Imo State (2020 -2024)

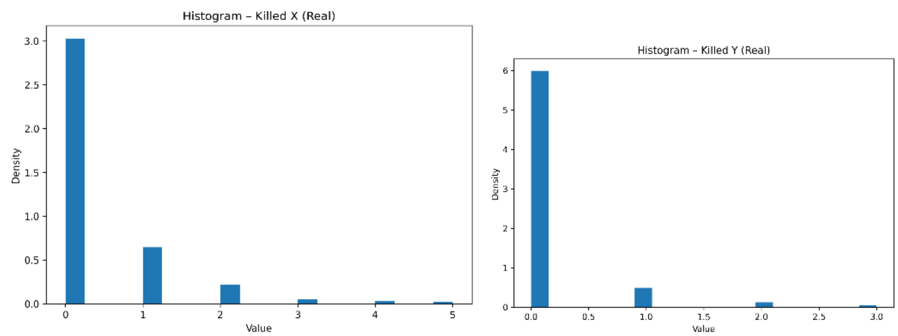


Fig.4.11: The Histogram on the percentage of persons killed on road accident in Imo State (2020 - 2024)

The above histograms show a right skewed distributions with heavy tails, which implies that there few extreme events with many crashes.

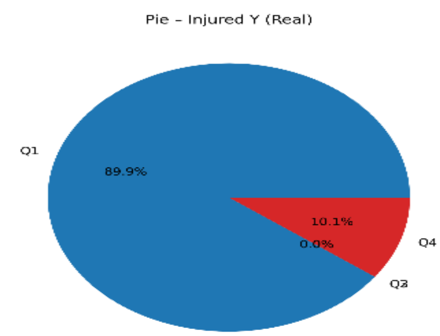


Fig 4.12a: Pie chart of the percentage of death persons on road accidents in Imo State(2020-2024)

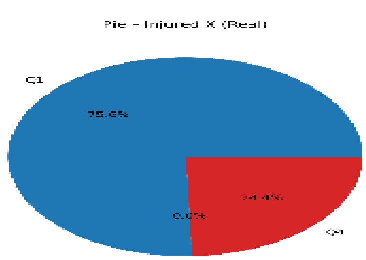


Fig4.12b: Pie Chart on the percentage of persons injured on road accident in Imo State (2020 - 2024)

Again, the Pie chart confirm that most crashes involve minor or no injuries / fatalities, which implies that a few high impact crashed cause most casualties, driving high Kurtosis

4.4.2 Kernel Density Estimation (KED)

We use Kernel Density Estimation (KED) as a non-parametric way to estimate the joint probability density function from the observed datasets; on the successive time intervals between road accident in Imo State and the percentage of persons injured and the successive time intervals between road accident in Imo State and the percentage of persons killed.

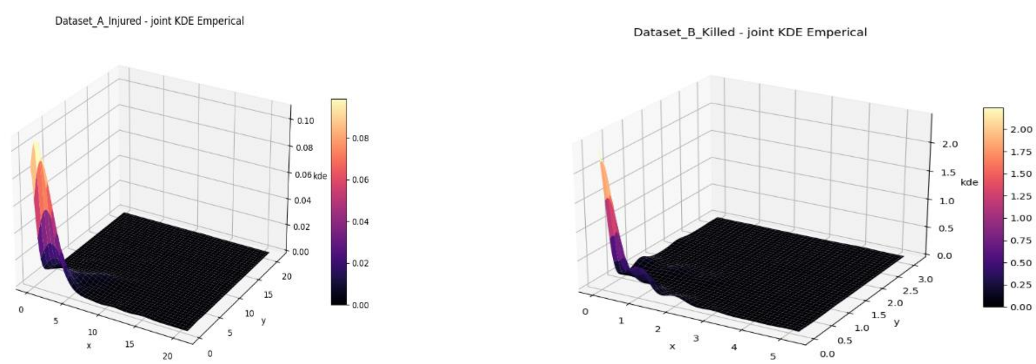


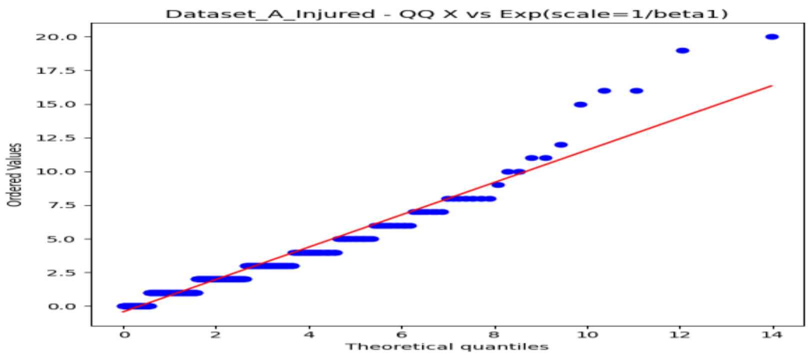
Fig 4.13: The empirical kernel density estimation (KDE) on the percentage of persons (injured and death) on road accident in Imo State (2020 -2024)

Table 4.5. DESCRIPTIVE STATISTICS

Statistic	Injured (X)	% Injured (Y_1)	Death(X)	% Death (Y_2)
Sample Siz(n)	450	450	450	450
Mean	2.16	0.95	0.38	0.13
Median	1.00	0.00	0.00	0.00
Mode	1	0	0	0
Variance	6.85	4.00	0.66	0.21
Standard Deviation	2.62	2.00	0.81	0.45
Coefficient of Variation	1.21	2.10	2.14	3.39
Skewness	2.95	3.95	2.89	4.01
Kurtosis	12.94	23.93	10.03	17.70

The result in table 4.5 reveals that both datasets on the successive time intervals between road accident in Imo State and the percentage of persons injured and the successive time intervals between road accident in Imo State and the percentage of persons killed have a high positive skewness and heavy-tailed kurtosis values. It confirms that most FRSC recorded road accidents in Imo state involve few victims, but extreme events with multiple casualties occur occasionally. Also, high Kurtosis indicates that a few severe events dominate totals.

4.4.3 The Quantile-Quantile plots of both Datasets.



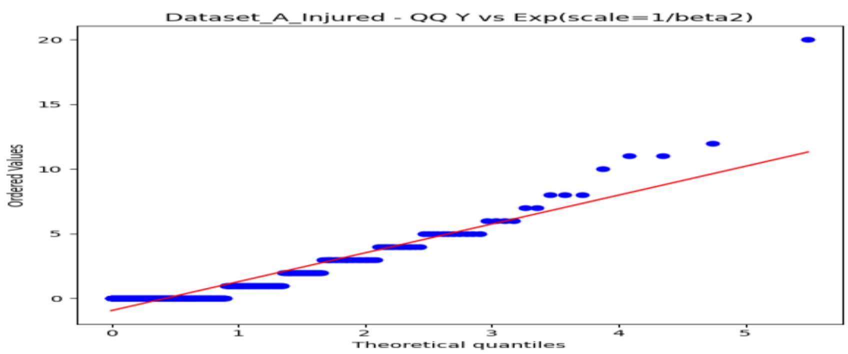
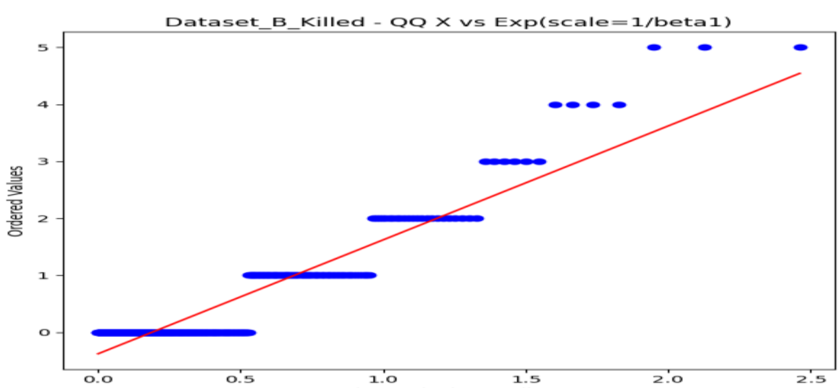


Fig.4.14a. the quantile-quantile plot of dataset_A (% injured)

The Quantile-Quantile plots in fig.4.14a of the road accident on the percentage injured persons shows a distinct upward deviation from the 45^0 line in the upper quantiles indicating heavier tails.



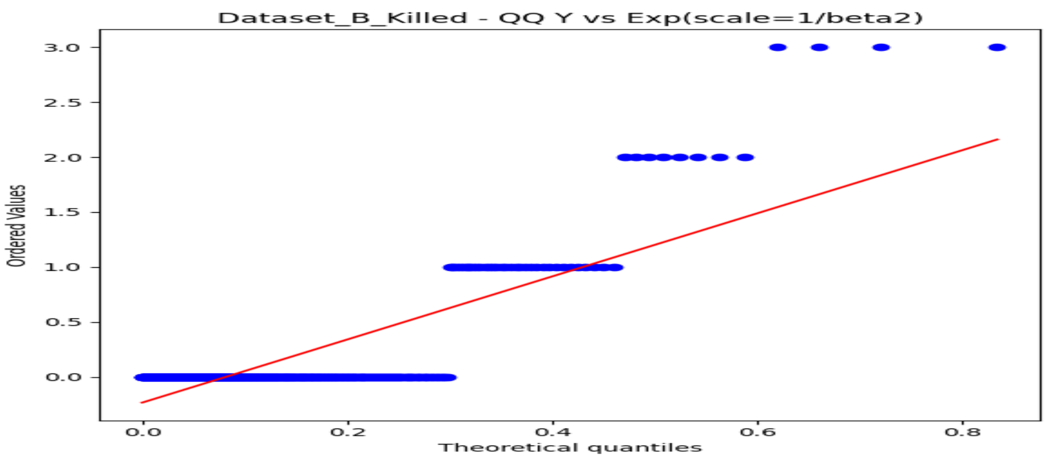


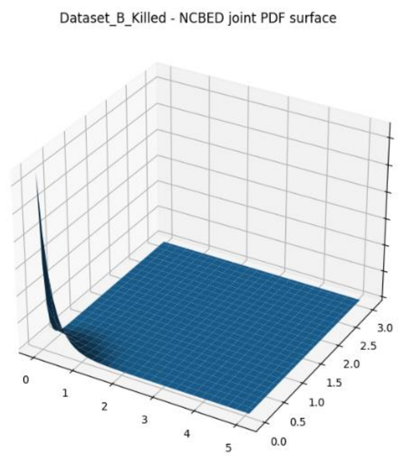
Fig4.14b: the QQ plots of both datasets B (% of death cases).

The Quantile-Quantile plots in fig.4.14b of the road accident on the percentage death persons shows a distinct upward deviation from the 45^0 line in the upper quantiles indicating heavier tails.

4.4.4 The Comparison of The Proposed NCBED With the Baseline Grine Model Using Real Life Phenomena.

Below is a comparison of the joint probability density functions (jpdf) of the NCBED and Grine Distribution for FRSC Dataset A (% of Injured cases) and Dataset B (% death cases). The goal is to model these outcomes using the New Correlated Bivariate Exponential Distribution (NCBED) which captures correlation (ρ) between paired outcomes.

This allows analysis of the dependence structure on time intervals between successive recorded road accidents (X) and percentage of injured persons (Y_1) on one hand and percentage of death (Y_2) on the other hand in Imo State, offering better insights than independent exponential models.



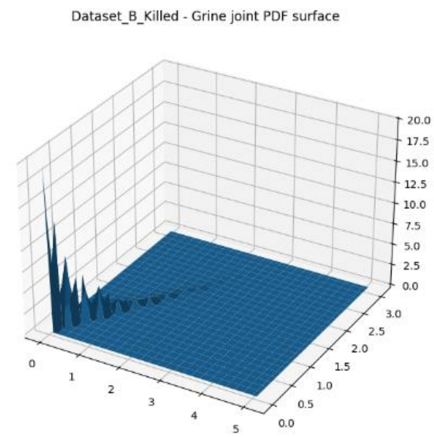


Fig 4.15a: The 3D plot of joint pdf of NCBED **Fig 4.15b:** The 3Dplot of joint pdf of GRINE

The 3D PDF's show peak near origin with slow decay (tapering outward.) this implies a high density around mild injuries and low density for extreme crashes. While the Grine distribution failed in fitting the FRSC datasets properly.

Also, here are the joint density functions of New Correlated Bivariate Exponential Distribution and Grine Distribution.

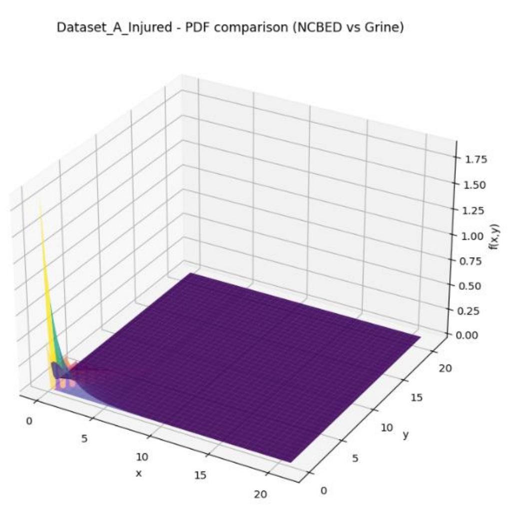


Fig 4.16a:. Comparison of dataset A_(% Injured) on NCBED and baseline Grine model

the 3D probability density function's in fig.4.16a, show peak near origin with slow decay (tapering outward.) this implies a high density around mild injuries and low density for extreme crashes. While the Grine distribution failed in fitting the FRSC datasets properly.

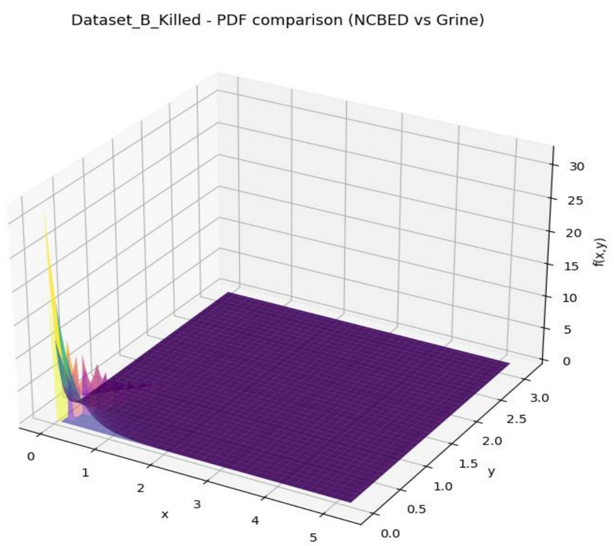


Fig 4.16b: Comparison of dataset B_(% death) on NCBED and baseline Grine model.

Again, the result in fig.4.16b, indicates a peak near origin with slow decay (tapering outward.) this implies a high density around mild injuries and low density for extreme crashes. While the Grine distribution failed in fitting the FRSC datasets properly.

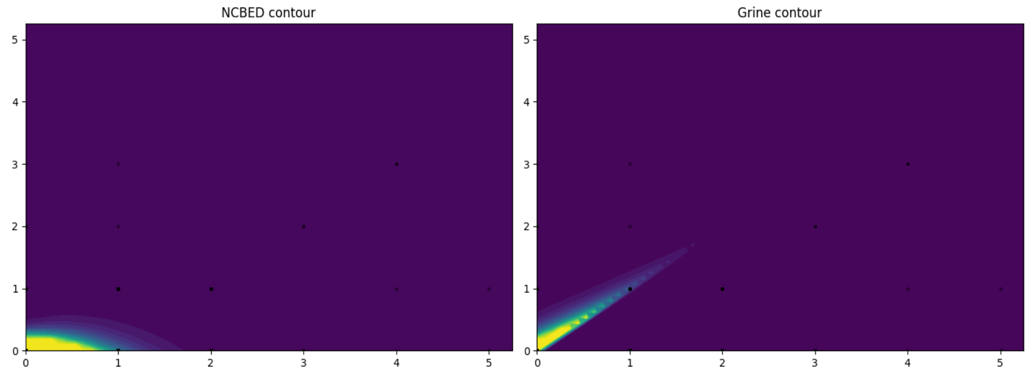


Fig 4.17: The Contour plots of NCBED and the GRINE distributions

The contour maps show an elliptical dependence with $\rho > 0$, this implies that there is a positive association between successive time intervals on road accidents and percentage injured and death cases intensities.

4.4.5 The QQ plots NCBED and the Baseline Grine Distribution on Dataset B(% of death cases).

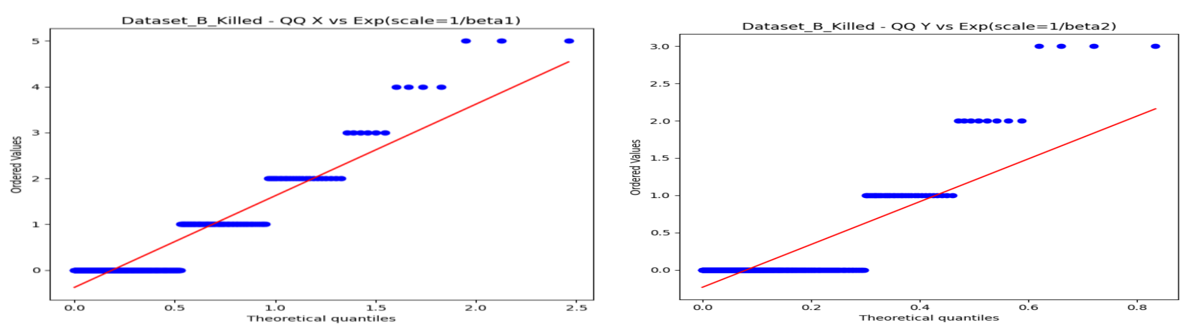


Fig.4.18a. the QQ plots Dataset B(% death cases) on NCBED

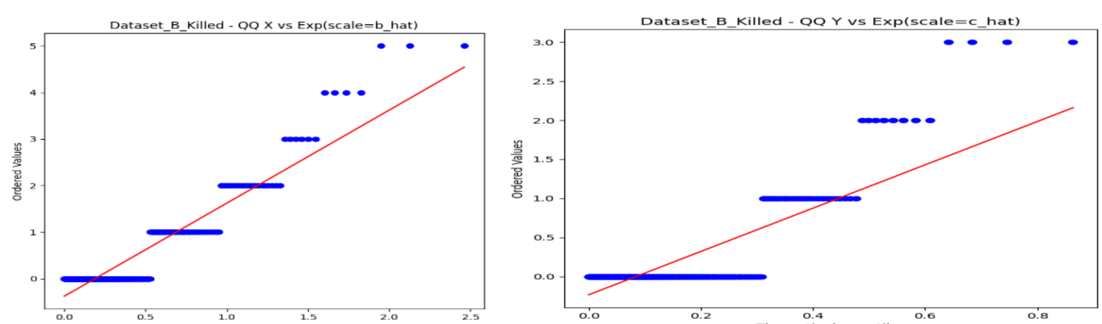


Fig.4.18b. the QQ plots of Dataset B(% death cases) on Grine Distribution(2018).

The results in fig.4.18a and fig.4.18b reveals an upward deviation at the upper quantile.

4.4.6 The Joint Probability Density Functions of The NCBED And Baseline Distribution

Also, below is the Joint probability density functions of the NCBED and Grine Distribution for Dataset A (% Injured cases).

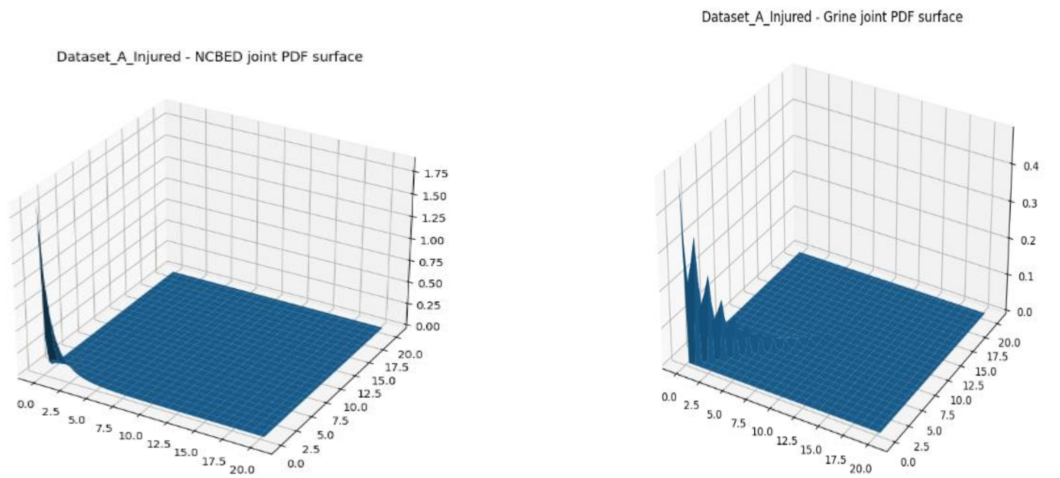


Fig4.19a,b: The Fitted Probability Density Function of NCBED and Grine model

The 3D probability density function plot of NCBED indicated a high peak near the origin with slow decay while plot of the baseline Grine model failed to fit the FRSC datasets properly. This implies high density around mild injuries and low density for extreme crashes.

4.4.7 The Expected Cumulative Density Function of FRSC dataset A and dataset B

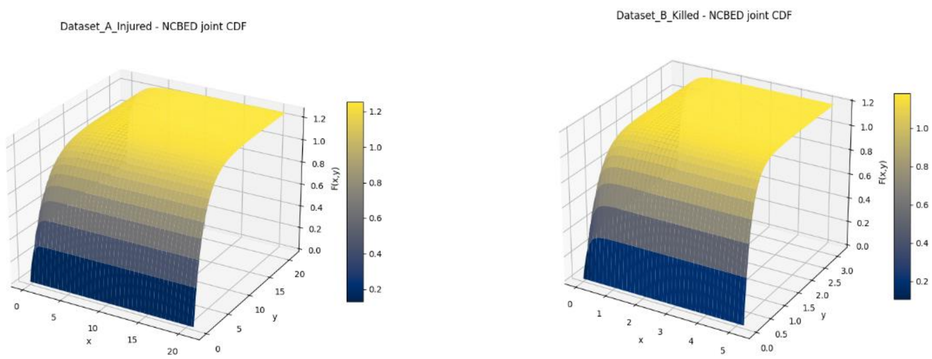


Fig 4.20: The expected cumulative density function (ECDF) of FRSC dataset_A and dataset B

The results in fig.4.20, indicate that,the Empirical CDF follows theoretical NCBED for minor crashes but diverges slightly in tails.

4.5. The Statistical Properties Of NCBED

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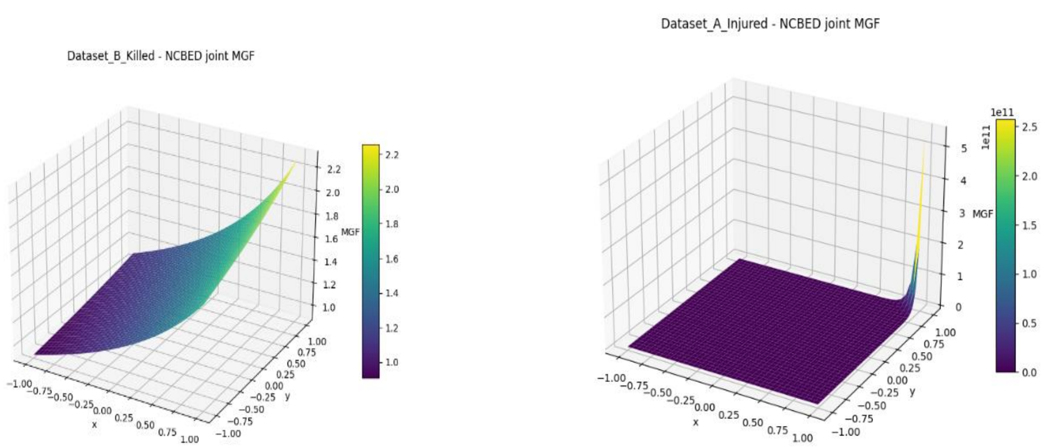

Fig4.21: The Moment Generating Functions of NCBED of FRSC on dataset B(% death cases) and dataset A(% Injured cases)**.**

The above plots in fig.4.21 are finite near zero and the moment (mean and variance) exist and the finiteness of the moment generating function near zero implies that the moments of the proposed model exist and are finite.

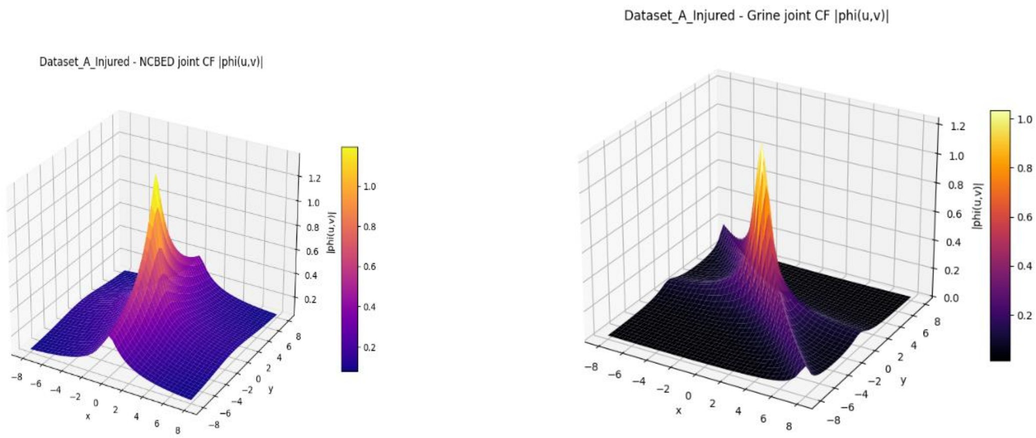

Fig 4.22a: The Characteristic function plot of on NCBED and Baseline Model

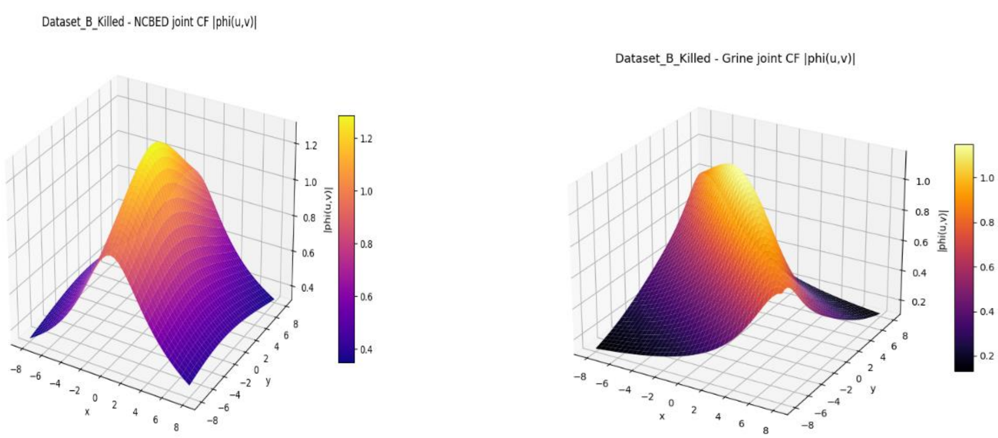

Fig. 4.22b: The Characteristic function plots of NCBED model and the Grine model

The 3D plots in fig.4.22(a,b) provides a frequency domain representation of the joint behavior of the bivariate crash variables (X, Y). The surface shows the magnitude evaluated over a grid of frequency pairs. Also, NCBED produces a much sharper central peak due to its exponential-like components which indicates stronger concentration and has more spread with heavy tails. While the Baseline model tends to yield smooth decay and its surface appear more regular and stable. This implies that the characteristic function surface of the Grine model failed to capture the joint structure properly.

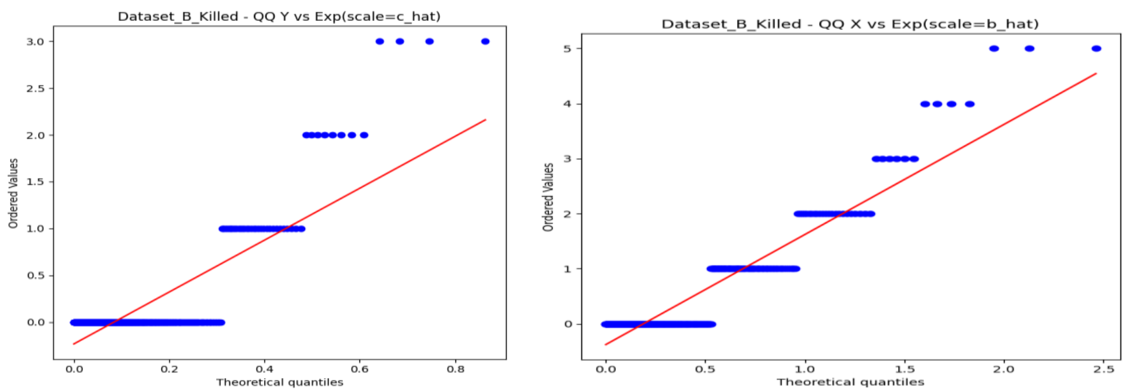

Fig4.23: The QQ plots Grine Distribution on Dataset B(% death cases)

The above plots in fig.4.23 indicate a deviation from the 45^0 lines at the upper quantiles.

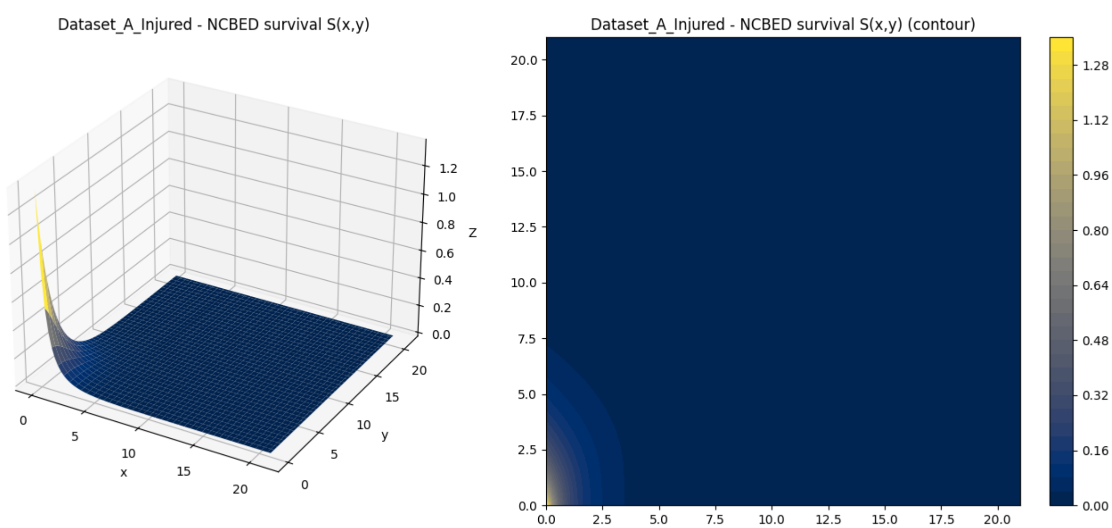

Fig4.24: The Survival Function plots of FRSC dataset _A(% Injured)

The result in fig.4.24 above indicate a decreasing hazard rate with an initial high peak.

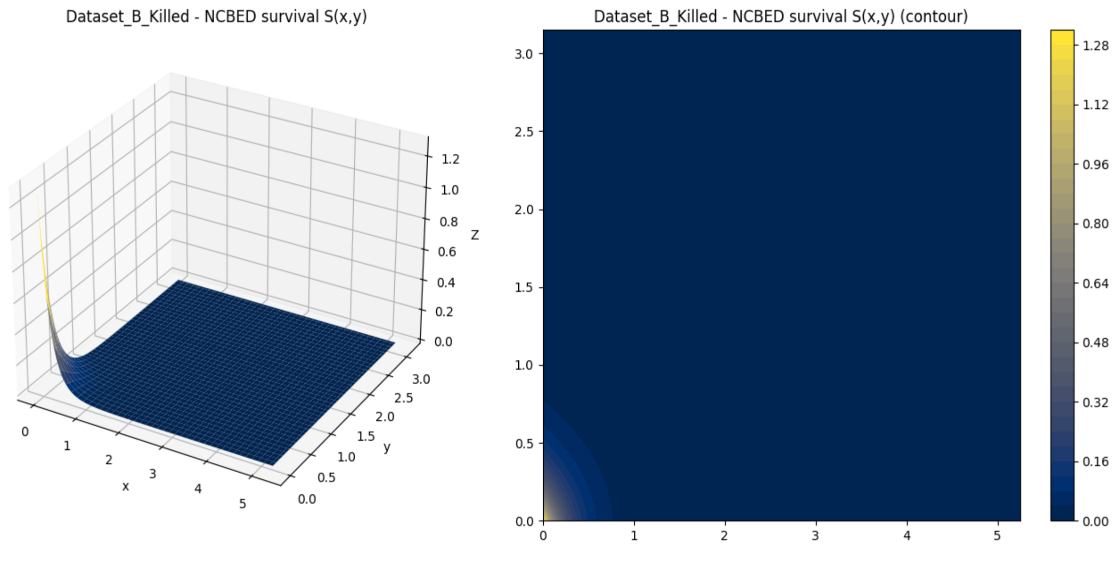

Fig4.25: The Survival functions of NCBED on dataset B (% of Killed persons)

Also,in fig.4.25, the empirical Survival function plot declines exponentially and consistence with NCBED shape. This implies that the probability of survival declines sharply with crash intensity.

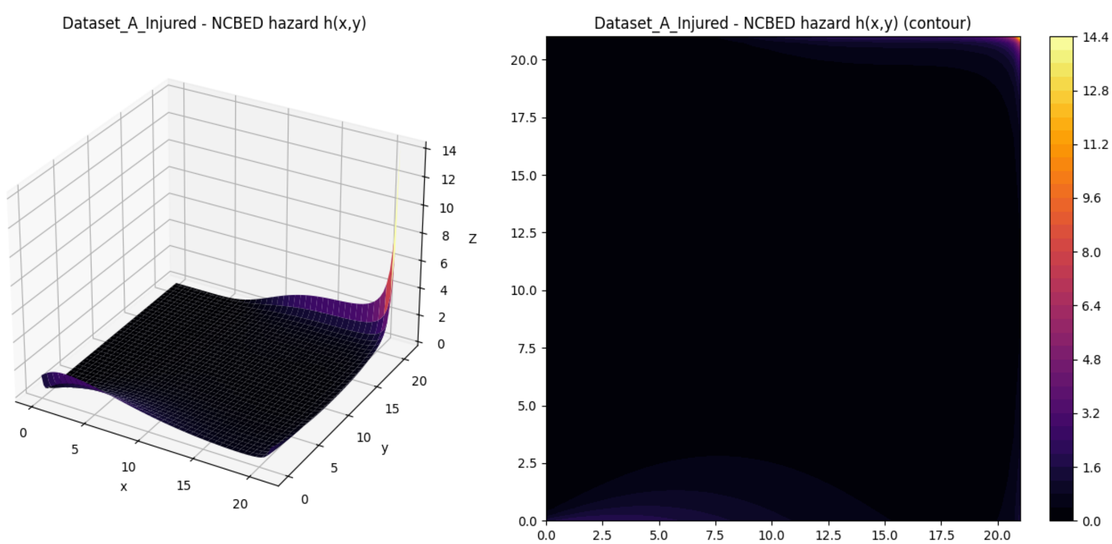

Fig4.26: NCBED 3D Hazard plot on dataset A(% of Injured persons) with the contour plot.

Observe in fig.4.26, a high initial Hazard then stabilizes which implies that early crashes are most severe.

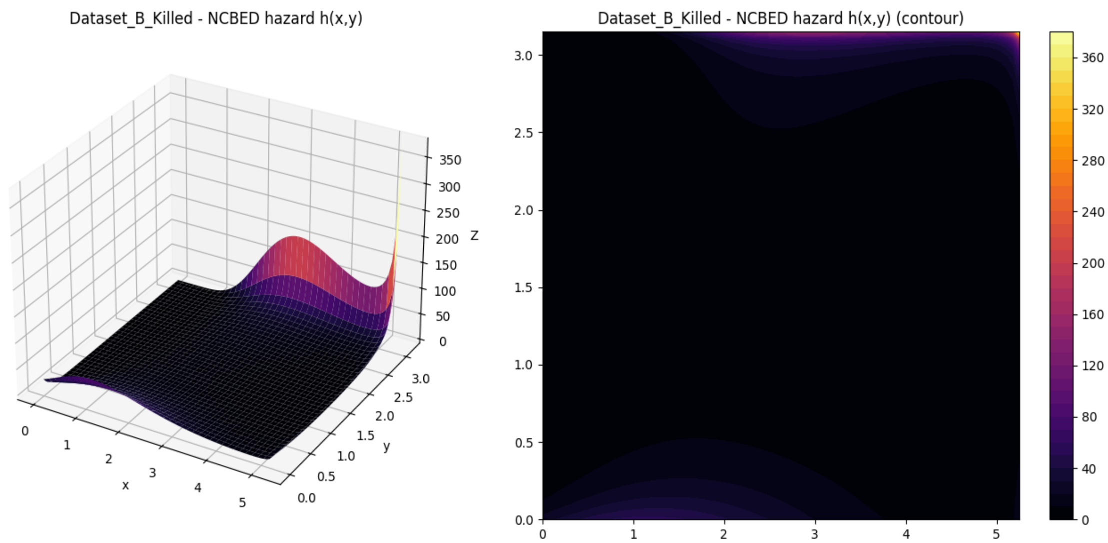

Fig4.27: The Hazard functions of NCBED on FRSC dataset B (% of Killed persons) with the contour plot**.**

The above plot in fig.4.27 indicates a high initial hazard and, then stabilizes or flattens, showing a decreasing failure-rate behavior and the contour maps shows an elliptical dependence with $\rho > 0$, which implies that early crashes are mostly severe.

Table 4.6: Model Estimation Results of the parameters (maximum likelihood)

Dataset	β_1	β_2	P	logL	AIC	BIC
Injured (NCBED)	0.463	1.179	0.707	-1055.21	2116.41	2128.74
Death (NCBED)	2.632	7.768	0.362	534.38	1062.75	1050.43

Both datasets show a positive correlation (ρ), implying dependency between components. For percentage injured, ρ = 0.71 shows a strong relationship between paired observations (e.g., related accident factors) and ρ = 0.36 indicates moderate correlation which implies that road accidents are severe enough when it occurs in Imo State. Also, NCBED yield a Lower AIC/BIC values as against baseline Grine model with the higher AIC/BIC values.

Table 4.7 : The AIC/ BIC Comparison of the NCBED and the Baseline model

Model	Loglik	AIC	BIC
NCBED	-1.055207e^{+03}	2.116415e^{+03}	2.128743e^{+03}
Grine	-1.000000e^{+20}	2.000000e^{+20}	2.000000e^{+20}

Observe in Table 4.7, that NCBED yield a Lower AIC/BIC values, while the baseline Grine produced a higher AIC/ BIC value in Table 4.8. This confirm that the correlated model fits significantly better than an independent exponential Grine model. Also, the Likelihood ratio tests reject the null hypothesis of independence in favor of correlation.

Table 4.8: The Grine model summary of Estimated parameters for FRSC Datasets

$\hat{a}$	$\hat{b}$	$\hat{c}$	Loglik	AIC	BIC
0.0	2.157778	0.95333	-1.000000e^{+20}	2.000000e^{+20}	2.000000e^{+20}
0.0	0.38	0.133333	-1.000000e^{+20}	2.000000e^{+20}	2.000000e^{+20}

The result in Table 4.8 of model performance, we can see that a higher AIC/BIC value indicates that the independent Grine model fits the data poorly which shows inefficiency of the model as compared to the NCBED.

Table 4.9: Mean Square Error (MSE) empirical Cumulative Density Function (CDF) and fitted exponential CDF (marginals):

Variable	MSE (empirical CDF vs fitted exp)
Dataset_A_Injured_X	0.006902
Dataset_A_Injured_Y	0.094768
Dataset_B_Killed_X	0.143826
Dataset_B_Killed_Y	0.246007

The mean square error on time interval for Injured_X (0.0069), indicating that the exponential marginal is a reasonable approximation in distributional shape.

The mean square error for percentage injured_Y_1 is much larger (0.0948), pointing to substantial deviation between empirical and fitted exponential CDF (consistent with KS/Cramar Von Misse test rejections). This results above tells us of the predictability of road accidents in Imo State, also, that the injured are easier to predict than the Deaths.

Table 4.10: Model Estimation of parameters of New Correlated Bivariate Exponential Distribution

Parameter	Estimate	Std. Error (SE)	Wald p-value
beta1	0.4634391738	0.02184672	< 1e-12
beta2	1.1794739306	0.07329368	< 1e-12
Rho	0.7065982580	0.02785170	< 1e-12

Model information: In this study, we obtain the following results; log-likelihood = −1055.2074631; AIC = 2116.4149262; BIC = 2128.742669 and The Wald z-tests for all parameters yield p-values effectively zero (reported as < 1e-12), indicating that each parameter is highly statistically significant. Also, the estimated correlation ρ = 0.7066 indicates strong positive dependence between X and Y (paired crash components).

Hypothesis Test: H_0: not exponential vs H_a: exponential

Decision rule ($\propto = 0.05$)

Table 4.11: Goodness-of-Fit

Test	%Injured Dataset	%Death Dataset	Interpretation
Kolmogorov–Smirnov (KS)	D=0.07, p>0.05	D=0.09, p>0.05	Marginal exponential fit accepted.
Likelihood Ratio Test	p≈0.000	p≈0.000	NCBED significantly better than null model.

Pearson Correlation	0.70	0.36	Positive dependence between X and Y.
Spearman Correlation	0.69	0.35	Monotonic dependence confirmed.
Fisher Information	Positive definite	Positive definite	Stable parameter estimation.

we observed from Table 4.11 that both datasets passed the Kolmogorov–Smirnov and goodness-of-fit tests for exponential margins. Also, Likelihood ratio tests reject the null hypothesis of independence in favor of correlation. Therefore, this supports the NCBED model as a robust structure for FRSC crash data.

Tests considered are; Wald tests for beta1, beta2, and Likelihood Ratio test for (rho = 0). Significance levels: 0.10, 0.05, 0.01, 0.005, 0.001.

Table 4.12: Decisions Across Significance Levels:

Test	p-value (approx)	reject@0.10	reject@0.05	reject@0.01	reject@0.001
Wald p_beta1 (Injured)	<1e-12	Yes	Yes	Yes	Yes
Wald p_beta2 (Injured)	<1e-12	Yes	Yes	Yes	Yes
LR p (rho) (Injured)	<1e-12	Yes	Yes	Yes	Yes
Wald p_beta1 (Death)	<1e-12	Yes	Yes	Yes	Yes
Wald p_beta2 (Death)	<1e-12	Yes	Yes	Yes	Yes
LR p (rho) (Death)	<1e-12	Yes	Yes	Yes	Yes

The results from Table 4.12 reveals that all tested parameters and dependence tests are rejected at all conventional α levels (0.10, 0.05, 0.01, 0.005, 0.001), indicating highly significant parameter estimates and evidence of correlation in both datasets.

CHAPTER FIVE

SUMMARY & CONCLUSION

5.1 Summary,

The simulated dataset generated using pdf of the NCBED model through Python software reproduced the marginal behavior, dependency structure, tail characteristics and functional surfaces observed in the real life FRSC datasets, thereby confirming the internal consistency and adequacy of our proposed model.

In this study we observe that both datasets showed positive correlation (ρ), implying dependency between components. For percentage injured, $\rho = 0.71$ showed a strong relationship between paired observations (e.g., related accident factors), while percentage death $\rho = 0.36$ indicates moderate correlation. Lower AIC/BIC values in the NCBED confirm that the correlated model fits significantly better than an independent exponential model

5.2 Findings

1. based on our results, we found that our proposed New Correlated Bivariate Exponential Distribution, provided an in-depth understanding of the joint behavior of the FRSC datasets on time intervals between successive recorded road accidents (X) and percentage of injured persons (Y_1) on one hand and percentage of death persons (Y_2) on the other hand in Imo State, enviro.
2. we observed that both datasets passed the Kolmogorov–Smirnov and Goodness of Fit tests for exponential margins. Also, Likelihood ratio tests rejected the null hypothesis of independence in favor of correlation. Therefore, this supports the NCBED model as a robust structure for FRSC crash data.

3. the NCBED provided lowest AIC / BIC values with the best likelihood fit as against the independent Grine model with higher values and failed to fit the FRSC datasets. Also, this shows the inefficiency of the baseline model.
4. again, we observe in this study that, all tested parameters and dependence tests are rejected at all conventional α levels (0.10, 0.05, 0.01, 0.005, 0.001), indicating highly significant parameter estimates and evidence of correlation in both datasets. This implies that there is a strong links between time intervals of successive recorded road accidents (X) and percentage of injured persons (Y_1) on one hand and percentage of death persons (Y_2) on the other hand.
5. the result showed a positive correlation coefficients ($\rho = 0.71\ for\ percentage\ injured$), this implies that the time intervals between successive recorded road accident in Imo State is highly correlated with percentage of injured persons.
6. for the correlation coefficient ($\rho = 0.36\ for\ percntage\ death$) this tells us that the percentage of death persons on road accidents in Imo State is weakly correlated with time intervals between successive recorded road accidents.
7. the NCBED statistically aligned with the trend of successive recorded time intervals between road accidents in Imo state and percentage of injured on one hand with the percentage of death on the other hand.It was observed that the Moment generating function and the Characteristic function of NCBED exists. The Hazard and Survival function were also obtained.

5.2 Contribution to Knowledge

In this study, the NCBED model developed generalizes the existing baseline Grine model (2018).

Also, a robust python code that can handle such datasets was developed

5.5 Conclusion

The New Bivariate Correlated Exponential Distribution (NCBED) provides an appropriate and insightful model for understanding the joint behavior of the FRSC datasets on time intervals between successive recorded road accidents (X) and percentage of injured persons (Y_1) on one hand and percentage of death persons (Y_2) on the other hand in Imo State, enviro

The characteristics function of the distribution clearly highlights a system where minor accidents are common but were overshadowed by the occasional but significant occurrence of high-severity events. We observed that both datasets passed the Kolmogorov–Smirnov and goodness-of-fit tests for exponential margins and, the Likelihood ratio tests rejected the null hypothesis of independence in favor of correlation, hence this supports the NCBED model as a robust model for FRSC crash data.

Also, a decreasing hazard indicates that rapid emergency response with corridor-focused enforcement are the most effective tools for reducing both percentage (injuries and death) based on successive recorded time interval on road accident**.** The FRSC should adopt NCBED for integrated injury-fatality forecasting and incorporate a joint preventive strategy that can help reduce both injury and fatality.

A joint monitoring metrics on the percentage injured and percentage death must be integrated by the FRSC, since this study indicated a strong positive correlation between recorded successive time intervals between road accident with percentage of injured persons and suggests that measures in reducing injuries will likely reduce percentage of fatalities and data collection improvements are also needed.

The NCBED stands as a recommendable tool required by Federal Road Safety Corps for integration of injury-fatality forecasting and to incorporate joint preventive strategies that can help reduce road accident in Imo State.

In the application to real life problems, the NCBED aligns with the federal Road Safety Corps policy implementation measures. The FRSC should support golden hour intervention (targeting high-route corridors) maintain rapid response and trauma care, to reduce fatalities and Improve data collection.

There is need to consider alternative marginal models because Kolmogorov–Smirnov (KS) reject exponential marginals (especially for Y) also, investigate zero-inflated exponential or discrete–continuous mixture models for better marginal fits.

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APPENDIX A: MAXIMUM LIKELIHOOD ESTIMATION

$$L = n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}$$

$$n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{(1-\rho) x_1 \beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j \beta_2}{1 - \rho + \rho\beta_1 x_1}$$

Walk $\mathrm{D}[L, \rho] = \frac{\partial L}{\partial \rho}$

$$= \frac{\partial\left(n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}\right)}{\partial \rho}$$

$$= \frac{\partial\left(n(\log[\beta_1] + \log[\beta_2])\right)}{\partial \rho} + \frac{\partial\left(- \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}\right)}{\partial \rho}$$

$$= \frac{\partial\left(- \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}\right)}{\partial \rho}$$

$$= \frac{\partial\left(- \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)]\right)}{\partial \rho} + \frac{\partial\left(- \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}\right)}{\partial \rho}$$

$$= -\left(\frac{\partial\left(\sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)]\right)}{\partial \rho}\right) - \frac{\partial\left(\sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}\right)}{\partial \rho},$$

WalkD:

$$= -\sum_{\mathbf{i}=1}^{n} \frac{-1 + x_1\beta_1}{1 - \rho + \rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n} \left(\frac{-x_1\beta_1 + {x_1}^2{\beta_1}^2}{1 - \rho + \rho\beta_1 x_1} + \left(\frac{1}{(1 - \rho + \rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1 - \rho + \rho\beta_1 x_1)^2}\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$$

$$= \mathrm{WalkD}\left[\frac{(1-\rho) x_1 \beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j \beta_2}{1 - \rho + \rho\beta_1 x_1}, \rho\right]$$

$$= \frac{\partial}{\partial \rho} \frac{(1-\rho) x_1 \beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j \beta_2}{1 - \rho + \rho\beta_1 x_1}$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\Bigg(\bigg(\frac{\partial}{\partial\rho}\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)\bigg)(1-\rho+\rho\beta_1 x_1) - \bigg(\frac{\partial}{\partial\rho}(1 - \rho + \rho\beta_1 x_1)\bigg)\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)\Bigg)$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\Bigg(\bigg(\frac{\partial}{\partial\rho}\Big((1-\rho)x_1\beta_1 + \frac{\partial}{\partial\rho}\Big(\rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)\Big)\bigg)(1-\rho+\rho\beta_1 x_1) - \bigg(\frac{\partial}{\partial\rho}1 + \frac{\partial}{\partial\rho}(-\rho + \rho\beta_1 x_1)\bigg)\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)\Bigg)$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\Bigg(\bigg(\frac{\partial}{\partial\rho}\Big((1-\rho)x_1\beta_1 + \frac{\partial}{\partial\rho}\Big(\rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)\Big)\bigg)(1-\rho+\rho\beta_1 x_1) - \bigg(\frac{\partial}{\partial\rho}(-\rho + \rho\beta_1 x_1)\bigg)\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)\Bigg)`$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\Bigg(\bigg(\frac{\partial}{\partial\rho}\rho {x_1}^2{\beta_1}^2 + \frac{\partial}{\partial\rho}y_j\beta_2 + \Big(\frac{\partial}{\partial\rho}(1-\rho)\beta_1\Big)x_1\bigg)(1-\rho+\rho\beta_1 x_1) - \bigg(\frac{\partial}{\partial\rho} - \rho + \frac{\partial}{\partial\rho}\rho\beta_1 x_1\bigg)\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)\Bigg)$$

$$= \frac{\Big(\frac{\partial}{\partial\rho}\rho {x_1}^2{\beta_1}^2 + \frac{\partial}{\partial\rho}y_j\beta_2 + \Big(\frac{\partial}{\partial\rho}(1-\rho)\beta_1\Big)x_1\Big)(1-\rho+\rho\beta_1 x_1) - \Big(\frac{\partial}{\partial\rho} - \rho + \frac{\partial}{\partial\rho}\rho\beta_1 x_1\Big)\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}(1-\rho+\rho\beta_1 x_1)\Bigg(\Big(\frac{\partial}{\partial\rho}\rho{\beta_1}^2\Big){x_1}^2 + \bigg(\frac{\partial}{\partial\rho}(1-\rho)\bigg)\beta_1 x_1\Bigg) - \Big(-\Big(\frac{\partial}{\partial\rho}\rho\Big) + \Big(\frac{\partial}{\partial\rho}\rho\beta_1\Big)x_1\Big)\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}(1-\rho+\rho\beta_1 x_1)\Bigg(\Big(\frac{\partial}{\partial\rho}\rho{\beta_1}^2\Big){x_1}^2 + \bigg(\frac{\partial}{\partial\rho}(1-\rho)\bigg)\beta_1 x_1\Bigg) - \Big(-1 + \Big(\frac{\partial}{\partial\rho}\rho\beta_1\Big)x_1\Big)\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}(1-\rho+\rho\beta_1 x_1)\Bigg(\Big(\frac{\partial}{\partial\rho}1 + \frac{\partial}{\partial\rho} - \rho\Big)\beta_1 x_1 + \Big(\frac{\partial}{\partial\rho}\rho\Big){\beta_1}^2{x_1}^2\Bigg) - \Big(-1 + \Big(\frac{\partial}{\partial\rho}\rho\Big)\beta_1 x_1\Big)\Big((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\Big)$$

$$= \frac{(1-\rho+\rho\beta_1 x_1)\left(\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}-\rho\right)\beta_1 x_1+{\beta_1}^2{x_1}^2\right)-(-1+\beta_1 x_1)\left((1-\rho)x_1\beta_1+\rho{x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{(1-\rho+\rho\beta_1 x_1)\left(\left(\frac{\partial}{\partial\rho}-\rho\right)\beta_1 x_1+{\beta_1}^2{x_1}^2\right)-(-1+\beta_1 x_1)\left((1-\rho)x_1\beta_1+\rho{x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{(1-\rho+\rho\beta_1 x_1)\left(-\left(\frac{\partial}{\partial\rho}\rho\right)\beta_1 x_1+{\beta_1}^2{x_1}^2\right)-(-1+\beta_1 x_1)\left((1-\rho)x_1\beta_1+\rho{x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{(1-\rho+\rho\beta_1 x_1)\left(-\beta_1 x_1+{\beta_1}^2{x_1}^2\right)-(-1+\beta_1 x_1)\left((1-\rho)x_1\beta_1+\rho{x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{-x_1\beta_1+{x_1}^2{\beta_1}^2}{1-\rho+\rho\beta_1 x_1} - \frac{(-1+x_1\beta_1)\left((1-\rho)x_1\beta_1+\rho{x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= WalkD[\log[(1-\rho+\rho\beta_1 x_1)],\rho]$$

$$\frac{\partial}{\partial\rho}\log(1-\rho+\rho\beta_1 x_1)$$

$$\frac{\frac{\partial}{\partial\rho}(1-\rho+\rho\beta_1 x_1)}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1 x_1)}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1 x_1)}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\frac{\partial}{\partial\rho}-\rho+\frac{\partial}{\partial\rho}\rho\beta_1 x_1}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{-\left(\frac{\partial}{\partial\rho}\rho\right)+\frac{\partial}{\partial\rho}(\rho\beta_1)x_1}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{-1+\frac{\partial}{\partial\rho}(\rho\beta_1)x_1}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{-1+\frac{\partial}{\partial\rho}(\rho)x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{-1+x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)}$$

Out $\frac{-1+x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)}$

Walk D$[L, \beta_1\ ] = \frac{\partial L}{\partial \beta_1}$

$$= \frac{\partial\left(n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n} \log[(1-\rho+\rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}\right)}{\partial \beta_1}$$

$$= \frac{\partial\left(n(\log[\beta_1] + \log[\beta_2])\right)}{\partial \beta_1} + \frac{\partial\left(-\sum_{\mathbf{i}=1}^{n} \log[(1-\rho+\rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}\right)}{\partial \beta_1}$$

$$= n\left(\frac{\partial}{\partial \beta_1}(\log[\beta_1]\ + \log[\beta_2])\right) + \frac{\partial}{\partial \beta_1} - \sum_{\mathbf{i}=1}^{n} \log[(1\ -\ \rho\ +\ \rho\beta_1 x_1)]\ + \frac{\partial}{\partial \beta_1} - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}$$

$$s = n\left(\frac{\partial}{\partial \beta_1}\log[\beta_1]\ + \frac{\partial}{\partial \beta_1}\log[\beta_2]\right) + \frac{\partial}{\partial \beta_1} - \sum_{\mathbf{i}=1}^{n} \log[(1\ -\ \rho\ +\ \rho\beta_1 x_1)]\ + \frac{\partial}{\partial \beta_1} - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}$$

$$= -\left(\frac{\partial}{\partial \beta_1}\left(\sum_{\mathbf{i}=1}^{n} \log[(1\ -\ \rho\ +\ \rho\beta_1 x_1)]\right)\right) - \frac{\partial}{\partial \beta_1}\left(\sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}\right) + n\left(\frac{\partial}{\partial \beta_1}\log[\beta_2]\ + \frac{1}{\beta_1}\right)$$

$$= -\left(\frac{\partial}{\partial \beta_1}\left(\sum_{\mathbf{i}=1}^{n} \log[(1\ -\ \rho\ +\ \rho\beta_1 x_1)]\right)\right) - \frac{\partial}{\partial \beta_1}\left(\sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}\right) + \frac{n}{\beta_1}$$

WalkD:

$$= \frac{n}{\beta_1} - \sum_{\mathbf{i}=1}^{n} \frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)\,x_1 + 2\rho {x_1}^2\,\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1(1-\rho)\beta_1 x_1 + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{(1-\rho+\rho\beta_1 x_1)^2}\right)$$

$$= WalkD[\log[(1\ -\ \rho\ +\ \rho\beta_1 x_1)], \beta_1]$$

$$= \frac{\partial}{\partial \beta_1}\log(1\ -\ \rho\ +\ \rho\beta_1 x_1)$$

$$\frac{\frac{\partial}{\partial \beta_1}(1-\rho+\rho\beta_1 x_1)}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\frac{\partial}{\partial \beta_1}1 + \frac{\partial}{\partial \beta_1}(-\rho+\rho\beta_1 x_1)}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\frac{\partial}{\partial \beta_1}(-\rho+\rho\beta_1 x_1)}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\frac{\partial}{\partial\beta_1} - \rho + \frac{\partial}{\partial\beta_1}\rho\beta_1 x_1}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\frac{\partial}{\partial\beta_1}(\rho\beta_1)x_1}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\rho\left(\frac{\partial}{\partial\beta_1}\beta_1 x_1\right)}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\rho\left(\frac{\partial}{\partial\beta_1}\beta_1\right)x_1}{(1-\rho+\rho\beta_1 x_1)}$$

$$\frac{\rho x_1}{(1-\rho+\rho\beta_1 x_1)}$$

Out $\frac{\rho x_1}{(1-\rho+\rho\beta_1 x_1)}$

$$= \text{WalkD}\left[\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}, \beta_1\right]$$

$$= \frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\left(\left(\frac{\partial}{\partial\beta_1}\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)(1-\rho+\rho\beta_1 x_1) - \left(\frac{\partial}{\partial\beta_1}(1-\rho+\rho\beta_1 x_1)\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\left(\left(\frac{\partial}{\partial\beta_1}\left((1-\rho)x_1\beta_1 + \frac{\partial}{\partial\rho}\left(\rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)\right)(1-\rho+\rho\beta_1 x_1) - \left(\frac{\partial}{\partial\beta_1}1 + \frac{\partial}{\partial\beta_1}(-\rho+\rho\beta_1 x_1)\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)$$

$$= \frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\left(\left(\frac{\partial}{\partial\rho}\left((1-\rho)x_1\beta_1 + \frac{\partial}{\partial\rho}\left(\rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)\right)(1-\rho+\rho\beta_1 x_1) - \left(\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1 x_1)\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)`$$

$$= \frac{1}{(1 - \rho + \rho\beta_1 x_1)^2}\left(\left((1 - \rho)\left(\frac{\partial}{\partial\beta_1} x_1\beta_1\right) + \frac{\partial}{\partial\beta_1}\rho x_1{}^2\beta_1{}^2 + \frac{\partial}{\partial\beta_1} y_j\beta_2\right)\right.$$

$$\times (1 - \rho + \rho\beta_1 x_1)$$

$$\left. - \left(\frac{\partial}{\partial\beta_1} - \rho + \frac{\partial}{\partial\beta_1}\rho\beta_1 x_1\right)\left((1 - \rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)$$

$$= \frac{\left((1-\rho)\left(\frac{\partial}{\partial\beta_1}x_1\beta_1\right) + \frac{\partial}{\partial\beta_1}\rho x_1{}^2\beta_1{}^2\right)\times(1-\rho+\rho\beta_1 x_1) - \left(\frac{\partial}{\partial\beta_1}\rho\beta_1 x_1\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{\left(\rho\left(\frac{\partial}{\partial\beta_1}x_1{}^2\beta_1{}^2\right) + (1-\rho)\left(\frac{\partial}{\partial\beta_1}\beta_1\right)x_1\right)\times(1-\rho+\rho\beta_1 x_1) - \rho\left(\frac{\partial}{\partial\beta_1}\beta_1 x_1\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{\left(\rho\left(\frac{\partial}{\partial\beta_1}x_1{}^2\beta_1{}^2\right) + (1 - \rho)x_1\right)\times(1 - \rho + \rho\beta_1 x_1) - \rho\left(\frac{\partial}{\partial\beta_1}\beta_1 x_1\right)\left((1 - \rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{\left((1 - \rho)x_1 + \rho\left(\frac{\partial}{\partial\beta_1}\beta_1{}^2\right)x_1{}^2\right)\times(1 - \rho + \rho\beta_1 x_1) - \rho\left(\frac{\partial}{\partial\beta_1}\beta_1\right)x_1\left((1 - \rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{\left((1 - \rho)x_1 + \rho\left(\frac{\partial}{\partial\beta_1}\beta_1{}^2\right)x_1{}^2\right)\times(1 - \rho + \rho\beta_1 x_1) - \rho x_1\left((1 - \rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{(1 - \rho + \rho\beta_1 x_1)\times\left((1 - \rho)x_1 + 2\,\rho x_1{}^2\beta_1\right) - \rho x_1\left((1 - \rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

Out $= \frac{(1-\rho)x_1 + 2\,\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2}$

Walk D$[L, \beta_2] = \frac{\partial L}{\partial\beta_2}$

$= \frac{\partial}{\partial\beta_2}\Big(n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] -$
$\sum_{\mathbf{i}=1}^{n}\frac{\beta_1 x_1(1-\rho) + \rho\beta_1{}^2 x_1{}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}\Big)$

$= \frac{\partial}{\partial\beta_2}n\big((\log[\beta_1] + \log[\beta_2])\big) + \frac{\partial}{\partial\beta_2}\Big(-\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] -$
$\sum_{\mathbf{i}=1}^{n}\frac{\beta_1 x_1(1-\rho) + \rho\beta_1{}^2 x_1{}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}\Big)$

$$= n\left(\frac{\partial}{\partial\beta_2}(\log[\beta_1] + \log[\beta_2])\right) + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2{x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}$$

$$= n\left(\frac{\partial}{\partial\beta_2}(\log[\beta_1] + \log[\beta_2])\right) + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2{x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}$$

$$= n\left(\frac{\partial}{\partial\beta_2}\log[\beta_1] + \frac{\partial}{\partial\beta_2}\log[\beta_2]\right) + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2{x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}$$

$$= -\left(\frac{\partial}{\partial\beta_2}\sum_{\mathbf{i}=1}^{n}\frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2{x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}\right) + n\left(\frac{\partial}{\partial\beta_2}\log[\beta_1] + \frac{1}{\beta_2}\right)$$

$$= -\left(\frac{\partial}{\partial\beta_2}\sum_{\mathbf{i}=1}^{n}\frac{\beta_1 x_1(1-\rho) + \rho{\beta_1}^2{x_1}^2 + \beta_2 y_j}{1-\rho+\rho\beta_1 x_1}\right) + \frac{n}{\beta_2}$$

WalkD:

$$\text{Out} = \frac{n}{\beta_2} - \sum_{\mathbf{i}=1}^{n}\frac{y_i}{1-\rho+\rho\beta_1 x_1}$$

$$= WalkD\left[\frac{(1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_i\beta_2}{1-\rho+\rho\beta_1 x_1}, \beta_2\right]$$

$$\frac{\partial}{\partial\beta_2}\frac{(1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_i\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= \frac{\frac{\partial}{\partial\beta_2}\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_i\beta_2\right)}{1-\rho+\rho\beta_1 x_1}$$

$$= \frac{\frac{\partial}{\partial\beta_2}(1-\rho)x_1\beta_1 + \frac{\partial}{\partial\beta_2}\left(\rho{x_1}^2{\beta_1}^2 + y_i\beta_2\right)}{1-\rho+\rho\beta_1 x_1}$$

$$= \frac{\frac{\partial}{\partial\beta_2}\left(\rho{x_1}^2{\beta_1}^2 + y_i\beta_2\right)}{1-\rho+\rho\beta_1 x_1}$$

$$= \frac{\frac{\partial}{\partial\beta_2}\rho{x_1}^2{\beta_1}^2 + \frac{\partial}{\partial\beta_2}y_i\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= \frac{\frac{\partial}{\partial \beta_2} y_i \beta_2}{1 - \rho + \rho\beta_1 x_1}$$

$$= \frac{\left(\frac{\partial}{\partial \beta_2} \beta_2\right) y_i}{1 - \rho + \rho\beta_1 x_1}$$

Out $= \frac{y_i}{1 - \rho + \rho\beta_1 x_1}$

To get Fisher's information, we have the following

Walk D[Walk D$[L, \rho], \rho] = \frac{\partial^2 L}{\partial \rho^2}$

$= \frac{\partial}{\partial \rho}\Big(n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}\Big) \quad = \frac{\partial}{\partial \rho} n(\log[\beta_1] + \log[\beta_2]) + \frac{\partial}{\partial \rho}\Big(- \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1-\rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}\Big)$

$$= \frac{\partial}{\partial \rho}\left(- \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1 - \rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}\right)$$

$$= \frac{\partial}{\partial \rho} - \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] + \frac{\partial}{\partial \rho} - \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1 - \rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}$$

$$= -\left(\frac{\partial}{\partial \rho} \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] \right) - \frac{\partial}{\partial \rho} \sum_{\mathbf{i}=1}^{n} \frac{\beta_1 x_1 (1 - \rho) + \rho{\beta_1}^2 {x_1}^2 + \beta_2 y_j}{1 - \rho + \rho\beta_1 x_1}$$

$= \frac{\partial}{\partial \rho}\left(- \sum_{\mathbf{i}=1}^{n} \frac{-1 + x_1\beta_1}{1 - \rho + \rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{-x_1\beta_1 + {x_1}^2{\beta_1}^2}{1 - \rho + \rho\beta_1 x_1} + \left(\frac{1}{(1 - \rho + \rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1 - \rho + \rho\beta_1 x_1)^2}\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)\right)$

$= \frac{\partial}{\partial \rho} - \sum_{\mathbf{i}=1}^{n} \frac{-1 + x_1\beta_1}{1 - \rho + \rho\beta_1 x_1} + \frac{\partial}{\partial \rho} - \sum_{\mathbf{i}=1}^{n}\left(\frac{-x_1\beta_1 + {x_1}^2{\beta_1}^2}{1 - \rho + \rho\beta_1 x_1} + \left(\frac{1}{(1 - \rho + \rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1 - \rho + \rho\beta_1 x_1)^2}\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$

$= -\left(\frac{\partial}{\partial\rho}\sum_{\mathbf{i}=1}^{n}\frac{-1+x_1\beta_1}{1-\rho+\rho\beta_1x_1}\right) - \frac{\partial}{\partial\rho}\sum_{\mathbf{i}=1}^{n}\left(\frac{-x_1\beta_1+{x_1}^2{\beta_1}^2}{1-\rho+\rho\beta_1x_1} + \left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$

Out $= -\sum_{\mathbf{i}=1}^{n}(-1 + x_1\beta_1)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) - \sum_{\mathbf{i}=1}^{n}\left(2\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(\frac{2}{(1-\rho+\rho\beta_1x_1)^3} - \frac{2x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^3}\right) - x_1\beta_1\left(\frac{2}{(1-\rho+\rho\beta_1x_1)^3} - \frac{2x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$

In $=$ Walk D $\left[\frac{-x_1\beta_1+{x_1}^2{\beta_1}^2}{1-\rho+\rho\beta_1x_1} + \left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right), \rho\right]$

$= \frac{\partial}{\partial\rho}\left(\frac{-x_1\beta_1+{x_1}^2{\beta_1}^2}{1-\rho+\rho\beta_1x_1} + \left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$

$= \frac{\partial}{\partial\rho}\frac{-x_1\beta_1+{x_1}^2{\beta_1}^2}{1-\rho+\rho\beta_1x_1} + \frac{\partial}{\partial\rho}\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$

$= \frac{\partial}{\partial\rho}\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right) + \left(\frac{\partial}{\partial\rho}\frac{1}{1-\rho+\rho\beta_1x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)$

$= \left(\frac{\partial}{\partial\rho}\frac{1}{1-\rho+\rho\beta_1x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(\frac{\partial}{\partial\rho}\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(\frac{\partial}{\partial\rho}\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$

$= \left(\frac{\partial}{\partial\rho}\frac{1}{1-\rho+\rho\beta_1x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(\frac{\partial}{\partial\rho}(1-\rho)x_1\beta_1 + \frac{\partial}{\partial\rho}\left(\rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho\beta_1x_1)^2} + \frac{\partial}{\partial\rho} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$

$$= \left(\frac{\partial}{\partial \rho}\frac{1}{1-\rho+\rho\beta_1 x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(\frac{\partial}{\partial \rho}\rho {x_1}^2{\beta_1}^2 + \frac{\partial}{\partial \rho}y_j\beta_2 + \left(\frac{\partial}{\partial \rho}(1 - \rho)\beta_1\right)x_1\right)\left(\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right) + \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{\partial}{\partial \rho}\frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= \left(\frac{\partial}{\partial \rho}\frac{1}{1-\rho+\rho\beta_1 x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(\frac{\partial}{\partial \rho}\rho {x_1}^2{\beta_1}^2 + \left(\frac{\partial}{\partial \rho}(1 - \rho)\beta_1\right)x_1\right)\left(\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right) + \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{\partial}{\partial \rho}\frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= \left(\frac{\partial}{\partial \rho}\frac{1}{1-\rho+\rho\beta_1 x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(\left(\frac{\partial}{\partial \rho}\rho {\beta_1}^2\right){x_1}^2 + \left(\frac{\partial}{\partial \rho}(1 - \rho)\right)x_1\beta_1\right)\left(\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right) + \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \left(\frac{\partial}{\partial \rho}\frac{\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right)x_1\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= \left(\frac{\partial}{\partial \rho}\frac{1}{1-\rho+\rho\beta_1 x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(\left(\frac{\partial}{\partial \rho}1 + \frac{\partial}{\partial \rho} - \rho\right)x_1\beta_1 + \left(\frac{\partial}{\partial \rho}\rho\right){x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right) + \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\right)x_1\beta_1\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= \left(\frac{\partial}{\partial \rho}\frac{1}{1-\rho+\rho\beta_1 x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(\left(\frac{\partial}{\partial \rho}1 + \frac{\partial}{\partial \rho} - \rho\right)x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right) + \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\right)x_1\beta_1\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= \left(\frac{\partial}{\partial \rho}\frac{1}{1-\rho+\rho\beta_1 x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(\left(\frac{\partial}{\partial \rho} - \rho\right)x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right) + \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \left(\frac{\partial}{\partial \rho}\frac{1}{(1-\rho+\rho\beta_1 x_1)^2}\right)x_1\beta_1\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= \left(\frac{\partial}{\partial\rho}\frac{1}{1-\rho+\rho\beta_1x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(-\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho\beta_1x_1)^2}\right)x_1\beta_1\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= \left(\frac{\partial}{\partial\rho}\frac{1}{1-\rho+\rho\beta_1x_1}\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right) + \left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho\beta_1x_1)^2}\right)x_1\beta_1\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= -\frac{\left(\frac{\partial}{\partial\rho}(1-\rho+\rho\beta_1x_1)\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)}{(1-\rho+\rho\beta_1x_1)^2} + \left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(-\frac{2\left(\frac{\partial}{\partial\rho}(1-\rho+\rho\beta_1x_1)\right)}{(1-\rho+\rho\beta_1x_1)^3} + \frac{2\left(\frac{\partial}{\partial\rho}(1-\rho+\rho\beta_1x_1)\right)x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= -\frac{\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1x_1)\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)}{(1-\rho+\rho\beta_1x_1)^2} + \left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(-\frac{2\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1x_1)\right)}{(1-\rho+\rho\beta_1x_1)^3} + \frac{2\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1x_1)\right)x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= -\frac{\left(\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1x_1)\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)}{(1-\rho+\rho\beta_1x_1)^2} + \left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(-\frac{2\left(\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1x_1)\right)}{(1-\rho+\rho\beta_1x_1)^3} + \frac{2\left(\frac{\partial}{\partial\rho}(-\rho+\rho\beta_1x_1)\right)x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= -\frac{\left(\frac{\partial}{\partial\rho}-\rho+\frac{\partial}{\partial\rho}\rho\beta_1x_1\right)\left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)}{(1-\rho+\rho\beta_1x_1)^2} + \left(-x_1\beta_1 + {x_1}^2{\beta_1}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right) + \left(-\frac{2\left(\frac{\partial}{\partial\rho}-\rho+\frac{\partial}{\partial\rho}\rho\beta_1x_1\right)}{(1-\rho+\rho\beta_1x_1)^3} + \frac{2\left(\frac{\partial}{\partial\rho}-\rho+\frac{\partial}{\partial\rho}\rho\beta_1x_1\right)x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)$$

$$= -\frac{\left(-\left(\frac{\partial}{\partial\rho}\rho\right)+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)}{(1-\rho+\rho\beta_1x_1)^2}+\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2}-\frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)+\left(-\frac{2\left(-\left(\frac{\partial}{\partial\rho}\rho\right)+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)}{(1-\rho+\rho\beta_1x_1)^3}+\frac{2x_1\left(-\left(\frac{\partial}{\partial\rho}\rho\right)+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\beta_1}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)$$

$$= -\frac{\left(-1+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)}{(1-\rho+\rho\beta_1x_1)^2}+\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2}-\frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)+\left(-\frac{2\times\left(-1+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)}{(1-\rho+\rho\beta_1x_1)^3}+\frac{2x_1\left(-1+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\beta_1}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)$$

$$= -\frac{\left(-1+\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1\right)\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)}{(1-\rho+\rho\beta_1x_1)^2}+\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2}-\frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)+\left(-\frac{2\times\left(-1+\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1\right)}{(1-\rho+\rho\beta_1x_1)^3}+\frac{2x_1\beta_1\left(-1+\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1\right)}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)$$

$$= -\frac{(-1+x_1\beta_1)\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)}{(1-\rho+\rho\beta_1x_1)^2}+\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2}-\frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)+\left(-\frac{2\times(-1+x_1\beta_1)}{(1-\rho+\rho\beta_1x_1)^3}+\frac{2x_1\beta_1(-1+x_1\beta_1)}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)$$

Out $= -\frac{(-1+x_1\beta_1)\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)}{(1-\rho+\rho\beta_1x_1)^2}+\left(-x_1\beta_1+x_1{}^2\beta_1{}^2\right)\left(\frac{1}{(1-\rho+\rho\beta_1x_1)^2}-\frac{x_1\beta_1}{(1-\rho+\rho\beta_1x_1)^2}\right)+\left(-\frac{2\times(-1+x_1\beta_1)}{(1-\rho+\rho\beta_1x_1)^3}+\frac{2x_1\beta_1(-1+x_1\beta_1)}{(1-\rho+\rho\beta_1x_1)^3}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)$

In Walk D $\left[\frac{-1+x_1\beta_1}{1-\rho+\rho x_1\beta_1},\rho\right]$

$$\frac{\partial}{\partial\rho}\frac{-1+x_1\beta_1}{1-\rho+\rho x_1\beta_1}$$

$$= \left(\frac{\partial}{\partial\rho}\frac{1}{1-\rho+\rho x_1\beta_1}\right)(-1+x_1\beta_1)$$

$$= -\frac{\frac{\partial}{\partial\rho}(1-\rho+\rho x_1\beta_1)(-1+x_1\beta_1)}{(1-\rho+\rho\beta_1x_1)^2}$$

$$= -\frac{\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}(-\rho+\rho x_1\beta_1)\right)(-1+x_1\beta_1)}{(1-\rho+\rho\beta_1x_1)^2}$$

$$= -\frac{\left(\frac{\partial}{\partial\rho}(-\rho + \rho x_1\beta_1)\right)(-1 + x_1\beta_1)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= -\frac{\left(\frac{\partial}{\partial\rho} - \rho + \frac{\partial}{\partial\rho}\rho x_1\beta_1\right)(-1 + x_1\beta_1)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= -\frac{\left(-\left(\frac{\partial}{\partial\rho}\ \rho\right) + \left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)(-1 + x_1\beta_1)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= -\frac{\left(-1 + \left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right) \times (-1 + x_1\beta_1)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= -\frac{(-1 + x_1\beta_1) \times \left(-1 + \left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= -\frac{(-1 + x_1\beta_1)^2}{(1 - \rho + \rho\beta_1 x_1)^2}$$

Out

$$= -\frac{(-1 + x_1\beta_1)^2}{(1 - \rho + \rho\beta_1 x_1)^2}$$

In Walk D $\left[\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2}{1-\rho+\rho x_1\beta_1}, \rho\right]$

$$\frac{\partial}{\partial\rho}\frac{(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1 - \rho + \rho x_1\beta_1}$$

$$= \frac{1}{(1-\rho+\rho x_1\beta_1)^2}\left(\left(\frac{\partial}{\partial\rho}\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)(1 - \rho + \rho x_1\beta_1) - \left(\frac{\partial}{\partial\rho}(1 - \rho + \rho x_1\beta_1)\right)\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$$

$$= \frac{1}{(1-\rho+\rho x_1\beta_1)^2}\left(\left(\frac{\partial}{\partial\rho}(1\,-\,\rho)x_1\beta_1+\frac{\partial}{\partial\rho}\left(\,\rho {x_1}^2{\beta_1}^2\,+y_j\beta_2\right)\right)(1\,-\,\rho\,+\,\rho x_1\beta_1)-\left(\frac{\partial}{\partial\rho}1+\right.\right.$$

$$\left.\left.\frac{\partial}{\partial\rho}(-\rho\,+\,\rho x_1\beta_1)\right)\left((1\,-\,\rho)x_1\beta_1\,+\rho {x_1}^2{\beta_1}^2\,+y_j\beta_2\right)\right)$$

$$= \frac{1}{(1-\rho+\rho x_1\beta_1)^2}\left(\left(\frac{\partial}{\partial\rho}(1\,-\,\rho)x_1\beta_1+\frac{\partial}{\partial\rho}\left(\,\rho {x_1}^2{\beta_1}^2\,+y_j\beta_2\right)\right)(1\,-\,\rho\,+\,\rho x_1\beta_1)-\right.$$

$$\left.\left(\frac{\partial}{\partial\rho}(-\rho\,+\,\rho x_1\beta_1)\right)\left((1\,-\,\rho)x_1\beta_1\,+\rho {x_1}^2{\beta_1}^2\,+y_j\beta_2\right)\right)$$

$$= \frac{1}{(1-\rho+\rho x_1\beta_1)^2}\left(\left(\frac{\partial}{\partial\rho}\rho {x_1}^2{\beta_1}^2\,+\frac{\partial}{\partial\rho}y_j\beta_2+\left(\frac{\partial}{\partial\rho}(1\,-\,\rho)\beta_1\right)x_1\right)(1\,-\,\rho\,+\,\rho x_1\beta_1)-\left(\frac{\partial}{\partial\rho}-\right.\right.$$

$$\left.\left.\rho\,+\,\frac{\partial}{\partial\rho}\rho x_1\beta_1\right)\left((1\,-\,\rho)x_1\beta_1\,+\rho {x_1}^2{\beta_1}^2\,+y_j\beta_2\right)\right)$$

$$= \frac{\left(\frac{\partial}{\partial\rho}\rho {x_1}^2{\beta_1}^2+\left(\frac{\partial}{\partial\rho}(1-\rho)\beta_1\right)x_1\right)(1-\rho+\rho x_1\beta_1)-\left(\frac{\partial}{\partial\rho}-\rho+\frac{\partial}{\partial\rho}\rho x_1\beta_1\right)\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{1}{(1-\rho+\rho x_1\beta_1)^2}\left((1\,-\,\rho\,+\,\rho x_1\beta_1)\left(\left(\frac{\partial}{\partial\rho}\rho{\beta_1}^2\right){x_1}^2\,+\left(\frac{\partial}{\partial\rho}(1\,-\,\rho)\right)x_1\beta_1\right)-\left(-\left(\frac{\partial}{\partial\rho}\,\rho\right)\,+\right.\right.$$

$$\left.\left.\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\left((1\,-\,\rho)x_1\beta_1\,+\rho {x_1}^2{\beta_1}^2\,+y_j\beta_2\right)\right)$$

$$= \frac{1}{(1-\rho+\rho x_1\beta_1)^2}\left((1\,-\,\rho\,+\,\rho x_1\beta_1)\left(\left(\frac{\partial}{\partial\rho}\rho{\beta_1}^2\right){x_1}^2\,+\left(\frac{\partial}{\partial\rho}(1\,-\,\rho)\right)x_1\beta_1\right)-\left(-1\,+\right.\right.$$

$$\left.\left.\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\left((1\,-\,\rho)x_1\beta_1\,+\rho {x_1}^2{\beta_1}^2\,+y_j\beta_2\right)\right)$$

$$= \frac{1}{(1-\rho+\rho x_1\beta_1)^2}\left((1\,-\,\rho\,+\,\rho x_1\beta_1)\left(\,\left(\frac{\partial}{\partial\rho}1+\,\frac{\partial}{\partial\rho}-\,\rho\right)x_1\beta_1+\left(\frac{\partial}{\partial\rho}\rho\right){x_1}^2{\beta_1}^2\right)-\left(-1\,+\right.\right.$$

$$\left.\left.\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1\right)\left((1\,-\,\rho)x_1\beta_1\,+\rho {x_1}^2{\beta_1}^2\,+y_j\beta_2\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)\left(\,\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}-\rho\right)x_1\beta_1+{x_1}^2{\beta_1}^2\right)-(-1+x_1\beta_1)\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{(1-\rho+\rho x_1\beta_1)\left(\left(\frac{\partial}{\partial\rho}-\rho\right)x_1\beta_1+x_1{}^2\beta_1{}^2\right)-(-1+x_1\beta_1)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{(1-\rho+\rho x_1\beta_1)\left(-\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1+x_1{}^2\beta_1{}^2\right)-(-1+x_1\beta_1)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1\beta_1+x_1{}^2\beta_1{}^2)-(-1+x_1\beta_1)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

Out = $\dfrac{(-x_1\beta_1+x_1{}^2\beta_1{}^2)}{(1-\rho+\rho x_1\beta_1)} - \dfrac{(-1+x_1\beta_1)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}$

In Walk D[log$(1 - \rho + \rho x_1\beta_1)\,,\rho$]

$$\frac{\partial}{\partial\rho}\log(1 - \rho + \rho x_1\beta_1)$$

$$= \frac{\frac{\partial}{\partial\rho}(1 - \rho + \rho x_1\beta_1)}{1 - \rho + \rho x_1\beta_1}$$

$$= \frac{\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}(-\rho + \rho x_1\beta_1)}{1 - \rho + \rho x_1\beta_1}$$

$$= \frac{\frac{\partial}{\partial\rho}(-\rho + \rho x_1\beta_1)}{1 - \rho + \rho x_1\beta_1}$$

$$= \frac{\frac{\partial}{\partial\rho}-\rho + \frac{\partial}{\partial\rho}\rho x_1\beta_1}{1 - \rho + \rho x_1\beta_1}$$

$$= \frac{-\left(\frac{\partial}{\partial\rho}\rho\right) + \left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1}{1 - \rho + \rho x_1\beta_1}$$

$$= \frac{-1 + \left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1}{1 - \rho + \rho x_1\beta_1}$$

$$= \frac{-1 + \left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1}{1 - \rho + \rho x_1\beta_1}$$

$$= \frac{-1 + x_1\beta_1}{1 - \rho + \rho x_1\beta_1}$$

$$\text{Out} = \frac{-1 + x_1\beta_1}{1 - \rho + \rho x_1\beta_1}$$

Walk D[Walk D[L, β_1], β_1] $= \frac{\partial^2 L}{\partial {\beta_1}^2}$

$$= \frac{\partial}{\partial\beta_1}\left(n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right)$$

$$= \frac{\partial}{\partial\beta_1}n(\log[\beta_1] + \log[\beta_2]) + \frac{\partial}{\partial\beta_1}\left(-\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right)$$

$$= n\left(\frac{\partial}{\partial\beta_1}(\log[\beta_1] + \log[\beta_2])\right) + \frac{\partial}{\partial\beta_1} - \sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] + \frac{\partial}{\partial\beta_1} - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= n\left(\frac{\partial}{\partial\beta_1}\log[\beta_1] + \frac{\partial}{\partial\beta_1}\log[\beta_2]\right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= -\left(\frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)]\right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1} + n\left(\frac{\partial}{\partial\beta_1}\log[\beta_2] + \frac{1}{\beta_1}\right)$$

$$= -\left(\frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)]\right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1} + \frac{n}{\beta_1}$$

WalkD:

$$\frac{\partial}{\partial\beta_1}\left(\frac{n}{\beta_1} - \sum_{\mathbf{i}=1}^{n}\frac{\rho x_i}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

$$= \frac{\partial}{\partial\beta_1}\frac{n}{\beta_1} + \frac{\partial}{\partial\beta_1}\left(-\sum_{\mathbf{i}=1}^{n}\frac{\rho x_i}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

$$= n\left(\frac{\partial}{\partial\beta_1}\frac{1}{\beta_1}\right) + \frac{\partial}{\partial\beta_1} - \sum_{\mathbf{i}=1}^{n}\frac{\rho x_i}{1-\rho+\rho\beta_1 x_1} + \frac{\partial}{\partial\beta_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= n\left(\frac{\partial}{\partial\beta_1}\frac{1}{\beta_1}\right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\frac{\rho x_i}{1-\rho+\rho\beta_1 x_1} - \frac{\partial}{\partial\beta_1}\ \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1+2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= -\left(\frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\frac{\rho x_i}{1-\rho+\rho\beta_1 x_1}\right) - \frac{\partial}{\partial\beta_1}\ \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1+2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right) - \frac{n}{\beta_1{}^2}$$

WalkD

Out $$= -\frac{n}{\beta_1{}^2} - \sum_{\mathbf{i}=1}^{n} - \frac{\rho^2 x_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} - \ \sum_{\mathbf{i}=1}^{n}\left(\frac{2\rho x_1{}^2}{1-\rho+\rho\beta_1 x_1} - \frac{2\rho x_1\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\rho^2 x_1{}^2(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

In Walk D $\left[\frac{(1-\rho)x_1+2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}, \beta_1\right]$

$$= \frac{\partial}{\partial\beta_1}\left(\frac{(1-\rho)x_1+2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= \frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1+2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} + \frac{\partial}{\partial\beta_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1+2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\partial}{\partial\beta_1}\frac{\rho x_i(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1+2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \rho\left(\frac{\partial}{\partial\beta_1}\frac{x_i(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= \frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1+2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i$$

$$= -\rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(\frac{\partial}{\partial\beta_1}\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)(1-\rho+\rho\beta_1 x_1) - \left(\frac{\partial}{\partial\beta_1}(1-\rho+\rho\beta_1 x_1)\right)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(\frac{\partial}{\partial\beta_1}(1-\rho)x_1 + \frac{\partial}{\partial\beta_1}2\rho x_1{}^2\beta_1\right)(1-\rho+\rho\beta_1 x_1) - \left(\frac{\partial}{\partial\beta_1}1 + \frac{\partial}{\partial\beta_1}(-\rho+\rho\beta_1 x_1)\right)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i +$$
$$\frac{\left(\frac{\partial}{\partial\beta_1}2\rho x_1{}^2\beta_1\right)(1-\rho+\rho\beta_1x_1) - \left(\frac{\partial}{\partial\beta_1}(-\rho+\rho\beta_1x_1)\right)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i +$$
$$\frac{2\left(\frac{\partial}{\partial\beta_1}\rho x_1{}^2\beta_1\right)(1-\rho+\rho\beta_1x_1) - \left(\frac{\partial}{\partial\beta_1}-\rho+\frac{\partial}{\partial\beta_1}\rho\beta_1x_1\right)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{2\left(\frac{\partial}{\partial\beta_1}\rho x_1{}^2\beta_1\right)(1-\rho+\rho\beta_1x_1) - \left(\frac{\partial}{\partial\beta_1}\rho\beta_1x_1\right)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{2\rho\left(\frac{\partial}{\partial\beta_1}x_1{}^2\beta_1\right)(1-\rho+\rho\beta_1x_1) - \rho\left(\frac{\partial}{\partial\beta_1}\beta_1x_1\right)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{2\rho\left(\frac{\partial}{\partial\beta_1}\beta_1\right)x_1{}^2(1-\rho+\rho\beta_1x_1) - \rho\left(\frac{\partial}{\partial\beta_1}\beta_1\right)x_1\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\rho\left(\frac{\partial}{\partial\beta_1}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{2\rho x_1{}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{2\rho x_1{}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{\frac{\partial}{\partial\beta_1}\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} +\right.$$
$$\left.\left(\frac{\partial}{\partial\beta_1}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)$$

$$= \frac{2\rho x_1{}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{\frac{\partial}{\partial\beta_1}(1-\rho)x_1\beta_1 + \frac{\partial}{\partial\beta_1}\left(\rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} +\right.$$
$$\left.\left(\frac{\partial}{\partial\beta_1}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)$$

$$= \frac{2\rho x_1{}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)\left(\frac{\partial}{\partial\beta_1}x_1\beta_1\right) + \frac{\partial}{\partial\beta_1}\rho x_1{}^2\beta_1{}^2 + \frac{\partial}{\partial\beta_1}y_j\beta_2}{(1-\rho+\rho x_1\beta_1)^2} +\right.$$
$$\left.\left(\frac{\partial}{\partial\beta_1}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)\right)$$

$$= \frac{2\rho{x_1}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho{x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)\left(\frac{\partial}{\partial\beta_1}x_1\beta_1\right) + \frac{\partial}{\partial\beta_1}\rho{x_1}^2{\beta_1}^2}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\left(\frac{\partial}{\partial\beta_1}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$$

$$= \frac{2\rho{x_1}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho{x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{\rho\left(\frac{\partial}{\partial\beta_1}{x_1}^2{\beta_1}^2\right) + (1-\rho)\left(\frac{\partial}{\partial\beta_1}\beta_1\right)x_1}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\left(\frac{\partial}{\partial\beta_1}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$$

$$= \frac{2\rho{x_1}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho{x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{\rho\left(\frac{\partial}{\partial\beta_1}{x_1}^2{\beta_1}^2\right) + (1-\rho)x_1}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\left(\frac{\partial}{\partial\beta_1}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$$

$$= \frac{2\rho{x_1}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho{x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + \rho\left(\frac{\partial}{\partial\beta_1}{\beta_1}^2\right){x_1}^2}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\left(\frac{\partial}{\partial\beta_1}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$$

$$= \frac{2\rho{x_1}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho{x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho{x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\left(\frac{\partial}{\partial\beta_1}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$$

$$= \frac{2\rho{x_1}^2(1-\rho+\rho\beta_1x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho{x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho{x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} - \right.$$

$$\left.\frac{2\left(\frac{\partial}{\partial\beta_1}(1-\rho+\rho\beta_1x_1)\right)\left((1-\rho)x_1\beta_1 + \rho{x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

$$= \frac{2\rho {x_1}^2(1-\rho+\rho\beta_1 x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(\frac{\partial}{\partial\beta_1}1 + \frac{\partial}{\partial\beta_1}(-\rho+\rho\beta_1 x_1)\right)\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

$$= \frac{2\rho {x_1}^2(1-\rho+\rho\beta_1 x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(\frac{\partial}{\partial\beta_1}(-\rho+\rho\beta_1 x_1)\right)\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

$$= \frac{2\rho {x_1}^2(1-\rho+\rho\beta_1 x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(\frac{\partial}{\partial\beta_1}-\rho + \frac{\partial}{\partial\beta_1}\rho\beta_1 x_1\right)\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

$$= \frac{2\rho {x_1}^2(1-\rho+\rho\beta_1 x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(\frac{\partial}{\partial\beta_1}\rho\beta_1 x_1\right)\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

$$= \frac{2\rho {x_1}^2(1-\rho+\rho\beta_1 x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\rho\left(\frac{\partial}{\partial\beta_1}\beta_1 x_1\right)\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

$$= \frac{2\rho {x_1}^2(1-\rho+\rho\beta_1 x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\rho\left(\frac{\partial}{\partial\beta_1}\beta_1\right)x_1\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

$$= \frac{2\rho {x_1}^2(1-\rho+\rho\beta_1 x_1) - \rho x_1\left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \rho x_i\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\rho x_1\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)$$

$$\text{Out} = \frac{2\rho {x_1}^2}{(1-\rho+\rho\beta_1 x_1)} - \frac{2\rho x_i\left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\rho^2 {x_1}^2\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}$$

Walk D[Walk D[L, β_2], β_2] $= \frac{\partial^2 L}{\partial {\beta_2}^2}$

$$= \frac{\partial}{\partial\beta_2}\left(n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n}\log[(1-\rho+\rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right)$$

$$= \frac{\partial}{\partial\beta_2}n(\log[\beta_1] + \log[\beta_2]) + \frac{\partial}{\partial\beta_2}\left(-\sum_{\mathbf{i}=1}^{n}\log[(1-\rho+\rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right)$$

$$= n\left(\frac{\partial}{\partial\beta_2}(\log[\beta_1] + \log[\beta_2])\right) + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\log[(1-\rho+\rho\beta_1 x_1)] + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= n\left(\frac{\partial}{\partial\beta_2}(\log[\beta_1] + \log[\beta_2])\right) + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= n\left(\frac{\partial}{\partial\beta_2}\log[\beta_1] + \frac{\partial}{\partial\beta_2}\log[\beta_2]\right) - \frac{\partial}{\partial\beta_2}\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= -\left(\frac{\partial}{\partial\beta_2}\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right) + n\left(\frac{\partial}{\partial\beta_2}\log[\beta_1] + \frac{1}{\beta_2}\right)$$

$$= -\left(\frac{\partial}{\partial\beta_2}\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right) + \frac{n}{\beta_2}$$

WalkD:

$$\frac{\partial}{\partial\beta_2}\left(\frac{n}{\beta_2} - \sum_{\mathbf{i}=1}^{n}\frac{y_i}{1-\rho+\rho x_1\beta_1}\right)$$

$$= \frac{\partial}{\partial\beta_2}\frac{n}{\beta_2} + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\frac{y_i}{1-\rho+\rho x_1\beta_1}$$

$$= \frac{\partial}{\partial\beta_2}\frac{n}{\beta_2}$$

$$= n\left(\frac{\partial}{\partial\beta_2}\frac{1}{\beta_2}\right)$$

$$= -\frac{n}{{\beta_2}^2}$$

$out \ = -\frac{n}{{\beta_2}^2}$

In Walk D $\left[\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho x_1\beta_1}, \beta_2\right]$

$\frac{\partial}{\partial\beta_2}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{1-\rho+\rho x_1\beta_1}$

$= \frac{\frac{\partial}{\partial\beta_2}\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2\right)}{1 - \rho + \rho x_1\beta_1}$

$= \frac{\frac{\partial}{\partial\beta_2}(1 - \rho)x_1\beta_1 + \frac{\partial}{\partial\beta_2}\left(\rho {x_1}^2{\beta_1}^2 + y_i\beta_2\right)}{1 - \rho + \rho x_1\beta_1}$

$= \frac{\frac{\partial}{\partial\beta_2}\left(\rho {x_1}^2{\beta_1}^2 + y_i\beta_2\right)}{1 - \rho + \rho x_1\beta_1}$

$= \frac{\frac{\partial}{\partial\beta_2}\rho {x_1}^2{\beta_1}^2 + \frac{\partial}{\partial\beta_2}y_i\beta_2}{1 - \rho + \rho x_1\beta_1}$

$= \frac{\frac{\partial}{\partial\beta_2}y_i\beta_2}{1 - \rho + \rho x_1\beta_1}$

$= \frac{\left(\frac{\partial}{\partial\beta_2}\beta_2\right)y_i}{1 - \rho + \rho x_1\beta_1}$

$= \frac{y_i}{1 - \rho + \rho x_1\beta_1}$

$out \ = \frac{y_i}{1 - \rho + \rho x_1\beta_1}$

Walk D[Walk D$[L, \beta_1], \beta_2] = \frac{\partial^2 L}{\partial\beta_1\partial\beta_2} = \frac{\partial^2 L}{\partial\beta_2\partial\beta_1}$

$= \frac{\partial}{\partial\beta_1}\Bigg(n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] -$
$\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\Bigg)$

$$= \frac{\partial}{\partial\beta_1} n(\log[\beta_1] \ + \log[\beta_2]) + \frac{\partial}{\partial\beta_1}\left(-\sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n} \frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right)$$

$$= n\left(\frac{\partial}{\partial\beta_1}(\log[\beta_1] \ + \log[\beta_2])\right) + \frac{\partial}{\partial\beta_1} - \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] + \frac{\partial}{\partial\beta_1} - \sum_{\mathbf{i}=1}^{n} \frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= n\left(\frac{\partial}{\partial\beta_1}\log[\beta_1] \ + \frac{\partial}{\partial\beta_1}\log[\beta_2]\right) - \frac{\partial}{\partial\beta_1} \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] - \frac{\partial}{\partial\beta_1} \sum_{\mathbf{i}=1}^{n} \frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= -\left(\frac{\partial}{\partial\beta_1} \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] \ \right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n} \frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1} + n\left(\frac{\partial}{\partial\beta_1}\log[\beta_2] + \frac{1}{\beta_1}\right)$$

$$= -\left(\frac{\partial}{\partial\beta_1} \sum_{\mathbf{i}=1}^{n} \log[(1 - \rho + \rho\beta_1 x_1)] \ \right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n} \frac{(1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1} + \frac{n}{\beta_1}$$

WalkD:

$$= \frac{\partial}{\partial\beta_2}\left(\frac{n}{\beta_1} - \sum_{\mathbf{i}=1}^{n} \frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

$$= \frac{\partial}{\partial\beta_2}\frac{n}{\beta_1} + \frac{\partial}{\partial\beta_2}\left(-\sum_{\mathbf{i}=1}^{n} \frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

$$= \frac{\partial}{\partial\beta_2}\left(-\sum_{\mathbf{i}=1}^{n} \frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

$$= \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n} \frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= -\left(\frac{\partial}{\partial\beta_2}\sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1 + \rho {x_1}^2 {\beta_1}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

WalkD:

$$\text{out} = -\sum_{\mathbf{i}=1}^{n} -\frac{\rho x_1 y_j}{(1 - \rho + \rho x_1 \beta_1)^2}$$

In Walk D $\left[\frac{(1-\rho)x_1+2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}, \beta_2\right]$

$$= \frac{\partial}{\partial\beta_2}\left(\frac{(1-\rho)x_1+2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= \frac{\partial}{\partial\beta_2}\frac{(1-\rho)x_1+2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} + \frac{\partial}{\partial\beta_2} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{\partial}{\partial\beta_2} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\beta_2}\frac{\rho x_i(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1 - \rho + \rho x_1\beta_1)^2}\right)$$

$$= -\rho\left(\frac{\partial}{\partial\beta_2}\frac{x_i(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1 - \rho + \rho x_1\beta_1)^2}\right)$$

$$= -\rho\left(\frac{\partial}{\partial\beta_2}\frac{(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1 - \rho + \rho x_1\beta_1)^2}\right)x_i$$

$$= -\frac{\rho\left(\frac{\partial}{\partial\beta_2}\left((1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2\right)\right)x_i}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= -\frac{\rho\left(\frac{\partial}{\partial\beta_2}(1 - \rho)x_1\beta_1 + \frac{\partial}{\partial\beta_2}\left(\rho {x_1}^2{\beta_1}^2 + y_i\beta_2\right)\right)x_i}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= -\frac{\rho\left(\frac{\partial}{\partial\beta_2}\left(\rho {x_1}^2{\beta_1}^2 + y_i\beta_2\right)\right)x_i}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= -\frac{\rho\left(\frac{\partial}{\partial\beta_2}\rho {x_1}^2{\beta_1}^2 + \frac{\partial}{\partial\beta_2}y_i\beta_2\right)x_i}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= -\frac{\rho\left(\frac{\partial}{\partial\beta_2}y_i\beta_2\right)x_i}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= -\frac{\rho\left(\frac{\partial}{\partial\beta_2}\beta_2\right)x_i y_i}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= -\frac{\rho x_i y_i}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$\text{out} = -\frac{\rho x_i y_i}{(1 - \rho + \rho x_1\beta_1)^2}$$

Walk D[Walk D[L, β_1], ρ] $= \frac{\partial^2 L}{\partial\beta_1\partial\rho}$

$$= \frac{\partial}{\partial\beta_1}\left(n(\log[\beta_1] + \log[\beta_2]) - \sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right)$$

$$= \frac{\partial}{\partial\beta_1}n(\log[\beta_1] + \log[\beta_2]) + \frac{\partial}{\partial\beta_1}\left(-\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}\right)$$

$$= n\left(\frac{\partial}{\partial\beta_1}(\log[\beta_1] + \log[\beta_2])\right) + \frac{\partial}{\partial\beta_1} - \sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] + \frac{\partial}{\partial\beta_1} - \sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= n\left(\frac{\partial}{\partial\beta_1}\log[\beta_1] + \frac{\partial}{\partial\beta_1}\log[\beta_2]\right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)] - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1}$$

$$= -\left(\frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)]\right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{1-\rho+\rho\beta_1 x_1} + n\left(\frac{\partial}{\partial\beta_1}\log[\beta_2] + \frac{1}{\beta_1}\right)$$

$$= -\left(\frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\log[(1 - \rho + \rho\beta_1 x_1)]\right) - \frac{\partial}{\partial\beta_1}\sum_{\mathbf{i}=1}^{n}\frac{(1 - \rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2}{1 - \rho + \rho\beta_1 x_1} + \frac{n}{\beta_1}$$

WalkD:

$$= \frac{\partial}{\partial\rho}\left(\frac{n}{\beta_1} - \sum_{\mathbf{i}=1}^{n}\frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1 + 2\rho x_1{}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

$$= \frac{\partial}{\partial\rho}\frac{n}{\beta_1} + \frac{\partial}{\partial\rho}\left(-\sum_{\mathbf{i}=1}^{n}\frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1+2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

$$= \frac{\partial}{\partial\rho}\left(-\sum_{\mathbf{i}=1}^{n}\frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1+2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)\right)$$

$$= \frac{\partial}{\partial\rho} - \sum_{\mathbf{i}=1}^{n}\frac{\rho x_1}{1-\rho+\rho\beta_1 x_1} + \frac{\partial}{\partial\beta_2} - \sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1+2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= -\left(\frac{\partial}{\partial\rho}\sum_{\mathbf{i}=1}^{n}\frac{\rho x_1}{1-\rho+\rho\beta_1 x_1}\right) - \frac{\partial}{\partial\beta_2}\sum_{\mathbf{i}=1}^{n}\left(\frac{(1-\rho)x_1+2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

WalkD:

Out $= -\sum_{\mathbf{i}=1}^{n}\left(\frac{x_1}{1-\rho+\rho\beta_1 x_1} + \rho x_1\left(\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right)\right) - \sum_{\mathbf{i}=1}^{n}\left(\frac{-x_1+2{x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_1\left(-x_1\beta_1+2{x_1}^2{\beta_1}^2\right)}{(1-\rho+\rho\beta_1 x_1)^2} + \left((1-\rho)x_1 + 2\rho {x_1}^2\beta_1\right)\left(\frac{1}{(1-\rho+\rho\beta_1 x_1)^2} - \frac{x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^2}\right) - \frac{x_1\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_j\beta_2\right)}{(1-\rho+\rho\beta_1 x_1)^2} - \rho x_1\left(\frac{2}{(1-\rho+\rho\beta_1 x_1)^3} - \frac{2x_1\beta_1}{(1-\rho+\rho\beta_1 x_1)^3}\right)\left((1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_j\beta_2\right)\right)$

In Walk D $\left[\frac{\rho x_1}{1-\rho+\rho x_1\beta_1}, \rho\right]$

$$\frac{\partial}{\partial\rho}\frac{\rho x_1}{1-\rho+\rho x_1\beta_1}$$

$$= \left(\frac{\partial}{\partial\rho}\frac{\rho}{1-\rho+\rho x_1\beta_1}\right)x_1$$

$$= \frac{x_1\left(-\rho\left(\frac{\partial}{\partial\rho}(1-\rho+\rho\beta_1 x_1)\right) + \left(\frac{\partial}{\partial\rho}\rho\right)(1-\rho+\rho\beta_1 x_1)\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{x_1\left(1-\rho-\rho\left(\frac{\partial}{\partial\rho}(1-\rho+\rho\beta_1 x_1)\right)+\rho\beta_1 x_1\right)}{(1-\rho+\rho\beta_1 x_1)^2}$$

$$= \frac{x_1\left(1 - \rho - \rho\left(\frac{\partial}{\partial\rho}1 + \frac{\partial}{\partial\rho}(-\rho + \rho\beta_1 x_1)\right) + \rho\beta_1 x_1\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{x_1\left(1 - \rho - \rho\left(\frac{\partial}{\partial\rho}(-\rho + \rho\beta_1 x_1)\right) + \rho\beta_1 x_1\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{x_1\left(1 - \rho - \rho\left(\frac{\partial}{\partial\rho} - \rho + \frac{\partial}{\partial\rho}\,\rho\beta_1 x_1\right) + \rho\beta_1 x_1\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{x_1\left(1 - \rho - \rho\left(-\left(\frac{\partial}{\partial\rho}\,\rho\right) + \left(\frac{\partial}{\partial\rho}\,\rho\beta_1\right)x_1\right) + \rho\beta_1 x_1\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{x_1\left(1 - \rho - \rho\left(-1 + \left(\frac{\partial}{\partial\rho}\,\rho\beta_1\right)x_1\right) + \rho\beta_1 x_1\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{x_1\left(1 - \rho + \rho\beta_1 x_1 - \rho\left(-1 + \left(\frac{\partial}{\partial\rho}\,\rho\right)x_1\beta_1\right)\right)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$= \frac{x_1\big(1 - \rho + \rho\beta_1 x_1 - \rho(-1 + x_1\beta_1)\big)}{(1 - \rho + \rho\beta_1 x_1)^2}$$

$$\text{out } = \frac{\rho x_1(-1 + x_1\beta_1)}{(1 - \rho + \rho\beta_1 x_1)^2} + \frac{x_1}{1 - \rho + \rho\beta_1 x_1}$$

In Walk D $\left[\frac{(1-\rho)x_1 + 2\rho {x_1}^2\beta_1}{1-\rho+\rho\beta_1 x_1} - \frac{\rho x_i(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2},\ \rho\right]$

$$= \frac{\partial}{\partial\rho}\left(\frac{(1 - \rho)x_1 + 2\rho {x_1}^2\beta_1}{1 - \rho + \rho\beta_1 x_1} - \frac{\rho x_i(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1 - \rho + \rho x_1\beta_1)^2}\right)$$

$$= \frac{\partial}{\partial\rho}\frac{(1 - \rho)x_1 + 2\rho {x_1}^2\beta_1}{1 - \rho + \rho\beta_1 x_1} + \frac{\partial}{\partial\rho} - \frac{\rho x_i(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= \frac{\partial}{\partial\rho}\frac{(1 - \rho)x_1 + 2\rho {x_1}^2\beta_1}{1 - \rho + \rho\beta_1 x_1} - \frac{\partial}{\partial\rho}\frac{\rho x_i(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= \frac{\partial}{\partial\rho}\frac{(1 - \rho)x_1 + 2\rho {x_1}^2\beta_1}{1 - \rho + \rho\beta_1 x_1} - \frac{\partial}{\partial\rho}\frac{\rho x_i(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1 - \rho + \rho x_1\beta_1)^2}$$

$$= \frac{\partial}{\partial\rho}\frac{(1 - \rho)x_1 + 2\rho {x_1}^2\beta_1}{1 - \rho + \rho\beta_1 x_1} - \left(\frac{\partial}{\partial\rho}\frac{\rho(1 - \rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1 - \rho + \rho x_1\beta_1)^2}\right)x_i$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(\frac{\partial}{\partial\rho}\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)\right)(1-\rho+\rho x_1\beta_1) - \left(\frac{\partial}{\partial\rho}(1-\rho+\rho x_1\beta_1)\right)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(\frac{\partial}{\partial\rho}(1-\rho)x_1+\frac{\partial}{\partial\rho}2\rho {x_1}^2\beta_1\right)(1-\rho+\rho x_1\beta_1) - \left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}(-\rho+\rho x_1\beta_1)\right)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(\frac{\partial}{\partial\rho}(1-\rho)x_1+\frac{\partial}{\partial\rho}2\rho {x_1}^2\beta_1\right)(1-\rho+\rho x_1\beta_1) - \left(\frac{\partial}{\partial\rho}(-\rho+\rho x_1\beta_1)\right)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(2\left(\frac{\partial}{\partial\rho}\rho {x_1}^2\beta_1\right)+\left(\frac{\partial}{\partial\rho}(1-\rho)\right)x_1\right)\times(1-\rho+\rho x_1\beta_1) - \left(\frac{\partial}{\partial\rho}-\rho+\frac{\partial}{\partial\rho}\rho x_1\beta_1\right)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}-\rho\right)x_1+2\left(\frac{\partial}{\partial\rho}\rho\beta_1\right){x_1}^2\right)(1-\rho+\rho x_1\beta_1) - \left(-\left(\frac{\partial}{\partial\rho}\rho\right)+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}-\rho\right)x_1+2\left(\frac{\partial}{\partial\rho}\rho\beta_1\right){x_1}^2\right)(1-\rho+\rho x_1\beta_1) - \left(-1+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho {x_1}^2{\beta_1}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i + \frac{\left(\left(\frac{\partial}{\partial\rho}-\rho\right)x_1+2\left(\frac{\partial}{\partial\rho}\rho\beta_1\right){x_1}^2\right)(1-\rho+\rho x_1\beta_1) - \left(-1+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i +$$

$$\frac{-\left(\left(-1+\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1\right)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)+(1-\rho+\rho x_1\beta_1)\left(-\left(\frac{\partial}{\partial\rho}\rho\right)x_1+2\left(\frac{\partial}{\partial\rho}\rho\right)x_1{}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= -\left(\frac{\partial}{\partial\rho}\frac{\rho(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right)x_i +$$

$$\frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} -$$

$$x_i\left(\rho\left(\frac{\partial}{\partial\rho}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right) + \frac{\left(\frac{\partial}{\partial\rho}\rho\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} -$$

$$x_i\left(\rho\left(\frac{\partial}{\partial\rho}\frac{(1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2}{(1-\rho+\rho x_1\beta_1)^2}\right) + \frac{\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} +\right.$$

$$\left.\rho\left(\frac{\frac{\partial}{\partial\rho}\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} +\right.$$

$$\left.\rho\left(\frac{\frac{\partial}{\partial\rho}(1-\rho)x_1\beta_1 + \frac{\partial}{\partial\rho}\left(\rho x_1{}^2\beta_1{}^2 + y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1 + \rho x_1{}^2\beta_1{}^2 + y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2{x_1}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \rho\left(\frac{\frac{\partial}{\partial\rho}\rho {x_1}^2{\beta_1}^2+\frac{\partial}{\partial\rho}y_i\beta_2+\left(\frac{\partial}{\partial\rho}(1-\rho)\beta_1\right)x_1}{(1-\rho+\rho x_1\beta_1)^2}+\left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2{x_1}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \rho\left(\frac{\frac{\partial}{\partial\rho}\rho {x_1}^2{\beta_1}^2+\left(\frac{\partial}{\partial\rho}(1-\rho)\beta_1\right)x_1}{(1-\rho+\rho x_1\beta_1)^2}+\left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2{x_1}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \rho\left(\frac{\left(\frac{\partial}{\partial\rho}\rho {\beta_1}^2\right){x_1}^2+\left(\frac{\partial}{\partial\rho}(1-\rho)\right)x_1\beta_1}{(1-\rho+\rho x_1\beta_1)^2}+\left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2{x_1}^2\beta_1)-\left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \rho\left(\frac{\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}-\rho\right)x_1\beta_1+\left(\frac{\partial}{\partial\rho}\rho\right){x_1}^2{\beta_1}^2}{(1-\rho+\rho x_1\beta_1)^2}+\left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}-\rho\right)x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{\left(\frac{\partial}{\partial\rho}-\rho\right)x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{-\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} + \left(\frac{\partial}{\partial\rho}\frac{1}{(1-\rho+\rho x_1\beta_1)^2}\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho x_1{}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(\frac{\partial}{\partial\rho}(1-\rho+\rho x_1\beta_1)\right)\left((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \Big((-1+x_1\beta_1)\big((1-\rho)x_1+2\rho x_1{}^2\beta_1\big)\Big)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(\frac{\partial}{\partial\rho}1+\frac{\partial}{\partial\rho}(-\rho+\rho x_1\beta_1)\right)\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^3}\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \Big((-1+x_1\beta_1)\big((1-\rho)x_1+2\rho x_1{}^2\beta_1\big)\Big)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(\frac{\partial}{\partial\rho}(-\rho+\rho x_1\beta_1)\right)\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^3}\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \Big((-1+x_1\beta_1)\big((1-\rho)x_1+2\rho x_1{}^2\beta_1\big)\Big)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(\frac{\partial}{\partial\rho}-\rho+\frac{\partial}{\partial\rho}\rho x_1\beta_1\right)\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^3}\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \Big((-1+x_1\beta_1)\big((1-\rho)x_1+2\rho x_1{}^2\beta_1\big)\Big)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(-\left(\frac{\partial}{\partial\rho}\rho\right)+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^3}\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2x_1{}^2\beta_1) - \Big((-1+x_1\beta_1)\big((1-\rho)x_1+2\rho x_1{}^2\beta_1\big)\Big)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^2} + \right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+x_1{}^2\beta_1{}^2}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(-1+\left(\frac{\partial}{\partial\rho}\rho\beta_1\right)x_1\right)\big((1-\rho)x_1\beta_1+\rho x_1{}^2\beta_1{}^2+y_i\beta_2\big)}{(1-\rho+\rho x_1\beta_1)^3}\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2{x_1}^2\beta_1) - \left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}+\right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+{x_1}^2{\beta_1}^2}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2\left(-1+\left(\frac{\partial}{\partial\rho}\rho\right)x_1\beta_1\right)\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)\right)$$

$$= \frac{(1-\rho+\rho x_1\beta_1)(-x_1+2{x_1}^2\beta_1) - \left((-1+x_1\beta_1)\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)\right)}{(1-\rho+\rho x_1\beta_1)^2} - x_i\left(\frac{\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}+\right.$$

$$\left.\rho\left(\frac{-x_1\beta_1+{x_1}^2{\beta_1}^2}{(1-\rho+\rho x_1\beta_1)^2} - \frac{2(-1+x_1\beta_1)\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3}\right)\right)$$

$$\text{out} = \frac{-x_1+2{x_1}^2\beta_1}{1-\rho+\rho x_1\beta_1} - \frac{(-1+x_1\beta_1)\times\left((1-\rho)x_1+2\rho {x_1}^2\beta_1\right)}{(1-\rho+\rho x_1\beta_1)^2} - \frac{\rho x_i(-x_1\beta_1+{x_1}^2{\beta_1}^2)}{(1-\rho+\rho x_1\beta_1)^2}+$$

$$\frac{2\rho x_1(-1+x_1\beta_1)\times\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^3} - \frac{x_1\left((1-\rho)x_1\beta_1+\rho {x_1}^2{\beta_1}^2+y_i\beta_2\right)}{(1-\rho+\rho x_1\beta_1)^2}$$

Appendix B: ESTIMATION OF PARAMETERS

Solve $A\beta + B\beta^2 + C\beta^3 + D = 0$ (1)

$$\beta_1 = \frac{\sqrt{\sqrt{(9ABC - 2B^2 - 27C^2D)^2 + 4(3AC - B^2)^3} + 9ABC - 2B^3 - 27C^2D}}{3\sqrt{2}C} - \frac{\sqrt{2}(3AC - B^2)}{3C\sqrt{\sqrt{(9ABC - 2B^2 - 27C^2D)^2 + 4(3AC - B^2)^3} + 9ABC - 2B^3 - 27C^2D}} - \frac{B}{3C} \quad C \neq 0 \tag{2}$$

$$\beta_1 = -\frac{(1 - i\sqrt{3})\sqrt{\sqrt{(9ABC - 2B^2 - 27C^2D)^2 + 4(3AC - B^2)^3} + 9ABC - 2B^3 - 27C^2D}}{6\sqrt{2}C} + \frac{(1 + i\sqrt{3})(3AC - B^2)}{3 \times 2^{\frac{2}{3}}C\sqrt{\sqrt{(9ABC - 2B^2 - 27C^2D)^2 + 4(3AC - B^2)^3} + 9ABC - 2B^3 - 27C^2D}} - \frac{B}{3C} \quad C \neq 0 \tag{3}$$

$$\beta_1 = -\frac{(1 + i\sqrt{3})\sqrt{\sqrt{(9ABC - 2B^2 - 27C^2D)^2 + 4(3AC - B^2)^3} + 9ABC - 2B^3 - 27C^2D}}{6\sqrt{2}C} + \frac{(1 - i\sqrt{3})(3AC - B^2)}{3 \times 2^{\frac{2}{3}}C\sqrt{\sqrt{(9ABC - 2B^2 - 27C^2D)^2 + 4(3AC - B^2)^3} + 9ABC - 2B^3 - 27C^2D}} - \frac{B}{3C} \quad C \neq 0 \tag{4}$$

$$\beta_1 = \frac{-\sqrt{A^2 - 4BD} - A}{2B} \quad C = 0 \quad B \neq 0 \tag{5}$$

$$\beta_1 = \frac{\sqrt{A^2 - 4BD} - A}{2B} \quad C = 0 \quad B \neq 0 \tag{6}$$

$$\beta_1 = -\frac{D}{A} \quad C = 0 \quad B = 0 \quad A \neq 0 \tag{7}$$

Possible intermediate steps.

Let $\beta_1 = \beta$ so, Solve for β

$A\beta + B\beta^2 + C\beta^3 + D = 0$ (8)

Eliminate the quadradic term by substituting $x = \frac{B}{3C} + \beta$:

$$D + A\left(x - \frac{B}{3C}\right) + B\left(x - \frac{B}{3C}\right)^2 + C\left(x - \frac{B}{3C}\right)^3 = 0 \tag{9}$$

Expand and collect in terms of x:

$$\frac{2B^3}{27C^2} - \frac{AB}{3C} + D + x\left(A - \frac{B^2}{3C}\right) + Cx^3 = 0 \tag{10}$$

Bring Equ (10) together using the common denominator $27C^2$

$$\frac{2B^3 - 9ABC + 27C^2D - 9B^2Cx + 27AC^2x + 27C^3x^3}{27C^2} = 0 \tag{11}$$

Multiply both sides by $27C^2$:

$$2B^3 - 9ABC + 27C^2D - 9B^2Cx + 27AC^2x + 27C^3x^3 = 0 \tag{12}$$

Collect in terms of x:

$$2B^3 - 9ABC + 27C^2D + x(27AC^2 - 9B^2C) + 27C^3x^3 = 0 \tag{13}$$

Divide both sides by $27C^2$:

$$\frac{2B^3 - 9ABC + 27C^2D}{27C^2} + \frac{x(27AC^2 - 9B^2C)}{27C^2} + x^3 = 0 \tag{14}$$

Change coordinates by substituting $x = y + \frac{\lambda}{y}$ where λ is a constant value that will be determined later:

$$\frac{2B^3 - 9ABC + 27C^2D}{27C^2} + \frac{(27AC^2 - 9B^2C)\left(y + \frac{\lambda}{y}\right)}{27C^2} + \left(y + \frac{\lambda}{y}\right)^3 = 0 \tag{15}$$

Multiply both sides by y^3 and collect in terms of y:

$$y^6 + y^4\left(-\frac{B^2}{3C^2} + \frac{A}{C} + 3\lambda\right) + y^3\left(\frac{2B^3}{27C^3} - \frac{AB}{3C^2} + \frac{D}{C}\right) + y^2\left(\lambda\left(\frac{A}{C} - \frac{B^2}{3C^2}\right) + 3\lambda^2\right) + \lambda^3 = 0$$

(16)

Substitute $\lambda = \frac{1}{3}\left(\frac{B^2}{3C^2} - \frac{A}{C}\right)$ and (17)

then $z = y^3$, yielding a quadratic equation in the variable z: (18)

$$\frac{(B^2 - 3AC)^3}{729C^6} + \frac{z(2B^3 - 9ABC + 27C^2D)}{27C^3} + z^2 = 0 \tag{19}$$

Find the positive solution to the quadratic equation:

$$z = \frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{54C^3} \tag{20}$$

Substitute back Equ (18):

$$y^3 = \frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{54C^3} \tag{21}$$

Taking cube root of Equ (21)

$$y = \frac{\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{C^3}}}{3\sqrt{2}} \tag{22}$$

$$y = -\frac{1}{3}\sqrt[3]{-\frac{1}{2}}\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{C^3}} \tag{23}$$

$$y = \frac{(-1)^{\frac{2}{3}}\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{C^3}}}{3\sqrt{2}} \tag{24}$$

Substitute the value of $x = \frac{\frac{B^2}{3C} - \frac{A}{C}}{3y} + y^2$ (25)

$$x = \frac{\sqrt[3]{2}\left(\frac{B^2}{3C} - \frac{A}{C}\right)}{\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{C^3}}} + \frac{\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{C^3}}}{3\sqrt[3]{2}} \tag{26}$$

$$x = \frac{(-1)^{\frac{2}{3}}\sqrt[3]{2}\left(\frac{B^2}{3C} - \frac{A}{C}\right)}{\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{C^3}}} - \frac{1}{3}\sqrt[3]{-\frac{1}{2}}\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}}{C^3}} \tag{27}$$

$$x = \frac{(-1)^{\frac{2}{3}}\sqrt[3]{\frac{-2B^3+9ABC-27C^2D+3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}}{C^3}}}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}\left(\frac{B^2}{3C}-\frac{A}{C}\right)}{\sqrt[3]{\frac{-2B^3+9ABC-27C^2D+3\sqrt{3}C^3\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}}{C^3}}} \qquad (28)$$

Bring each solution to a common denominator and simplify:

$$x = \frac{2B^2+C\left(\sqrt[3]{2}C\left(\left(\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}\right)^{\frac{2}{3}}\right)^{\frac{2}{3}}-6A\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}}} \qquad (29)$$

$$x = \frac{2(-1)^{\frac{2}{3}}B^2-6(-1)^{\frac{2}{3}}AC-\sqrt[3]{-2}C^2\left(\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}\right)^{\frac{2}{3}}}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}}} \qquad (30)$$

$$x = \frac{\sqrt{-1}\left(C\left(6A+\sqrt[3]{-2}C\left(\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}\right)^{\frac{2}{3}}\right)-2B^2\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}}} \qquad (31)$$

Substitute back for $\beta = x - \frac{B}{3C}$: (32)

$$\beta = \frac{2B^2 + C\left(\sqrt[3]{2}C\left(\frac{-2B^3 + 9ABC - 27C^2D + 3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}} - 9D\right)}{C^3}\right)^{\frac{2}{3}} - 6A\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}} - 9D\right)}{C^3}}} - \frac{B}{3C} \tag{33}$$

$$\beta = \frac{2(-1)^{\frac{2}{3}}B^2 - 6(-1)^{\frac{2}{3}}AC - \sqrt[3]{-2}C^2\left(\frac{-2B^3 + 9ABC - 27C^2D + 3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}} - 9D\right)}{C^3}\right)^{\frac{2}{3}}}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}} - 9D\right)}{C^3}}} - \frac{B}{3C} \tag{34}$$

$$\beta = \frac{\sqrt{-1}\left(C\left(6A + \sqrt[3]{-2}C\left(\frac{-2B^3 + 9ABC - 27C^2D + 3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}} - 9D\right)}{C^3}\right)^{\frac{2}{3}}\right) - 2B^2\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3 + 9ABC - 27C^2D + 3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}} - 9D\right)}{C^3}}} - \frac{B}{3C} \tag{35}$$

Note

$$\frac{-2B^3 + 9ABC - 27C^2D + 3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}} - 9D\right)}{C^3} = -\frac{2B^3}{C^3} + \frac{9AB}{C^2} - \frac{27D}{C} + 3\sqrt{3}\sqrt{\frac{-(A^2B^2) + 4A^3C + 4B^3D - 18ABCD + 27C^2D^2}{C^4}}: \tag{36}$$

$$\beta = \frac{2B^2 + C\left(\sqrt[3]{2}C\left(-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}\right)^{\frac{2}{3}}-6A\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}}} - \frac{B}{3C}$$

(37)

$$\beta = \frac{2(-1)^{\frac{2}{3}}B^2 - 6(-1)^{\frac{2}{3}}AC - \sqrt[3]{-2}C^2\left(\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}\right)^{\frac{2}{3}}}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}}} - \frac{B}{3C}$$

(38)

$$\beta = \frac{\sqrt{-1}\left(C\left(6A+\sqrt[3]{-2}C\left(\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}\right)^{\frac{2}{3}}\right)-2B^2\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}}} - \frac{B}{3C} \quad (39)$$

Substituting Equ (36) again:

$$\beta = \frac{2B^2 + C\left(\sqrt[3]{2}C\left(-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}\right)^{\frac{2}{3}}-6A\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}}} - \frac{B}{3C} \quad (40)$$

$$\beta = \frac{2(-1)^{\frac{2}{3}}B^2 - 6(-1)^{\frac{2}{3}}AC - \sqrt[3]{-2}C^2\left(-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}\right)^{\frac{2}{3}}}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}}} - \frac{B}{3C} \quad (41)$$

$$\beta = \frac{\sqrt{-1}\left(C\left(6A+\sqrt[3]{-2}C\left(\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}\right)^{\frac{2}{3}}\right)-2B^2\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{\frac{-2B^3+9ABC-27C^2D+3C^2\left(\sqrt{3}C\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}-9D\right)}{C^3}}}-\frac{B}{3C} \tag{42}$$

Substituting Equ (36) again:

Answer:

$$\beta = \frac{2B^2+C\left(\sqrt[3]{2}C\left(-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}\right)^{\frac{2}{3}}-6A\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}}}-\frac{B}{3C} \tag{43}$$

$$\beta = \frac{2(-1)^{\frac{2}{3}}B^2-6(-1)^{\frac{2}{3}}AC-\sqrt[3]{-2}C^2\left(-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}\right)^{\frac{2}{3}}}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}}}-\frac{B}{3C} \tag{44}$$

$$\beta = \frac{\sqrt{-1}\left(C\left(6A+\sqrt[3]{-2}C\left(-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}\right)^{\frac{2}{3}}\right)-2B^2\right)}{3\times 2^{\frac{2}{3}}C^2\sqrt[3]{-\frac{2B^3}{C^3}+\frac{9AB}{C^2}-\frac{27D}{C}+3\sqrt{3}\sqrt{\frac{-(A^2B^2)+4A^3C+4B^3D-18ABCD+27C^2D^2}{C^4}}}}-\frac{B}{3C} \tag{45}$$

APPENDIX C: SIMULATED DATASETS

SIMULATED DATASET ON INJURED_A

X	Y_1
1.012575	1.075901
6.495162	0.910362
2.84124	0.449026
1.969924	1.809936
0.366012	0.081073
0.36595	0.466644
0.129119	0.007739
4.339782	0.578326
1.983172	0.04433
2.65676	0.101506
0.04488	0.084298
7.559886	1.307455
3.854711	1.204987
0.515034	0.60782
0.43302	2.261664
0.43719	0.324547
0.782741	0.237587
1.605228	1.512795
1.220301	0.183861
0.742755	2.326261
2.042055	0.009393
0.324172	2.402608
0.745544	0.031079
0.984543	1.58649
1.313944	0.548355
3.318519	4.179125
0.480614	0.053104
1.557976	0.599704
1.936612	2.656201
0.102632	0.501009
2.018237	0.761171
0.40342	0.820276
0.14514	0.4112
6.416547	1.091383
7.273059	1.061132
3.565358	1.988695
0.783893	0.032845
0.22177	0.224827
2.487376	2.381268

1.251705	1.608632
0.280839	0.416064
1.474941	0.715197
0.075509	0.219455
5.17957	0.204876
0.646162	0.436137
2.343895	0.34205
0.806029	0.619657
1.584045	0.060233
1.707282	2.750857
0.441024	2.960109
7.536689	1.491119
3.219928	0.642723
6.05276	0.394867
4.859635	1.606601
1.96585	0.881952
5.501106	0.179621
0.19993	1.646138
0.470685	1.196944
0.099866	2.023967
0.849153	0.917342
1.061906	0.683321
0.683066	0.380079
3.807517	2.363992
0.952068	1.435148
0.71164	0.032526
1.688258	0.02003
0.327762	0.324086
3.496638	1.422006
0.167175	2.964764
9.352109	0.255261
3.192392	0.752982
0.478032	0.332082
0.011949	2.3576
3.646416	1.596343
2.647796	1.461312
2.817326	0.513503
3.18318	0.447135
0.165996	0.216953
0.95782	0.041439
0.26573	1.367247
4.290798	1.526703
2.106637	6.305609
0.867034	4.043097
0.141697	0.549833

0.803746	1.036059
0.848683	2.05372
2.822101	1.539041
2.189902	0.220529
4.708807	0.564965
1.37896	0.101532
0.274841	2.106455
2.695332	0.749893
3.086464	0.215055
1.777763	0.840054
3.180322	0.803141
1.469044	0.32751
1.59606	0.089794
1.203637	0.808013
0.055558	0.49563
0.246348	1.010201
0.068906	0.495832
2.183084	1.483001
0.814337	0.56799
1.532963	0.610567
5.138233	2.049775
0.618718	0.361021
1.139913	0.103982
3.039762	0.024109
0.560601	0.979607
0.172847	0.658088
0.738261	0.857374
0.379355	0.164842
5.728783	0.151556
3.562241	0.012587
2.165314	0.334451
4.426716	0.82116
3.512789	0.426185
0.445571	0.397653
4.813591	2.232481
1.67249	0.320567
3.554605	0.61951
4.885724	1.466517
0.825846	0.357635
0.25158	0.664503
0.558188	1.380472
1.202005	2.166816
3.67648	0.137899
4.253717	2.371583
0.015054	0.455969

1.542541	0.221738
1.165788	0.444891
0.541962	2.72152
0.275505	0.464015
0.888806	0.283396
6.177969	1.082949
0.84236	0.194664
1.57831	0.058702
2.619725	0.110424
0.975259	0.09796
7.698502	0.208914
7.081836	0.176958
0.625887	0.71593
1.483813	0.148418
0.772333	0.299293
0.723393	1.597504
0.081099	0.434083
2.029369	0.845149
1.507248	0.140049
0.114041	0.144597
0.704785	0.029318
5.154622	0.181511
0.590929	0.227782
0.337758	0.133747
1.450611	0.068521
9.157673	0.195897
0.598016	0.431019
2.406253	0.182124
3.094006	0.375461
0.585476	0.488118
2.811038	0.951561
0.989389	0.028699
2.158861	1.244071
2.166055	0.765955
1.655844	0.063798
0.204188	1.40815
3.891858	2.236811
0.834649	0.04465
0.445434	0.224065
0.089828	1.10944
1.928572	1.051165
2.44228	0.161172
0.036093	0.158294
1.548485	0.343642
0.554168	0.461165

2.235722	0.749799
0.413439	0.317522
2.533685	0.493825
1.055062	0.989039
5.956198	0.039475
0.319231	0.199511
0.900078	0.888974
0.259892	1.541209
5.580415	0.730776
4.52773	0.705641
0.643723	0.07934
2.327726	0.464847
3.667103	0.658841
1.748083	0.209012
1.627565	0.234167
0.597438	0.33054
0.210871	0.013811
4.909202	0.373145
4.977498	0.229401
2.163535	0.307344
0.893419	0.090718
0.92691	1.576556
2.793164	0.729798
4.906988	1.090867
4.706391	1.468378
3.265925	0.579441
2.216704	0.070705
0.18965	0.523982
0.380403	0.608469
4.937485	1.316623
2.012111	0.433097
0.019937	0.091884
0.230875	0.227627
2.350166	0.354104
0.010949	0.698565
0.378292	0.582137
1.716936	0.330629
2.54038	3.426586
2.277404	0.726893
0.547966	0.188379
2.687329	0.086333
0.584376	0.115667
0.849375	0.200197
2.961238	0.143753
2.263017	0.161086

4.082414	0.300513
2.312731	0.14906
1.812626	1.716705
0.212232	0.056974
0.989159	0.531974
0.66494	0.370252
0.60353	2.819389
7.794548	0.152356
1.077565	0.365081
4.803324	3.318615
2.152022	1.552891
3.417537	1.443082
1.507066	0.220868
1.856022	0.142098
1.463604	0.815529
0.468701	1.834573
2.765752	0.65816
0.711154	0.595864
0.053117	0.221584
2.237544	1.142731
0.420623	0.142906
6.08726	0.418341
6.640684	0.626409
5.315698	0.705054
0.997512	0.198763
0.033612	0.08219
5.686864	0.971182
1.206063	0.247183
7.338249	1.059798
7.150293	0.199983
4.137288	0.590145
0.75258	0.536001
1.049308	0.038229
4.10997	0.368296
0.822427	0.102188
0.400739	0.045127
1.755861	3.465049
5.93667	0.41025
2.569533	1.323761
1.821404	0.222316
0.220585	0.779812
2.059662	1.09824
9.948568	1.549159
0.325653	0.437587
1.576244	0.394854

4.528323	0.398623
2.913069	2.1691
2.576543	1.41661
2.615843	2.682855
0.961272	0.094807
0.74996	0.924875
3.57624	2.396243
3.584771	0.172089
4.35428	0.063007
5.274929	1.339915
1.545167	0.634315
1.502209	1.365067
3.454488	0.128271
2.265057	1.237473
2.612095	0.180116
3.427882	0.151947
4.762905	0.170085
0.890044	1.198122
1.016175	0.784377
0.212964	0.50438
1.863052	0.337182
0.078983	1.41698
1.352076	0.36519
1.688015	1.271383
0.728531	0.406918
1.928257	0.360542
0.066837	0.419377
0.082132	0.242366
3.73155	1.198643
0.96363	0.499081
0.293218	0.180912
1.593848	1.711869
3.171168	0.40379
0.524594	0.544786
2.104304	1.827932
0.192497	0.665963
0.114502	0.084177
1.635396	2.099323
1.678555	0.740186
2.189143	0.316495
2.794232	0.12157
8.034595	2.084017
1.567172	0.71953
0.841574	0.540392
3.421483	1.906356

0.681536	1.090233
1.247158	0.119713
0.176301	0.253913
0.055406	0.192735
7.093423	1.615033
3.900754	0.030097
2.569138	0.672769
1.134686	1.038402
0.410639	1.443952
0.367082	0.287965
0.621452	1.203366
1.719294	0.088073
2.705524	1.508995
2.329081	0.106891
0.708639	0.355829
6.685008	1.812377
2.889297	0.132585
1.743979	0.195936
2.04132	0.983914
1.173911	0.922068
0.614235	0.715933
0.949451	0.845073
3.060116	0.646915
0.031284	0.195424
0.266227	0.289929
0.101619	0.13559
0.089723	1.616466
4.173573	0.789901
2.624372	0.410098
1.386984	0.455358
0.222158	2.005934
1.459773	0.122878
1.384105	0.648083
0.410397	0.48621
1.227556	0.687109
1.096876	0.01317
2.064391	1.582202
2.175283	2.085536
0.100039	0.563354
1.012825	0.854913
2.121362	1.975302
1.509232	0.909743
4.188994	0.149498
2.319553	0.672574
0.383766	0.64254

0.157911	0.374689
2.219042	1.037042
0.057977	1.837892
1.901749	1.976095
6.079001	0.640617
1.848743	0.091082
1.060116	3.01158
2.224292	1.420368
1.322621	0.097386
1.702083	1.903036
6.124037	2.189357
1.052837	0.52566
7.010806	1.051304
5.087117	0.496451
0.470171	0.038987
0.15511	0.277139
0.22921	1.107329
0.039681	0.003129
0.214062	0.276411
2.479012	0.402274
0.159351	0.523428
0.828924	1.787482
4.021068	0.37883
0.050809	0.287403
3.634836	1.155626
0.714403	0.423145
0.27134	0.173925
2.57456	0.480442
2.139215	0.117407
4.530065	0.180322
2.866157	0.562192
3.510689	0.464531
0.714944	1.731856
0.421485	0.310589
2.996623	0.714968
3.547832	0.858495
10.04876	0.022908
1.148107	0.787591
1.003891	0.140431
3.232247	2.718772
0.899217	0.114502
5.761554	0.555263
4.218101	0.080775
1.209121	4.187352
2.998841	0.574065

3.030881	0.746432
0.234847	0.047365
5.02426	1.344051
1.518442	0.174594
3.77898	1.995364
0.83233	0.162634
4.87396	0.202469
1.063758	0.026774
0.023513	0.430134
5.087831	0.81132
0.206554	0.046284
0.829996	1.058148
6.466789	0.670618
6.490476	0.827324
1.838418	0.440156
2.156113	0.396493
1.283907	0.598347
0.748797	0.118866
0.859843	0.142496
2.408774	1.559894
3.011903	2.406191
3.383811	0.395962
3.363605	0.267116
0.206363	0.703322
1.471709	0.388195
0.127917	0.017426
1.720741	0.127554
1.257023	0.916636
4.718229	1.015644
0.932572	0.019585
0.268656	0.171579
0.332961	0.180344
3.093019	0.923634
2.077733	0.015329
0.230038	0.074973
0.189572	1.094559
2.604883	0.157235
0.163011	0.718338
3.722561	0.236681
2.643273	0.083969
0.183084	0.189459
0.191295	0.871576
9.311785	3.014107
1.011646	1.270765
0.999169	0.362423

3.615513	0.951459
6.348526	0.251763
9.21105	0.532597
3.020666	1.868404
1.018515	0.009383
0.188145	0.060802

SIMULATED DATASET DEATH_B

X	Y_2
0.086544	0.485032
0.009986	0.017019
0.074346	0.077442
0.324848	0.113403
0.203199	0.139344
0.828809	0.107637
0.631558	0.001233
0.155327	0.045002
0.111527	0.082325
0.177336	0.010518
0.331365	0.050924
0.115885	0.003815
0.363392	0.009716
0.195568	0.058043
0.298221	0.016682
0.212757	0.096541
0.129528	0.132492
1.10116	0.22551
0.535585	0.027484
0.056057	0.020473
0.753291	0.014251
0.248183	0.117351
0.835373	0.161107
0.597401	0.004002
0.205639	0.316515
0.008439	0.005907
0.116197	0.088066
0.289688	0.144398
0.372726	0.243573
0.110758	0.141332
0.055731	0.180715
0.66895	0.05252
1.54506	0.264174
0.27699	0.009742

0.069946	0.252205
0.118044	0.08966
0.006893	0.188078
0.912373	0.080536
0.046523	0.115012
0.319081	0.087934
0.118937	0.1487
0.299992	0.00998
0.391365	0.007433
0.657458	0.035857
0.085858	0.019525
0.004106	0.227139
0.054666	0.027631
0.855141	0.491666
0.768924	0.053193
0.337873	0.023743
0.340747	0.184258
0.406161	0.128756
0.071605	0.077246
0.912863	0.10843
0.2015	0.14595
0.179405	0.029673
0.271725	0.653591
0.017864	0.408083
0.067535	0.117618
0.497442	0.02717
0.032095	0.126757
0.343205	0.00886
0.104536	0.00351
0.18313	0.034117
0.126502	0.07032
0.163228	0.231171
0.471456	0.157699
0.130928	0.153842
0.310319	0.065093
0.240028	0.049161
0.404651	0.055519
1.025648	0.603562
0.489779	0.004989
0.089866	0.226827
0.011764	0.095722
0.112954	0.065199
0.335726	0.152515
0.019606	0.073959
0.254713	0.240042

0.337348	0.272748
0.151078	0.062313
0.547244	0.039985
0.041858	0.09995
0.029007	0.270503
0.483742	0.028795
0.254069	0.113267
0.433013	0.120256
0.211903	0.152102
0.105054	0.015906
0.634948	0.158171
0.596586	0.298885
0.440591	0.023892
0.117962	0.030657
0.331306	0.419202
0.166296	0.022782
0.035669	0.214359
0.925671	0.090949
0.054637	0.031711
1.114241	0.287064
0.219321	0.067961
0.076027	0.081089
0.289904	0.052058
0.766154	0.116307
0.489297	0.02176
0.610056	0.062029
0.399293	0.417069
0.437659	0.035996
0.702515	0.136422
0.106667	0.044363
0.24963	0.172274
0.092843	0.015778
1.632303	0.561595
1.070783	0.083966
0.014938	0.029836
0.454065	0.009234
0.963135	0.025181
0.073956	0.082207
0.31164	0.048281
0.917563	0.236436
0.012825	0.062436
0.443919	0.03452
0.131049	0.108805
0.958925	0.165891
1.315506	0.026957

1.07216	0.025481
0.238728	0.004325
0.73558	0.171362
0.691246	0.138733
0.142726	0.07311
0.65566	0.23476
0.014004	0.181498
0.33682	0.103948
0.097063	0.228199
0.047711	0.0257
0.029736	0.01319
0.442533	0.037937
0.154233	0.006692
0.479096	0.092117
0.0251	0.306254
0.140654	0.004557
0.287956	0.015226
0.580832	0.074783
0.142536	0.308513
0.365118	0.045133
0.806341	0.092408
0.355294	0.005033
0.098489	0.018082
0.009174	0.477595
0.757924	0.433442
0.007984	0.000548
0.771321	0.392874
0.279541	0.118959
1.039042	0.278945
0.595417	0.075597
2.295776	0.142449
0.160379	0.076491
0.541256	0.135435
0.190894	0.017248
0.242749	0.003526
0.366723	0.043727
0.768297	0.157904
1.537548	0.035246
0.542991	0.137939
0.200859	0.402552
0.203156	0.011335
0.496802	0.136287
0.101316	0.066086
0.043473	0.225738
0.162623	0.022234

0.125743	0.133552
0.130499	0.206332
0.098803	0.32585
0.01597	0.127417
0.006698	0.076064
1.633944	0.154391
0.207296	0.233747
0.180121	0.096734
0.422723	0.003709
0.091436	0.300584
1.112186	0.164027
0.573143	0.140005
0.034782	0.027022
0.20074	0.123572
0.784647	0.065053
1.075275	0.146409
0.233948	0.013036
0.352932	0.127056
0.067869	0.833056
1.756293	0.008209
0.097866	0.425599
1.062073	0.072382
0.38948	0.244214
0.34752	0.180467
0.266948	0.097451
0.097382	0.150992
0.072126	0.236169
0.092498	0.058236
0.076623	0.044412
0.561575	0.136293
0.160044	0.187994
0.022126	0.159419
1.291225	0.234328
0.799271	0.074563
0.975787	0.231167
1.960762	0.022601
0.070941	0.088078
0.187378	0.000662
0.527248	0.048208
0.442205	0.055078
0.062058	0.056409
0.628297	0.149258
0.094387	0.055353
0.094089	0.066998
0.285932	0.031696

0.33377	0.055133
0.322225	0.030353
0.03563	0.008457
0.77959	0.120048
0.114637	0.124629
0.051508	0.107824
0.815475	0.081471
1.157015	0.067724
0.735809	0.255799
0.615773	0.09165
0.395458	0.077961
0.297236	0.003043
0.033795	0.046421
0.194966	0.054896
0.173166	0.058147
0.111693	0.097987
0.477283	0.092714
0.254352	0.108724
0.031387	0.160924
0.092354	0.188574
0.426933	0.152318
0.029406	0.345889
0.707501	0.00235
0.253815	0.025299
0.243257	0.00088
0.333285	0.123024
0.64658	0.287746
0.158723	0.031794
0.420837	0.314487
0.309743	0.007299
0.11536	0.307841
0.78315	0.05628
0.592921	0.013323
0.398932	0.079563
0.705944	0.037891
0.749992	0.043231
0.457598	0.044446
0.673666	0.205979
0.443982	0.093519
0.423296	0.044829
0.357961	0.111767
0.518862	0.154426
0.06413	0.035607
0.79007	0.069427
0.762946	0.016807

0.011023	0.022129
0.648994	0.144124
0.051233	0.02234
0.151567	0.084744
0.505285	0.150961
0.065083	0.012646
0.632625	0.105264
0.662713	0.037497
0.26299	0.383287
0.002379	0.073045
0.125639	0.185644
0.356324	0.094723
1.475446	0.006824
0.371043	0.119242
0.111718	0.341433
0.373261	0.109484
0.288354	0.199976
0.562015	0.266815
0.042017	0.028835
0.531558	0.029303
0.28939	0.110414
1.22421	0.076306
0.155358	0.208575
0.371859	0.273869
0.998442	0.059513
0.040163	0.0301
1.027999	0.208423
0.432398	0.038694
0.026087	0.192049
0.132964	0.062489
0.457355	0.13331
0.025893	0.01115
0.324073	0.115157
0.157628	0.068472
0.360211	0.104806
0.017387	0.020392
0.762058	0.172606
1.348083	0.295412
1.288533	0.035785
0.514288	0.012326
0.051752	0.00267
0.527286	0.126181
0.009245	0.103405
0.008308	0.087562
0.145194	0.02999

0.249062	0.057513
0.546427	0.11133
0.426976	0.082602
0.219252	0.507984
0.118718	0.016584
2.172982	0.224123
0.206265	0.059586
0.222946	0.064505
0.066352	0.141503
0.588155	0.146739
0.43936	0.571564
0.092633	0.015457
0.031926	0.012222
0.423712	0.154897
0.394672	0.103197
0.118531	0.036224
1.118944	0.011624
0.060814	0.010011
0.210269	0.258552
1.067851	0.029557
0.202111	0.044893
0.377876	0.030606
0.188211	0.050256
0.119019	0.008135
1.535088	0.114357
0.195519	0.008035
0.833781	0.211688
0.097037	0.029968
0.088998	0.087287
0.011746	0.234878
0.391627	0.125662
0.170711	0.086954
0.741866	0.050458
0.23802	0.046874
1.280455	0.161846
0.076206	0.576264
0.75373	0.139911
0.556575	0.100864
0.54726	0.161829
0.691805	0.079665
0.531552	0.007627
0.365444	0.098131
0.052247	0.352813
0.012279	0.021347
0.941893	0.157742

0.356056	0.026591
0.591295	0.095582
0.243926	0.011783
0.046337	0.066722
0.049666	0.157428
0.429647	0.051398
0.208944	0.125824
0.083109	0.178173
0.251211	0.304032
0.024644	0.029687
0.323896	0.060025
0.116357	0.018687
0.593165	0.60931
0.13799	0.297015
0.22555	0.089648
0.004341	0.204074
0.027928	0.081756
0.185079	0.11191
0.242794	0.010815
0.340286	0.166325
0.128055	0.015476
0.440938	0.211007
0.730446	0.195566
0.562024	0.152662
0.015012	0.004081
0.2432	0.041846
0.041166	0.034003
0.102913	0.050324

APPENDIX D: Dataset on Cesarean-section and Normal Delivery 2014-2024 (Federal University of Technology Owerri) Medicals.

Card_No	Time_Delivery	Sex_Baby	X	Y
com_20	2pm	m	3.5kg	9.5
com_58	7am	m	3.8kg	10.0
com_70	5pm	m	3.4kg	9.0
com_1	12noon	m	3.7kg	9.5
com_100	1pm	f	3.7kg	9.5
com_3	4pm	m	3.6kg	9.5
com_10	12pm	f	3.3kg	9.5
com_458	7pm	f	3.5kg	9.0
com_200	11:06am	f	3.2kg	9.0
com_99	5:35am	f	3.4kg	9.5
com_7	4:14am	m	3.5kg	10
com_11	3:06pm	m	3.7kg	10
com_23	2:06am	m	3.7kg	9.5
sp_730	1.:15am	m	3.7kg	9.0
sp_112	3:02pm	m	3.0kg	9.0
sp_779	12:15am	f	3.5kg	9.0
sp_909	7:45pm	m	3.0kg	9.0
23/111	3:04am	f	3.3kg	9.0
22/099	8:06pm	f	2.9kg	9.0
22/156	12:01am	f	3.0kg	9.5
21/777	3:06pm	f	3.0kg	9.5
21/106	2:15am	f	3.0kg	9.5
21/222	12:35pm	f	3.0kg	10.0
com_24	1:00am	f	3.0kg	10.0
com_50	12:00noon	m	3.5kg	10.0
com_07	4:06am	m	3.3kg	9.5
com_101	2:16pm	m	3.6kg	9.5
com_25	8:00am	f	3.6kg	9.0
com_56	9:15am	m	3.5kg	9.0
com_28	11:20am	m	3.0kg	9.5
sp_72	10:30am	m	2.9kg	9.5
sp_119	6:20am	f	3.0kg	9.5
sp_514	2:08am	f	3.0kg	9.5
com_202	7:00am	f	3.0kg	9.5
com_60	2:00pm	f	3.2kg	9.5
com_71	7:00am	f	3.0kg	9.5
com_73	5:00am	f	2.8kg	9.5
com_55	2:00am	f	3.0kg	9.5
sp_113	8:00pm	f	3.0kg	10.0
23/112	7:00am	f	3.0kg	10.0

21/107	1:00pm	m	3.0kg	9.5
com_80	6:00pm	m	3.0kg	9.5
com_95	5:00am	f	3.0kg	10.0
com_98	3:00pm	f	3.0kg	9.5
21/220	4:00pm	f	3.0kg	9.5
21/230	8:00am	f	3.0kg	10.0
23/100	1:00am	m	3.5kg	10.0
sp_303	2:00pm	m	3.2kg	10.0
com_203	11:00pm	f	3.8kg	10.0
sp_65	8:00pm	m	3.0kg	10.0
21/109	2:00pm	m	3.0kg	10.0
22/114	8:00pm	m	3.0kg	9.5
sp_107	7:00am	m	3.0kg	9.5
com_56	3:00pm	m	3.0kg	10.0
com_08	12:00pm	f	3.0kg	10.0
sp_1020	7:00pm	m	3.0kg	10.0
23/021	1:00pm	f	3.0kg	9.5
com_90	2:00pm	m	2.9kg	9.5
com_100	6:00am	m	3.3kg	9.5
sp_117	3:00am	f	3.0kg	9.0
sp_103	1:00pm	f	3.6kg	10.0
com_108	2:00am	f	3.0kg	10.0
com_04	3:00am	f	3.0kg	10.0
com_09	2:00pm	f	3.0kg	9.5
com_12	4:00pm	m	3.0kg	9.5
com_14	7:00am	m	3.0kg	9.0
sp_100	8:00am	m	3.0kg	9.5
sp_130	12:00pm	m	3.5kg	10.0
sp_160	9:00pm	f	3.8kg	9.5
sp_200	8:00pm	f	3.5kg	9.5
com_201	10:00am	f	3.8kg	10.0
21/170	11:00pm	m	3.0kg	9.5
sp_800	10:00am	m	3.0kg	9.5
com_300	1:00am	f	3.8kg	9.5
com_24	2:00pm	m	4.0kg	9.5
com_29	6:00pm	m	3.0kg	9.5
Jp_102	9:15am	f	2.9kg	10.0
Jp_100	4:10pm	f	3.0kg	9.5
com_57	10:30pm	m	3.5kg	9.5
com_67	6:20pm	m	3.0kg	10.0
sp_68	11:20am	m	3.5kg	9.5
Jp_999	1:15am	f	3.0kg	9.5
22/108	8:06pm	m	3.5kg	9.5
sp_75	3:00pm	m	3.5kg	9.5

sp_1038	7:00am	m	2.8kg	9.5
com_107	5:35pm	f	3.0kg	9.5
23/224	4:30pm	m	3.0kg	9.5
com_19	2:00am	m	3.0kg	9.5
sp_1039	1:00am	f	3.0kg	9.5
com_12	12:00noon	f	3.5kg	10.0
com_140	1:00pm	m	2.8kg	9.5
23/777	05:00am	m	3.0kg	9.5
com_204	5:00pm	m	3.5kg	9.5
com_103	6:00am	m	3.0kg	9.5
com_105	7:10pm	f	3.0kg	9.5
sp_105	5:am	f	3.0kg	9.5
sp_1040	11:06am	f	2.8kg	9.5
23/370	2:06pm	m	3.0kg	9.5
sp_400	7:00am	m	3.5kg	9.5
sp_403	11:00pm	m	3.0kg	9.5
sp_73	4:am	f	3.5kg	9.5
sp_1038	2:00am	m	3.0kg	9.5
Jp_109	5:00pm	m	3.0kg	9.5
com_210	12:00pm	m	3.0kg	9.0
21/333	7:00pm	m	3.0kg	9.0
22/157	8:50am	f	3.2kg	9.5
com_352	12:00pm	m	3.0kg	9.5
sp_111	5:00am	m	3.5kg	9.5
com_157	1:00am	f	3.5kg	9.5
sp_405	9:10pm	f	3.8kg	9.5
23/100	5:00pm	m	3.5kg	10.0
com_119	8:10am	m	3.0kg	9.5
com_300	10:00pm	f	3.5kg	9.5
Jp_910	5:00am	m	3.0kg	9.5
com_25	12:00noon	m	3.5kg	9.5
com_55	1:00am	f	3.0kg	9.5
Jp_36	7:00pm	f	3.5kg	9.5
com_301	2:00am	m	3.0kg	10.0
sp_2330	3:00pm	f	3.5kg	9.5
22/003	3:00am	m	3.0kg	9.5
21/104	1:00pm	f	3.5kg	9.5
Jp_307	5:00pm	m	3.0kg	9.0
com_109	1:00pm	m	3.5kg	9.5
com_145	1:00am	m	3.0kg	9.5
com_130	2:00pm	f	3.5kg	9.5
com_135	5:00pm	f	3.0kg	9.5
sp_1520	7:00am	m	3.0kg	9.5
sp_2020	5:10am	m	3.0kg	9.5

sp_2030	7:10pm	m	3.3kg	9.5
sp_2035	5:00pm	f	3.5kg	9.5
sp_2055	11:30pm	f	3.5kg	9.5
com_9	9:00pm	m	3.0kg	9.5
sp_305	5:00am	m	3.0kg	9.5
Jp_333	9:10am	m	3.5kg	9.5
Jp_2013	8:06pm	m	3.0kg	9.5
Jp_106	9:00am	f	3.5kg	9.5
com_1010	2:00pm	m	3.0kg	9.5
com_120	1:00am	m	3.0kg	9.5
com_170	5:00pm	m	3.5kg	9.5
sp_660	1:00pm	f	3.0kg	9.5
sp_6655	2:00am	f	3.0kg	9.5
com_190	1:00pm	f	3.5kg	9.5
com_192	2:am	m	3.5kg	9.5
com_152	3:00pm	m	3.0kg	9.5
sp_706	1:00pm	f	3.0kg	9.5
Jp_500	2:00am	f	3.0kg	9.5
Jp_6060	5:00pm	m	3.5kg	9.5
com_6065	1:30am	f	3.8kg	9.5
sp_105	1:pm	f	3.0kg	9.5
sp_88	2:00am	m	3.5kg	9.5
sp_89	11:30pm	m	3.0kg	9.5
com_61	5:10pm	m	3.0kg	9.5
com_62	5:00am	f	3.0kg	9.5
sp_106	7:00pm	f	4.0kg	9.5
com_25	5:00pm	m	3.0kg	9.5
com_35	1:00am	m	3.5kg	9.5
com_27	2:10pm	m	3.5kg	9.5
com_28	3:00am	f	3.0kg	9.5
com_40	1:50am	m	3.5kg	9.5
com_66	5:00pm	m	3.0kg	9.5
com_84	10:00am	m	3.0kg	9.5
com_76	7:00pm	f	3.0kg	9.5
com_115	1:00pm	f	3.5kg	9.5
com_91	5:00am	m	3.0kg	9.5
com_98	1:00pm	f	2.8kg	9.5
com_102	2:00pm	f	2.5kg	9.5
com_114	5:00pm	f	3.8kg	9.5
com_115	1:00am	f	3.0kg	9.5
com_205	8:00am	m	3.0kg	9.5
com_200	8:00pm	m	3.0kg	9.5
com_215	5:00am	m	3.0kg	9.5
com_226	1:00pm	m	3.8kg	10.0

com_107	7:00am	f	3.0kg	9.5
com_265	7:00pm	m	3.0kg	9.5
com_250	5:00pm	f	3.5kg	9.5
com_260	1:00pm	m	3.0kg	9.5
com_85	2:00am	m	3.0kg	9.5
com_185	5:00pm	m	3.8kg	9.5
com_190	5:00am	m	3.0kg	9.5
com_198	2:00pm	f	3.0kg	9.5
com_280	1:00am	m	3.0kg	9.5
com_286	2:00am	m	3.5kg	9.5
com_290	7:00am	m	3.0kg	9.5
com_222	7:00pm	m	3.5kg	9.5
com_201	2:00am	f	3.0kg	9.0
com_245	7:00pm	m	2.5kg	9.0
com_301	1:05am	f	3.0kg	9.0
com_300	2:10am	f	3.9kg	8.5
com_266	3:00pm	m	2.5kg	9.5
com_250	4:15am	m	3.0kg	9.5
com_211	2:15pm	f	2.8kg	9.0
com_246	7:00am	m	3.0kg	8.5
com_304	6:05am	m	2.4kg	9.5
com_302	8:15am	m	2.4kg	0.0
com_302	3:00am	f	3.0kg	9.5
com_305	5:00am	f	2.4kg	9.5
com_306	2:00pm	f	3.0kg	9.0
com_307	10:00am	m	3.3kg	8.5
com_308	12:30am	f	3.3kg	10.0
com_309	9:00am	m	3.0kg	9.5
com_310	10:12am	f	2.8kg	9.0
com_311	1:00am	m	4.2kg	8.5
com_312	11:00pm	f	3.0kg	9.5
com_313	2:30am	m	3.2kg	9.0
com_250	4:00pm	f	3.1kg	8.5
com_314	4:30pm	f	2.4kg	10.0
com_315	3:30pm	f	2.9kg	10.0
com_316	1103pm	m	2.0kg	9.0
com_317	12:00noon	f	2.8kg	9.0
com_318	12:30am	f	3.0kg	9.5
com_319	2:12pm	f	3.0kg	10.0
com_320	4:10pm	f	3.0kg	10.0
com_321	5:30pm	m	3.0kg	9.0
com_322	6:00pm	f	2.8kg	9.5
com_323	10:00pm	m	2.9kg	10.0
com_324	12:05pm	m	3.0kg	9.5

com_325	7:05am	m	4.0kg	9.5
com_326	10:15pm	m	3.0kg	10.0
com_327	8:10am	f	3.0kg	9.0
com_328	6:20am	f	2.5kg	9.0
com_329	12:00noon	f	3.0kg	9.0
com_330	10:00am	f	3.5kg	9.5
com_331	9:20am	m	3.2kg	9.0
com_332	11:00pm	f	3.0kg	8.5
com_333	7:00am	f	3.0kg	9.0
com_334	3:00am	m	3.0kg	9.5
com_335	1:00pm	f	2.5kg	8.5
com_336	8:00am	f	2.3kg	9.0
com_337	7:00pm	f	3.0kg	9.0
com_338	12:00noon	m	3.0kg	9.0
com_339	6:00am	m	4.0kg	9.5
com_340	5:00pm	f	3.8kg	9.0
com_341	10:00pm	f	3.4kg	8.5
com_342	1:00am	f	3.6kg	8.5
com_343	2:00pm	f	2.7kg	9.0
com_344	7:00am	m	3.0kg	9.5
com_345	2:00am	f	3.0kg	10.0
com_346	9:00am	m	3.2kg	10.0
com_347	6:00pm	f	3.3kg	10.0
com_348	7:30am	m	3.0kg	10.0
com_349	4:00am	m	2.8kg	10.0
com_350	7:00am	f	2.6kg	10.0
com_351	6:00pm	f	3.0kg	9.0
com_352	11:21pm	f	3.0kg	9.0
com_353	8:30pm	f	3.0kg	9.5
com_354	1:10pm	f	2.7kg	9.5
com_355	12:00noon	f	3.0kg	9.0
com_356	3:00am	m	3.0kg	9.5
com_357	6:10am	m	2.0kg	9.0
com_358	11:00pm	f	3.0kg	9.0
com_359	6:40pm	f	3.0kg	8.5
com_360	8:00am	f	3.0kg	9.0
com_361	10:11pm	m	3.0kg	9.5
com_362	7:00pm	f	3.5kg	10.0
com_363	11:00am	m	3.0kg	9.0
com_364	3:00am	f	3.0kg	9.5
com_365	10:40am	f	2.5kg	9.0
com_366	3:05pm	f	2.5kg	8.5
com_367	11:20pm	m	3.0kg	9.0
com_368	4:30pm	f	3.0kg	9.5

com_369	6:00am	f	4.0kg	9.0
com_370	7:31am	m	3.0kg	9.0
com_371	8:05pm	f	3.0kg	9.0
com_372	6:06pm	m	2.5kg	9.0
com_373	9:30pm	f	3.0kg	9.0
com_374	8:50am	f	3.0kg	9.0
com_375	3:05pm	f	3.0kg	9.0
com_376	11:00pm	f	3.0kg	9.0
com_377	10:00am	m	3.0kg	9.0
com_378	5:00am	f	3.0kg	9.5
com_379	12:00noon	f	3.0kg	9.5
com_380	6:00pm	m	3.0kg	10.0
com_381	7:30pm	f	2.5kg	9.0
com_382	11:10pm	f	3.0kg	9.5
com_383	7:20pm	f	3.0kg	9.0
com_384	3:00am	f	3.0kg	9.5
com_385	6:00pm	m	2.5kg	9.0
com_386	7:30am	f	3.2kg	9.0
com_387	5:00pm	m	2.8kg	9.0
com_388	10:12am	f	3.0kg	9.0
com_389	7:06am	f	3.0kg	9.0
com_390	10:07pm	f	3.0kg	9.0
com_391	11:05am	f	3.0kg	9.0
com_392	12:00noon	f	3.0kg	9.5
com_393	8:00pm	f	3.0kg	10.0
com_394	7:00am	f	2.5kg	9.0
com_395	10:12am	m	2.5kg	9.0
com_396	11:00pm	f	3.0kg	9.0
com_397	8:00am	f	3.0kg	9.0
com_388	4:20am	m	3.0kg	8.5
com_589	2:10am	f	3.0kg	9.0
com_590	7:40pm	f	3.0kg	9.5
com_591	11:10pm	f	3.0kg	9.0
com_792	6:30am	m	3.0kg	9.0
com_893	10:10pm	f	2.5kg	9.0
com_994	8:30am	m	3.0kg	9.0
com_895	6:00pm	f	3.0kg	9.0
com_716	3:05am	m	3.0kg	8.5
com_597	6:40am	f	3.0kg	9.0
com_598	12:12pm	f	2.5kg	9.5
com_399	5:00pm	m	3.0kg	9.0
com_400	12:00am	f	3.0kg	9.0
com_401	2:30am	f	3.0kg	8.5
com_402	7:05am	f	3.0kg	9.0

com_403	11:00pm	f	2.5kg	9.0
com_404	8:30am	f	2.8kg	9.5
com_405	6:10pm	m	2.5kg	9.0
com_406	3:05am	f	3.0kg	9.0
com_407	7:00pm	f	3.0kg	9.0
com_408	11:15pm	m	3.0kg	9.0
com_409	10:36am	f	3.0kg	9.0
com_410	9:11pm	f	3.0kg	9.0
com_411	11:15pm	f	3.0kg	9.0
com_412	6:05pm	m	2.5kg	9.0
com_413	11:10pm	f	3.0kg	9.5
com_414	4:04am	m	3.0kg	9.0
com_415	6:00am	m	3.0kg	8.5
com_416	2:00pm	m	3.0kg	9.0
com_417	12:00noon	f	3.0kg	9.0
com_418	10:00pm	f	3.0kg	9.0
com_419	11:00am	f	2.6kg	8.0
com_420	7:00am	m	3.0kg	8.5
com_421	10:20am	f	3.0kg	10.0
com_422	6:30am	m	3.0kg	9.0
com_423	7:00pm	f	3.7kg	9.0
com_424	8:00pm	f	2.5kg	8.5
com_425	10:00am	f	3.0kg	10.0
com_426	6:00am	m	3.2kg	9.5
com_427	11:00pm	m	3.0kg	9.0
com_428	2:00am	f	2.5kg	9.0
com_429	4:00pm	m	3.5kg	8.5
com_430	8:am	f	3.0kg	8.5
com_431	7:40pm	m	3.0kg	9.5
com_432	6:05pm	m	2.5kg	10.0
com_433	8:30am	f	3.0kg	10.0
com_434	7:15pm	f	3.0kg	9.0
com_435	9:12am	f	3.0kg	9.0
com_436	6:06pm	f	3.0kg	10.0
com_437	10:00am	f	3.0kg	10.0
com_438	2:04pm	f	2.5kg	9.5
com_439	7:00am	f	3.0kg	9.0
com_440	6:06pm	f	2.5kg	9.0
com_441	4:00am	f	3.0kg	9.0
com_442	7:00pm	f	3.0kg	9.0
com_443	4:20pm	f	4.0kg	10.0
com_444	7:00am	m	2.5kg	10.0
com_445	8:00pm	f	3.0kg	9.0
com_446	12:12noon	f	2.5kg	9.0

com_447	3:05am	f	3.0kg	8.5
com_448	2:00pm	m	3.0kg	9.0
com_449	10:00pm	m	2.5kg	9.5
com_450	10:30am	f	4.0kg	10.0
com_451	3:00am	f	3.0kg	10.0
com_452	7:00pm	f	2.4kg	10.0
com_651	6:00am	f	3.0kg	9.0
com_453	1:00am	f	3.2kg	9.0
com_454	12:15noon	m	3.0kg	9.5
com_455	2:00pm	f	2.8kg	9.5
com_856	8:00am	f	3.0kg	9.0
com_456	9:00pm	m	3.0kg	9.0
com_457	11:00pm	f	2.7kg	8.5
com_458	8:00am	f	3.0kg	9.0
com_459	10:00pm	f	3.0kg	10.0
com_460	6:00am	f	3.8kg	10.0
com_461	7:00pm	f	3.5kg	10.0
com_462	9:00pm	m	2.5kg	9.5
com_463	2:00am	m	2.8kg	9.0
com_464	7:30am	m	2.0kg	8.5
com_465	11:15pm	f	3.0kg	9.0
com_466	1:15am	m	3.0kg	9.5
com_467	6:06am	m	3.0kg	9.0
com_468	9:25pm	m	3.0kg	9.0
com_469	4:30pm	f	3.0kg	9.5
com_470	6:02am	m	2.5kg	10.0
com_471	10:20am	f	2.5kg	10.0
com_472	11:13am	m	3.0kg	9.0
com_473	3:40am	f	3.0kg	9.0

APPENDIX E

FEDERAL ROAD SAFETY (FRSC)DATASET_A(%INJURED) IN IMO STATE 2020 -2024

Date_Accident	Type_Accident	%_num_male_Killed	%_num_fem_Killed	Y_1	X
2020-01-03 00:00:00	Serious	0	0	0	Na
2020-01-05 00:00:00	Minor	0	0	0	0.110416667
2020-01-04 00:00:00	Serious	0	0	0	0.229166667
2020-01-07 00:00:00	Serious	0	0	0	02.20
2020-01-09 00:00:00	Serious	0	0	0	13.04
2020-01-13 00:00:00	Serious	0	0	0	0.425
2020-01-15 00:00:00	Fatal	0	3.333333	3.333333333	01.09
2020-01-18 00:00:00	Fatal	0.584795	1.666667	2.251461988	05.23
2020-01-19 00:00:00	Serious	0	0	0	01.01
2020-01-28 00:00:00	Serious	0	0	0	11.35
2020-01-29 00:00:00	Serious	0	0	0	01.34
2020-01-29 00:00:00	Fatal	1.169591	0	1.169590643	00.40
2020-02-05 00:00:00	Serious	0	0	0	01.20
2020-02-06 00:00:00	Serious	0	0	0	09.00
2020-02-07 00:00:00	Serious	0	0	0	04.20
2020-02-11 00:00:00	Fatal	0.584795	3.333333	3.918128655	04.45
2020-02-12 00:00:00	Fatal	0.584795	0	0.584795322	01.25
2020-02-18 00:00:00	Serious	0	0	0	06.15
2020-02-18 00:00:00	Serious	0	0	0	02.03
2020-02-19 00:00:00	Fatal	0.584795	0	0.584795322	13.09

2020-02-21 00:00:00	Fatal	1.169591	0	1.16959064 3	14.30
2020-03-05 00:00:00	Serious	0	0	0	02.20
2020-03-19 00:00:00	Fatal	0.584795	0	0.58479532 2	01.30
2020-03-22 00:00:00	Serious	0	0	0	04.10
2020-03-26 00:00:00	Minor	0	0	0	07.25
2020-03-26 00:00:00	Serious	0	0	0	02.25
2020-03-28 00:00:00	Serious	0	0	0	14.24
2020-03-31 00:00:00	Serious	0	0	0	08.00
2020-04-19 00:00:00	Serious	0	0	0	04.00
2020-04-26 00:00:00	Serious	0	0	0	05.06
2020-05-15 00:00:00	Serious	0	0	0	00.19
2020-05-29 00:00:00	Serious	0	0	0	00.10
2020-05-29 00:00:00	Serious	0	0	0	05.00
2020-05-30 00:00:00	Minor	0	0	0	05.00
2020-06-01 00:00:00	Minor	0	0	0	16.00
2020-06-02 00:00:00	Serious	0	0	0	11.25
2020-06-07 00:00:00	Serious	0	0	0	00.05
2020-06-12 00:00:00	Fatal	0.584795	0	0.58479532 2	07.12
2020-06-14 00:00:00	Serious	0	0	0	07.22
2020-06-19 00:00:00	Minor	0	0	0	00.25
2020-06-21 00:00:00	Serious	0	0	0	01.35
2020-06-23 00:00:00	Serious	0	0	0	04.10
2020-06-29 00:00:00	Serious	0	0	0	07.05
2020-07-04 00:00:00	Serious	0	0	0	00.05

2020-07-04 00:00:00	Serious	0	0	0	03.35
2020-07-07 00:00:00	Fatal	1.169591	0	1.169590643	05.57
2020-07-15 00:00:00	Fatal	0.584795	1.666667	2.251461988	01.08
2020-07-16 00:00:00	Fatal	0.584795	0	0.584795322	4.05
2020-07-25 00:00:00	Fatal	0.584795	0	0.584795322	5.13
2020-07-26 00:00:00	Serious	0	0	0	04.08
2020-07-31 00:00:00	Fatal	0.584795	0	0.584795322	04.10
2020-08-02 00:00:00	Serious	0	0	0	07.00
2020-08-02 00:00:00	Minor	0	0	0	02.00
2020-08-11 00:00:00	Serious	0	0	0	05.25
2020-08-11 00:00:00	Serious	0	0	0	11.30
2020-08-17 00:00:00	Fatal	0.584795	5	5.584795322	07.30
2020-08-26 00:00:00	Serious	0	0	0	03.15
2020-09-05 00:00:00	Serious	0	0	0	06.15
2020-09-07 00:00:00	Fatal	1.169591	1.666667	2.83625731	05.00
2020-09-11 00:00:00	Serious	0	0	0	05.05
2020-09-13 00:00:00	Fatal	0.584795	0	0.584795322	04.00
2020-09-15 00:00:00	Fatal	0.584795	0	0.584795322	08.10
2020-09-17 00:00:00	Minor	0	0	0	10.10
2020-09-18 00:00:00	Fatal	0.584795	0	0.584795322	03.15
2020-09-18 00:00:00	Fatal	0.584795	1.666667	2.251461988	09.20
2020-09-22 00:00:00	Fatal	0.584795	1.666667	2.251461988	08.55
2020-09-24 00:00:00	Fatal	1.169591	0	1.169590643	01.50
2020-09-28 00:00:00	Fatal	1.169591	0	1.169590643	12.50

2020-09-30 00:00:00	Serious	0	0	0	05.40
2020-10-03 00:00:00	Serious	0	0	0	06.21
2020-10-07 00:00:00	Serious	0	0	0	03.19
2020-10-08 00:00:00	Serious	0	0	0	02.19
2020-10-08 00:00:00	Serious	0	0	0	02.26
2020-10-10 00:00:00	Serious	0	0	0	17.31
2020-10-11 00:00:00	Serious	0	0	0	22.19
2020-10-15 00:00:00	Serious	0	0	0	00.23
2020-10-15 00:00:00	Serious	0	0	0	02.17
2020-11-10 00:00:00	Fatal	0.584795	0	0.584795322	03:15
2020-11-10 00:00:00	Fatal	0.584795	0	0.584795322	01.20
2020-11-10 00:00:00	Fatal	0.584795	0	0.584795322	00.20
2020-11-12 00:00:00	Fatal	0.584795	0	0.584795322	01.40
2020-11-15 00:00:00	Fatal	0	0	0	05.05
2020-11-15 00:00:00	Serious	0	0	0	04.30
2020-11-17 00:00:00	Serious	0	0	0	11.55
2020-11-23 00:00:00	Fatal	0	3.333333	3.333333333	07.20
2020-11-30 00:00:00	Minor	0	0	0	07.20
2020-12-01 00:00:00	Fatal	0.584795	0	0.584795322	01.10
2020-12-01 00:00:00	Serious	0	0	0	08.35
2020-12-08 00:00:00	Serious	0	0	0	02.05
2020-12-09 00:00:00	Serious	0	0	0	12.35
2020-12-10 00:00:00	Fatal	0.584795	0	0.584795322	03.15
2020-12-10 00:00:00	Serious	0	0	0	06.10

2020-12-18 00:00:00	Fatal	0.584795	1.666667	2.251461988	04.11
2020-12-27 00:00:00	Serious	0	0	0	01.21
2020-12-30 00:00:00	Serious	0	0	0	00.10
2020-12-31 00:00:00	Serious	0	0	0	07.30
2021-01-02 00:00:00	Serious	0	0	0	01.02
2021-01-09 00:00:00	Minor	0	0	0	05.24
2021-01-12 00:00:00	Serious	0	0	0	02.13
2021-01-21 00:00:00	Fatal	2.923977	1.666667	4.590643275	04.05
2021-01-27 00:00:00	Serious	0	0	0	02.10
2021-02-01 00:00:00	Fatal	1.754386	0	1.754385965	05.25
2021-02-04 00:00:00	Fatal	0	1.666667	1.666666667	01.05
2021-02-10 00:00:00	Fatal	0.584795	0	0.584795322	07.44
2021-02-09 00:00:00	Fatal	0.584795	0	0.584795322	08.49
2021-02-15 00:00:00	Serious	0	0	0	06.05
2021-02-26 00:00:00	Fatal	1.169591	0	1.169590643	05.15
2021-03-20 00:00:00	Serious	0	0	0	01.25
2021-03-21 00:00:00	Serious	0	0	0	02.00
2021-04-01 00:00:00	Serious	0	0	0	06.50
2021-04-01 00:00:00	Serious	0	0	0	10.00
2021-04-04 00:00:00	Serious	0	0	0	06.50
2021-04-07 00:00:00	Fatal	1.169591	0	1.169590643	07.30
2021-04-08 00:00:00	Fatal	1.169591	0	1.169590643	12.20
2021-04-11 00:00:00	Serious	0	0	0	11.05
2021-04-13 00:00:00	Fatal	0.584795	0	0.584795322	08.00

2021-04-16 00:00:00	Minor	0	0	0	00.13
2021-04-16 00:00:00	Serious	0	0	0	04.28
2021-04-18 00:00:00	Serious	0	0	0	03.35
2021-04-18 00:00:00	Minor	0	0	0	04.15
2021-04-20 00:00:00	Serious	0	0	0	05.34
2021-04-23 00:00:00	Serious	0	0	0	17.39
2021-04-26 00:00:00	Serious	0	0	0	10.55
2021-04-26 00:00:00	Serious	0	0	0	01.35
2021-05-01 00:00:00	Serious	0	0	0	00.05
2021-05-01 00:00:00	Fatal	1.169591	0	1.169590643	02.25
2021-05-05 00:00:00	Serious	0	0	0	05.25
2021-05-07 00:00:00	Serious	0	0	0	11.25
2021-05-08 00:00:00	Fatal	1.754386	3.333333	5.087719298	12.30
2021-05-11 00:00:00	Serious	0	0	0	7.05
2021-05-12 00:00:00	Serious	0	0	0	11.10
2021-05-17 00:00:00	Serious	0	0	0	6.30
2021-05-26 00:00:00	Serious	0	0	0	0.50
2021-06-01 00:00:00	Serious	0	0	0	1:30
2021-06-02 00:00:00	Minor	0	0	0	4.20
2021-06-11 00:00:00	Serious	0	0	0	9.20
2021-06-13 00:00:00	Fatal	0.584795	0	0.584795322	11.05
2021-06-21 00:00:00	Fatal	1.754386	0	1.754385965	03.00
2021-06-21 00:00:00	Minor	0	0	0	01.25
2021-06-22 00:00:00	Fatal	0.584795	0	0.584795322	1.10

2021-06-26 00:00:00	Serious	0	0	0	2.20
2021-07-03 00:00:00	Serious	0	0	0	03.10
2021-07-10 00:00:00	Minor	0	0	0	02.21
2021-07-18 00:00:00	Serious	0	0	0	03.29
2021-07-24 00:00:00	Serious	0	0	0	05.00
2021-07-28 00:00:00	Serious	0	0	0	08.10
2021-07-31 00:00:00	Serious	0	0	0	03.10
2021-08-03 00:00:00	Fatal	0	1.666667	1.666666667	05.35
2021-08-06 00:00:00	Minor	0	0	0	05.37
2021-08-08 00:00:00	Fatal	0.584795	0	0.584795322	09.02
2021-08-08 00:00:00	Serious	0	0	0	10.55
2021-08-10 00:00:00	Serious	0	0	0	05.05
2021-08-12 00:00:00	Serious	0	0	0	04.40
2021-08-13 00:00:00	Minor	0	0	0	08.00
2021-08-13 00:00:00	Serious	0	0	0	00.15
2021-08-14 00:00:00	Fatal	1.169591	0	1.169590643	17.00
2021-08-26 00:00:00	Fatal	2.339181	0	2.339181287	19.10
2021-08-27 00:00:00	Serious	0	0	0	04.15
2021-09-05 00:00:00	Fatal	1.169591	1.666667	2.83625731	11.40
2021-09-05 00:00:00	Fatal	0.584795	0	0.584795322	11.20
2021-09-11 00:00:00	Serious	0	0	0	01.35
2021-09-15 00:00:00	Serious	0	0	0	02:05
2021-09-16 00:00:00	Serious	0	0	0	00.30
2021-09-20 00:00:00	Fatal	0.584795	0	0.584795322	11.35

2021-09-25 00:00:00	Fatal	0.584795	0	0.584795322	09.25
2021-09-25 00:00:00	Serious	0	0	0	09.10
2021-09-29 00:00:00	Fatal	1.169591	0	1.169590643	04.10
2021-09-30 00:00:00	Serious	0	0	0	00.25
2021-10-03 00:00:00	Serious	0	0	0	07.40
2021-10-09 00:00:00	Fatal	1.169591	1.666667	2.83625731	03.30
2021-10-10 00:00:00	Serious	0	0	0	01.25
2021-10-13 00:00:00	Serious	0	0	0	04.10
2021-10-15 00:00:00	Minor	0	0	0	03.30
2021-10-15 00:00:00	Serious	0	0	0	06.40
2021-10-17 00:00:00	Fatal	0	0	0	02.35
2021-10-29 00:00:00	Serious	0	0	0	09.15
2021-11-12 00:00:00	Serious	0	0	0	10.00
2021-11-13 00:00:00	Serious	0	0	0	04.26
2021-11-14 00:00:00	Fatal	1.169591	0	1.169590643	11.16
2021-11-14 00:00:00	Serious	0	0	0	06.55
2021-11-14 00:00:00	Serious	0	0	0	05.25
2021-11-16 00:00:00	Fatal	0.584795	0	0.584795322	00.40
2021-11-20 00:00:00	Fatal	0	0	0	04.00
2021-11-20 00:00:00	Serious	0	0	0	02.30
2021-11-21 00:00:00	Fatal	2.339181	1.666667	4.005847953	02.35
2021-11-27 00:00:00	Serious	0	0	0	05.07
2021-11-27 00:00:00	Serious	0	0	0	01.02
2021-11-30 00:00:00	Fatal	1.169591	1.666667	2.83625731	03.25

2021-12-03 00:00:00	Serious	0	0	0	12.05
2021-12-04 00:00:00	Serious	0	0	0	06.35
2021-12-05 00:00:00	Serious	0	0	0	04.10
2021-12-07 00:00:00	Fatal	1.169591	0	1.169590643	01.45
2021-12-09 00:00:00	Serious	0	0	0	00.55
2021-12-19 00:00:00	Fatal	0.584795	0	0.584795322	09.15
2021-12-20 00:00:00	Serious	0	0	0	00.15
2021-12-23 00:00:00	Serious	0	0	0	05.25
2021-12-23 00:00:00	Serious	0	0	0	09.25
2021-12-24 00:00:00	Fatal	0	1.666667	1.666666667	10.25
2021-12-25 00:00:00	Serious	0	0	0	08.40
2021-12-25 00:00:00	Fatal	0	1.666667	1.666666667	00.30
2021-12-25 00:00:00	Serious	0	0	0	06.35
2021-12-25 00:00:00	Serious	0	0	0	03.35
2021-12-26 00:00:00	Serious	0	0	0	02.20
2021-12-27 00:00:00	Serious	0	0	0	04.10
2021-12-28 00:00:00	Serious	0	0	0	04.00
2021-12-31 00:00:00	Serious	0	0	0	06.30
2022-01-02 00:00:00	Serious	0	0	0	04.25
2022-01-02 00:00:00	Serious	0	0	0	01.27
2022-01-03 00:00:00	Serious	0	0	0	10.32
2022-01-04 00:00:00	Serious	0	0	0	04.33
2022-01-05 00:00:00	Serious	0	0	0	01.15
2022-01-11 00:00:00	Minor	0	0	0	04.00

2022-01-16 00:00:00	Serious	0	0	0	06.34
2022-01-20 00:00:00	Serious	0	0	0	10.21
2022-01-23 00:00:00	Serious	0	0	0	05.15
2022-01-30 00:00:00	Serious	0	0	0	08.00
2022-02-05 00:00:00	Minor	0	0	0	04.20
2022-02-06 00:00:00	Serious	0	0	0	01.14
2022-02-09 00:00:00	Serious	0	0	0	08.18
2022-02-11 00:00:00	Fatal	0.584795	0	0.584795322	09.12
2022-02-15 00:00:00	Serious	0	0	0	09.00
2022-03-03 00:00:00	Fatal	1.169591	0	1.169590643	08.25
2022-03-11 00:00:00	Serious	0	0	0	07.15
2022-03-11 00:00:00	Fatal	0.584795	0	0.584795322	10.20
2022-03-12 00:00:00	Fatal	0	1.666667	1.666666667	07.55
2022-03-13 00:00:00	Serious	0	0	0	01.25
2022-03-15 00:00:00	Serious	0	0	0	08.10
2022-03-19 00:00:00	Serious	0	0	0	09.04
2022-03-22 00:00:00	Fatal	0.584795	0	0.584795322	02.21
2022-03-23 00:00:00	Fatal	0	1.666667	1.666666667	04.25
2022-03-31 00:00:00	Serious	0	0	0	03.25
2022-04-05 00:00:00	Minor	0	0	0	04.25
2022-04-07 00:00:00	Fatal	0.584795	0	0.584795322	03.35
2022-04-19 00:00:00	Fatal	0.584795	0	0.584795322	02.05
2022-04-23 00:00:00	Serious	0	0	0	01.25
2022-04-29 00:00:00	Serious	0	0	0	02.25

2022-05-01 00:00:00	Serious	0	0	0	05.05
2022-05-02 00:00:00	Serious	0	0	0	05.15
2022-05-07 00:00:00	Serious	0	0	0	02.50
2022-05-19 00:00:00	Minor	0	0	0	06.40
2022-05-19 00:00:00	Serious	0	0	0	06.00
2022-05-29 00:00:00	Serious	0	0	0	08.00
2022-05-30 00:00:00	Fatal	1.754386	0	1.754385965	05.15
2022-06-09 00:00:00	Serious	0	0	0	08.05
2022-06-10 00:00:00	Fatal	0.584795	0	0.584795322	02.30
2022-06-11 00:00:00	Serious	0	0	0	08.00
2022-06-21 00:00:00	Serious	0	0	0	09.35
2022-06-22 00:00:00	Serious	0	0	0	04.25
2022-06-22 00:00:00	Serious	0	0	0	06.30
2022-06-30 00:00:00	Fatal	0	1.666667	1.666666667	03.40
2022-07-08 00:00:00	Serious	0	0	0	01.20
2022-07-13 00:00:00	Serious	0	0	0	11.20
2022-07-14 00:00:00	Serious	0	0	0	12.00
2022-07-16 00:00:00	Serious	0	0	0	09.20
2022-07-24 00:00:00	Minor	0	0	0	07.10
2022-08-05 00:00:00	Fatal	0.584795	0	0.584795322	06.20
2022-08-09 00:00:00	Serious	0	0	0	01.14
2022-08-10 00:00:00	Minor	0	0	0	02.19
2022-08-20 00:00:00	Fatal	0.584795	0	0.584795322	04.00
2022-08-26 00:00:00	Serious	0	0	0	07.35

2022-08-30 00:00:00	Serious	0	0	0	02.10
2022-09-01 00:00:00	Serious	0	0	0	08.12
2022-09-09 00:00:00	Serious	0	0	0	01.37
2022-09-10 00:00:00	Serious	0	0	0	04.45
2022-09-14 00:00:00	Fatal	1.169591	0	1.169590643	06.00
2022-09-19 00:00:00	Fatal	0	0	0	02.30
2022-09-21 00:00:00	Fatal	0	3.333333	3.333333333	15.15
2022-09-26 00:00:00	Minor	0	0	0	21.07
2022-10-02 00:00:00	Serious	0	0	0	12.02
2022-10-02 00:00:00	Serious	0	0	0	00.15
2022-10-03 00:00:00	Minor	0	0	0	08.05
2022-10-09 00:00:00	Serious	0	0	0	03.30
2022-10-19 00:00:00	Serious	0	0	0	05.36
2022-10-26 00:00:00	Serious	0	0	0	01.16
2022-10-28 00:00:00	Serious	0	0	0	07.04
2022-11-02 00:00:00	Fatal	0.584795	0	0.584795322	06.02
2022-11-03 00:00:00	Fatal	0	1.666667	1.666666667	00.26
2022-11-05 00:00:00	Fatal	0.584795	0	0.584795322	01.39
2022-11-06 00:00:00	Fatal	0	5	5	06.30
2022-11-11 00:00:00	Serious	0	0	0	04.15
2022-11-12 00:00:00	Fatal	2.923977	0	2.923976608	08.15
2022-11-15 00:00:00	Serious	0	0	0	12.10
2022-11-16 00:00:00	Serious	0	0	0	02.05
2022-11-19 00:00:00	Fatal	0.584795	0	0.584795322	01.49

2022-11-22 00:00:00	Serious	0	0	0	10.39
2022-11-23 00:00:00	Serious	0	0	0	14.08
2022-11-25 00:00:00	Serious	0	0	0	06.05
2022-11-26 00:00:00	Fatal	0.584795	0	0.584795322	01.05
2022-12-12 00:00:00	Serious	0	0	0	00.10
2022-12-16 00:00:00	Serious	0	0	0	02.50
2022-12-17 00:00:00	Serious	0	0	0	06.50
2022-12-18 00:00:00	Serious	0	0	0	02.45
2022-12-18 00:00:00	Serious	0	0	0	08.00
2022-12-20 00:00:00	Serious	0	0	0	08.15
2022-12-24 00:00:00	Serious	0	0	0	08.45
2022-12-27 00:00:00	Fatal	1.169591	0	1.169590643	01.35
2023-01-18 00:00:00	Serious	0	0	0	04.05
2023-01-20 00:00:00	Fatal	0.584795	0	0.584795322	03.20
2023-01-29 00:00:00	Fatal	0	3.333333	3.333333333	05.20
2023-01-29 00:00:00	Serious	0	0	0	05.00
2023-02-11 00:00:00	Fatal	0.584795	0	0.584795322	03.45
2023-02-12 00:00:00	Minor	0	0	0	04.50
2023-02-17 00:00:00	Serious	0	0	0	09.20
2023-02-22 00:00:00	Serious	0	0	0	03.00
2023-03-05 00:00:00	Serious	0	0	0	06.20
2023-03-05 00:00:00	Fatal	0.584795	0	0.584795322	03.20
2023-03-08 00:00:00	Fatal	0.584795	0	0.584795322	07.30
2023-03-11 00:00:00	Fatal	0.584795	0	0.584795322	05.20

2023-03-22 00:00:00	Fatal	0	1.666667	1.666666667	05.15
2023-03-23 00:00:00	Serious	0	0	0	03.25
2023-03-26 00:00:00	Fatal	0.584795	0	0.584795322	12.00
2023-04-16 00:00:00	Serious	0	0	0	01.30
2023-04-17 00:00:00	Serious	0	0	0	13.55
2023-04-22 00:00:00	Minor	0	0	0	08.45
2023-04-28 00:00:00	Serious	0	0	0	00.40
2023-04-29 00:00:00	Serious	0	0	0	04.30
2023-05-06 00:00:00	Serious	0	0	0	01.05
2023-05-07 00:00:00	Serious	0	0	0	04.05
2023-05-07 00:00:00	Fatal	0.584795	0	0.584795322	01.10
2023-05-16 00:00:00	Serious	0	0	0	03.25
2023-05-20 00:00:00	Minor	0	0	0	04.37
2023-05-28 00:00:00	Serious	0	0	0	01.22
2023-06-07 00:00:00	Fatal	0	0	0	01.00
2023-06-08 00:00:00	Serious	0	0	0	02.08
2023-06-17 00:00:00	Minor	0	0	0	04.08
2023-06-19 00:00:00	Serious	0	0	0	03.00
2023-06-16 00:00:00	Fatal	1.754386	0	1.754385965	01.08
2023-06-20 00:00:00	Minor	0	0	0	04.28
2023-06-21 00:00:00	Serious	0	0	0	07.50
2023-07-03 00:00:00	Serious	0	0	0	00.15
2023-07-07 00:00:00	Serious	0	0	0	04.35
2023-07-11 00:00:00	Serious	0	0	0	02.20

2023-07-13 00:00:00	Serious	0	0	0	04.20
2023-07-21 00:00:00	Minor	0	0	0	06.55
2023-07-28 00:00:00	Fatal	0.584795	1.666667	2.251461988	06.00
2023-07-31 00:00:00	Fatal	0	1.666667	1.666666667	16.55
2023-08-01 00:00:00	Fatal	0	1.666667	1.666666667	10.00
2023-08-04 00:00:00	Serious	0	0	0	03.20
2023-08-06 00:00:00	Minor	0	0	0	01.10
2023-08-17 00:00:00	Serious	0	0	0	04.00
2023-08-30 00:00:00	Minor	0	0	0	06.05
2023-09-06 00:00:00	Fatal	0.584795	0	0.584795322	11.30
2023-09-11 00:00:00	Serious	0	0	0	06.25
2023-09-15 00:00:00	Serious	0	0	0	07.00
2023-09-20 00:00:00	Fatal	1.169591	1.666667	2.83625731	07.20
2023-09-23 00:00:00	Serious	0	0	0	07.30
2023-09-24 00:00:00	Fatal	0.584795	0	0.584795322	06.20
2023-09-26 00:00:00	Fatal	0	1.666667	1.666666667	06.30
2023-09-29 00:00:00	Serious	0	0	0	04.30
2025-10-01 00:00:00	Minor	0	0	0	03.20
2023-10-06 00:00:00	Serious	0	0	0	06.00
2023-10-07 00:00:00	Serious	0	0	0	09.00
2023-10-09 00:00:00	Fatal	0	1.666667	1.666666667	10.00
2023-10-12 00:00:00	Serious	0	0	0	06.00
2023-10-14 00:00:00	Minor	0	0	0	01.15
2023-10-14 00:00:00	Minor	0	0	0	05.45

2023-10-26 00:00:00	Fatal	0.584795	0	0.584795322	06.00
2023-10-29 00:00:00	Fatal	0.584795	0	0.584795322	05.00
2023-11-02 00:00:00	Minor	0	0	0	06.30
2023-11-05 00:00:00	Fatal	0	3.333333	3.333333333	03.00
2023-11-05 00:00:00	Fatal	0	1.666667	1.666666667	04.10
2023-11-14 00:00:00	Serious	0	0	0	02.10
2023-11-19 00:00:00	Fatal	0.584795	0	0.584795322	00.00
2023-11-19 00:00:00	Serious	0	0	0	08.30
2023-11-23 00:00:00	Minor	0	0	0	03.48
2023-11-26 00:00:00	Serious	0	0	0	06.52
2023-11-29 00:00:00	Serious	0	0	0	09.00
2023-11-07 00:00:00	Fatal	2.339181	5	7.339181287	01.20
2023-12-15 00:00:00	Serious	0	0	0	10.00
2023-12-15 00:00:00	Fatal	0.584795	0	0.584795322	11.00
2023-12-17 00:00:00	Serious	0	0	0	00.35
2023-12-17 00:00:00	Serious	0	0	0	06.46
2023-12-24 00:00:00	Serious	0	0	0	03.36
2023-12-25 00:00:00	Serious	0	0	0	01.25
2023-12-27 00:00:00	Minor	0	0	0	02.30
2023-12-29 00:00:00	Fatal	0.584795	1.666667	2.251461988	04.20
2023-12-29 00:00:00	Serious	0	0	0	07.03
2024-02-01	Serious	0	0	0	05.18
2024-04-01	Minor	0	0	0	01.15
2024-06-01	Minor	0	0	0	00.10
2024-10-01	Fatal	0	1.666667	1.666666667	03.10

2024-01-28	Fatal	0.584795	0	0.584795322	02.20
2024-01-29	Minor	0	0	0	01.00
2024-03-01	Minor	0	0	0	06.10
2024-06-01	Serious	0	0	0	02.50
2024-07-01	Fatal	0.584795	0	0.584795322	04.50
2024-08-01	Minor	0	0	0	06.30
2024-01-17	Fatal	0.584795	0	0.584795322	03.00
2024-01-24	Minor	0	0	0	03.35
2024-01-26	Minor	0	0	0	05.05
2024-02-08	Minor	0.584795	0	0.584795322	00.30
2024-02-27	Fatal	0.584795	0	0.584795322	05.25
2024-02-07	Minor	0	0	0	02.25
2024-02-10	Fatal	0.584795	0	0.584795322	07.30
2024-05-03	Minor	0	0	0	03.00
2024-07-03	Minor	0	0	0	07.16
2024-12-03	Minor	0	0	0	04.16
2024-03-26	Minor	0	0	0	08.30
2024-11-03	Minor	0	0	0	06.30
2024-02-03	Fatal	0.584795	0	0.584795322	06.00
2024-08-03	Serious	0	0	0	02.00
2024-03-28	Fatal	0.584795	0	0.584795322	05.30
2024-04-26	Serious	0	0	0	04.20
2024-12-04	Serious	0	0	0	00.40
2024-04-04	Minor	0	0	0	00.20
2024-04-19	Fatal	2.923977	0	2.923976608	04.10
2024-01-04	Fatal	0.584795	1.666667	2.251461988	08.15
2024-05-18	Fatal	1.754386	3.333333	5.087719298	04.15
2024-07-06	Minor	0	0	0	00.00
2024-06-20	Minor	0	0	0	02.20
2024-05-06	Serious	0	0	0	03.50
2024-09-06	Fatal	0.584795	0	0.584795322	07.50
2024-06-30	Minor	0	0	0	06.00
2024-06-21	Fatal	0.584795	1.666667	2.251461988	06.20

2024-02-07	Serious	0	0	0	04.20
2024-04-07	Minor	0	0	0	00.00
2024-08-07	Serious	0	0	0	01.40
2024-07-21	Minor	0	0	0	02.00
2024-07-24	Minor	0	0	0	01.05
2024-07-17	Serious	0	0	0	03.25
2024-07-30	Serious	0	0	0	04.20
2024-07-07	Fatal	0.584795	0	0.58479532 2	01.03
2024-08-26	Minor	0	0	0	1.18
2024-08-31	Minor	0	0	0	1.15
2024-08-25	Serious	0	0	0	3.30
2024-08-28	Serious	0	0	0	2.00
2024-08-29	Serious	0	0	0	5.10
2024-09-13	Minor	0	0	0	4.05
2024-01-10	Fatal	0.584795	0	0.58479532 2	1.05
2024-10-24	Minor	0	0	0	2.30
2024-10-25	Serious	0	0	0	2.20
2024-10-28	Serious	0	0	0	01.20
2024-10-28	Serious	0	0	0	07.00
2024-03-10	Minor	0	0	0	01.35
2024-10-19	Fatal	0.584795	0	0.58479532 2	03.35
2024-10-31	Fatal	1.169591	1.66666 7	2.83625731	04.20
2024-12-10	Serious	0	0	0	01.30
2024-11-19	Serious	0	0	0	01.10
2024-11-26	Minor	0	0	0	07.30
2024-05-11	Fatal	2.339181	5	7.33918128 7	03.30
2024-11-24	Minor	0	0	0	04.15
2024-11-25	Minor	0.584795	0	0.58479532 2	04.30
2024-11-17	Minor	0	0	0	02.05
2024-11-24	Minor	0	0	0	08.15
2024-11-12	Serious	0	0	0	08.15
2024-10-28	Serious	0	0	0	08.28
2024-04-12	Minor	0	0	0	04.13
2024-12-15	Serious	0	0	0	05.10
2024-12-30	Fatal	1.169591	0	1.16959064 3	06.10
2024-12-15	Minor	0	0	0	Na

FEDERAL ROAD SAFETY (FRSC)DATASET_B(% DEATH) IN IMO STATE 2020 -2024

Date_Accident	Type_Accident	%_num_male_Killed	%_num_fem_Killed	Y_2	X
2020-01-03 00:00:00	Serious	0	0	0	Na
2020-01-05 00:00:00	Minor	0	0	0	0.110416667
2020-01-04 00:00:00	Serious	0	0	0	0.229166667
2020-01-07 00:00:00	Serious	0	0	0	02.20
2020-01-09 00:00:00	Serious	0	0	0	13.04
2020-01-13 00:00:00	Serious	0	0	0	0.425
2020-01-15 00:00:00	Fatal	0	3.333333	3.333333333	01.09
2020-01-18 00:00:00	Fatal	0.584795	1.666667	2.251461988	05.23
2020-01-19 00:00:00	Serious	0	0	0	01.01
2020-01-28 00:00:00	Serious	0	0	0	11.35
2020-01-29 00:00:00	Serious	0	0	0	01.34
2020-01-29 00:00:00	Fatal	1.169591	0	1.169590643	00.40
2020-02-05 00:00:00	Serious	0	0	0	01.20
2020-02-06 00:00:00	Serious	0	0	0	09.00
2020-02-07 00:00:00	Serious	0	0	0	04.20
2020-02-11 00:00:00	Fatal	0.584795	3.333333	3.918128655	04.45
2020-02-12 00:00:00	Fatal	0.584795	0	0.584795322	01.25
2020-02-18 00:00:00	Serious	0	0	0	06.15
2020-02-18 00:00:00	Serious	0	0	0	02.03
2020-02-19 00:00:00	Fatal	0.584795	0	0.584795322	13.09

2020-02-21 00:00:00	Fatal	1.169591	0	1.1695906 43	14.30
2020-03-05 00:00:00	Serious	0	0	0	02.20
2020-03-19 00:00:00	Fatal	0.584795	0	0.5847953 22	01.30
2020-03-22 00:00:00	Serious	0	0	0	04.10
2020-03-26 00:00:00	Minor	0	0	0	07.25
2020-03-26 00:00:00	Serious	0	0	0	02.25
2020-03-28 00:00:00	Serious	0	0	0	14.24
2020-03-31 00:00:00	Serious	0	0	0	08.00
2020-04-19 00:00:00	Serious	0	0	0	04.00
2020-04-26 00:00:00	Serious	0	0	0	05.06
2020-05-15 00:00:00	Serious	0	0	0	00.19
2020-05-29 00:00:00	Serious	0	0	0	00.10
2020-05-29 00:00:00	Serious	0	0	0	05.00
2020-05-30 00:00:00	Minor	0	0	0	05.00
2020-06-01 00:00:00	Minor	0	0	0	16.00
2020-06-02 00:00:00	Serious	0	0	0	11.25
2020-06-07 00:00:00	Serious	0	0	0	00.05
2020-06-12 00:00:00	Fatal	0.584795	0	0.5847953 22	07.12
2020-06-14 00:00:00	Serious	0	0	0	07.22
2020-06-19 00:00:00	Minor	0	0	0	00.25
2020-06-21 00:00:00	Serious	0	0	0	01.35
2020-06-23 00:00:00	Serious	0	0	0	04.10
2020-06-29 00:00:00	Serious	0	0	0	07.05
2020-07-04 00:00:00	Serious	0	0	0	00.05

2020-07-04 00:00:00	Serious	0	0	0	03.35
2020-07-07 00:00:00	Fatal	1.169591	0	1.169590643	05.57
2020-07-15 00:00:00	Fatal	0.584795	1.666667	2.251461988	01.08
2020-07-16 00:00:00	Fatal	0.584795	0	0.584795322	04.05
2020-07-25 00:00:00	Fatal	0.584795	0	0.584795322	05.13
2020-07-26 00:00:00	Serious	0	0	0	04.08
2020-07-31 00:00:00	Fatal	0.584795	0	0.584795322	04.10
2020-08-02 00:00:00	Serious	0	0	0	07.00
2020-08-02 00:00:00	Minor	0	0	0	02.00
2020-08-11 00:00:00	Serious	0	0	0	05.25
2020-08-11 00:00:00	Serious	0	0	0	11.30
2020-08-17 00:00:00	Fatal	0.584795	5	5.584795322	07.30
2020-08-26 00:00:00	Serious	0	0	0	03.15
2020-09-05 00:00:00	Serious	0	0	0	06.15
2020-09-07 00:00:00	Fatal	1.169591	1.666667	2.83625731	05.00
2020-09-11 00:00:00	Serious	0	0	0	05.05
2020-09-13 00:00:00	Fatal	0.584795	0	0.584795322	04.00
2020-09-15 00:00:00	Fatal	0.584795	0	0.584795322	08.10
2020-09-17 00:00:00	Minor	0	0	0	10.10
2020-09-18 00:00:00	Fatal	0.584795	0	0.584795322	03.15
2020-09-18 00:00:00	Fatal	0.584795	1.666667	2.251461988	09.20
2020-09-22 00:00:00	Fatal	0.584795	1.666667	2.251461988	08.55
2020-09-24 00:00:00	Fatal	1.169591	0	1.169590643	01.50
2020-09-28 00:00:00	Fatal	1.169591	0	1.169590643	12.50

2020-09-30 00:00:00	Serious	0	0	0	05.40
2020-10-03 00:00:00	Serious	0	0	0	06.21
2020-10-07 00:00:00	Serious	0	0	0	03.19
2020-10-08 00:00:00	Serious	0	0	0	02.19
2020-10-08 00:00:00	Serious	0	0	0	02.26
2020-10-10 00:00:00	Serious	0	0	0	17.31
2020-10-11 00:00:00	Serious	0	0	0	22.19
2020-10-15 00:00:00	Serious	0	0	0	00.23
2020-10-15 00:00:00	Serious	0	0	0	02.17
2020-11-10 00:00:00	Fatal	0.584795	0	0.584795322	03.15
2020-11-10 00:00:00	Fatal	0.584795	0	0.584795322	01.20
2020-11-10 00:00:00	Fatal	0.584795	0	0.584795322	00.20
2020-11-12 00:00:00	Fatal	0.584795	0	0.584795322	01.40
2020-11-15 00:00:00	Fatal	0	0	0	05.05
2020-11-15 00:00:00	Serious	0	0	0	04.30
2020-11-17 00:00:00	Serious	0	0	0	11.55
2020-11-23 00:00:00	Fatal	0	3.333333	3.333333333	07.20
2020-11-30 00:00:00	Minor	0	0	0	07.20
2020-12-01 00:00:00	Fatal	0.584795	0	0.584795322	01.10
2020-12-01 00:00:00	Serious	0	0	0	08.35
2020-12-08 00:00:00	Serious	0	0	0	02.05
2020-12-09 00:00:00	Serious	0	0	0	12.35
2020-12-10 00:00:00	Fatal	0.584795	0	0.584795322	03.15
2020-12-10 00:00:00	Serious	0	0	0	06.10

2020-12-18 00:00:00	Fatal	0.584795	1.666667	2.251461988	04.11
2020-12-27 00:00:00	Serious	0	0	0	01.21
2020-12-30 00:00:00	Serious	0	0	0	00.10
2020-12-31 00:00:00	Serious	0	0	0	07.30
2021-01-02 00:00:00	Serious	0	0	0	01.02
2021-01-09 00:00:00	Minor	0	0	0	05.24
2021-01-12 00:00:00	Serious	0	0	0	02.13
2021-01-21 00:00:00	Fatal	2.923977	1.666667	4.590643275	04.05
2021-01-27 00:00:00	Serious	0	0	0	02.10
2021-02-01 00:00:00	Fatal	1.754386	0	1.754385965	05.25
2021-02-04 00:00:00	Fatal	0	1.666667	1.666666667	01.05
2021-02-10 00:00:00	Fatal	0.584795	0	0.584795322	07.44
2021-02-09 00:00:00	Fatal	0.584795	0	0.584795322	08.49
2021-02-15 00:00:00	Serious	0	0	0	06.05
2021-02-26 00:00:00	Fatal	1.169591	0	1.169590643	05.15
2021-03-20 00:00:00	Serious	0	0	0	01.25
2021-03-21 00:00:00	Serious	0	0	0	02.00
2021-04-01 00:00:00	Serious	0	0	0	06.50
2021-04-01 00:00:00	Serious	0	0	0	10.00
2021-04-04 00:00:00	Serious	0	0	0	06.50
2021-04-07 00:00:00	Fatal	1.169591	0	1.169590643	07.30
2021-04-08 00:00:00	Fatal	1.169591	0	1.169590643	12.20
2021-04-11 00:00:00	Serious	0	0	0	11.05
2021-04-13 00:00:00	Fatal	0.584795	0	0.584795322	08.00

2021-04-16 00:00:00	Minor	0	0	0	00.13
2021-04-16 00:00:00	Serious	0	0	0	04.28
2021-04-18 00:00:00	Serious	0	0	0	03.35
2021-04-18 00:00:00	Minor	0	0	0	04.15
2021-04-20 00:00:00	Serious	0	0	0	05.34
2021-04-23 00:00:00	Serious	0	0	0	17.39
2021-04-26 00:00:00	Serious	0	0	0	10.55
2021-04-26 00:00:00	Serious	0	0	0	01.35
2021-05-01 00:00:00	Serious	0	0	0	00.05
2021-05-01 00:00:00	Fatal	1.169591	0	1.169590643	02.25
2021-05-05 00:00:00	Serious	0	0	0	05.25
2021-05-07 00:00:00	Serious	0	0	0	11.25
2021-05-08 00:00:00	Fatal	1.754386	3.333333	5.087719298	12.30
2021-05-11 00:00:00	Serious	0	0	0	07.05
2021-05-12 00:00:00	Serious	0	0	0	11.10
2021-05-17 00:00:00	Serious	0	0	0	06.30
2021-05-26 00:00:00	Serious	0	0	0	00.50
2021-06-01 00:00:00	Serious	0	0	0	01.30
2021-06-02 00:00:00	Minor	0	0	0	04.20
2021-06-11 00:00:00	Serious	0	0	0	09.20
2021-06-13 00:00:00	Fatal	0.584795	0	0.584795322	11.05
2021-06-21 00:00:00	Fatal	1.754386	0	1.754385965	03.00
2021-06-21 00:00:00	Minor	0	0	0	01.25
2021-06-22 00:00:00	Fatal	0.584795	0	0.584795322	01.10

2021-06-26 00:00:00	Serious	0	0	0	02.20
2021-07-03 00:00:00	Serious	0	0	0	03.10
2021-07-10 00:00:00	Minor	0	0	0	02.21
2021-07-18 00:00:00	Serious	0	0	0	03.29
2021-07-24 00:00:00	Serious	0	0	0	05.00
2021-07-28 00:00:00	Serious	0	0	0	08.10
2021-07-31 00:00:00	Serious	0	0	0	03.10
2021-08-03 00:00:00	Fatal	0	1.666667	1.666666667	05.35
2021-08-06 00:00:00	Minor	0	0	0	05.37
2021-08-08 00:00:00	Fatal	0.584795	0	0.584795322	09.02
2021-08-08 00:00:00	Serious	0	0	0	10.55
2021-08-10 00:00:00	Serious	0	0	0	05.05
2021-08-12 00:00:00	Serious	0	0	0	04.40
2021-08-13 00:00:00	Minor	0	0	0	08.00
2021-08-13 00:00:00	Serious	0	0	0	00.15
2021-08-14 00:00:00	Fatal	1.169591	0	1.169590643	17.00
2021-08-26 00:00:00	Fatal	2.339181	0	2.339181287	19.10
2021-08-27 00:00:00	Serious	0	0	0	04.15
2021-09-05 00:00:00	Fatal	1.169591	1.666667	2.83625731	11.40
2021-09-05 00:00:00	Fatal	0.584795	0	0.584795322	11.20
2021-09-11 00:00:00	Serious	0	0	0	01.35
2021-09-15 00:00:00	Serious	0	0	0	02.05
2021-09-16 00:00:00	Serious	0	0	0	00.30
2021-09-20 00:00:00	Fatal	0.584795	0	0.584795322	11.35

2021-09-25 00:00:00	Fatal	0.584795	0	0.5847953 22	09.25
2021-09-25 00:00:00	Serious	0	0	0	09.10
2021-09-29 00:00:00	Fatal	1.169591	0	1.1695906 43	04.10
2021-09-30 00:00:00	Serious	0	0	0	00.25
2021-10-03 00:00:00	Serious	0	0	0	07.40
2021-10-09 00:00:00	Fatal	1.169591	1.666667	2.8362573 1	03.30
2021-10-10 00:00:00	Serious	0	0	0	01.25
2021-10-13 00:00:00	Serious	0	0	0	04.10
2021-10-15 00:00:00	Minor	0	0	0	03.30
2021-10-15 00:00:00	Serious	0	0	0	06.40
2021-10-17 00:00:00	Fatal	0	0	0	02.35
2021-10-29 00:00:00	Serious	0	0	0	09.15
2021-11-12 00:00:00	Serious	0	0	0	10.00
2021-11-13 00:00:00	Serious	0	0	0	04.26
2021-11-14 00:00:00	Fatal	1.169591	0	1.1695906 43	11.16
2021-11-14 00:00:00	Serious	0	0	0	06.55
2021-11-14 00:00:00	Serious	0	0	0	05.25
2021-11-16 00:00:00	Fatal	0.584795	0	0.5847953 22	00.40
2021-11-20 00:00:00	Fatal	0	0	0	04.00
2021-11-20 00:00:00	Serious	0	0	0	02.30
2021-11-21 00:00:00	Fatal	2.339181	1.666667	4.0058479 53	02.35
2021-11-27 00:00:00	Serious	0	0	0	05.07
2021-11-27 00:00:00	Serious	0	0	0	01.02
2021-11-30 00:00:00	Fatal	1.169591	1.666667	2.8362573 1	03.25

2021-12-03 00:00:00	Serious	0	0	0	12.05
2021-12-04 00:00:00	Serious	0	0	0	06.35
2021-12-05 00:00:00	Serious	0	0	0	04.10
2021-12-07 00:00:00	Fatal	1.169591	0	1.169590643	01.45
2021-12-09 00:00:00	Serious	0	0	0	00.55
2021-12-19 00:00:00	Fatal	0.584795	0	0.584795322	09.15
2021-12-20 00:00:00	Serious	0	0	0	00.15
2021-12-23 00:00:00	Serious	0	0	0	05.25
2021-12-23 00:00:00	Serious	0	0	0	09.25
2021-12-24 00:00:00	Fatal	0	1.666667	1.666666667	10.25
2021-12-25 00:00:00	Serious	0	0	0	08.40
2021-12-25 00:00:00	Fatal	0	1.666667	1.666666667	00.30
2021-12-25 00:00:00	Serious	0	0	0	06.35
2021-12-25 00:00:00	Serious	0	0	0	03.35
2021-12-26 00:00:00	Serious	0	0	0	02.20
2021-12-27 00:00:00	Serious	0	0	0	04.10
2021-12-28 00:00:00	Serious	0	0	0	04.00
2021-12-31 00:00:00	Serious	0	0	0	06.30
2022-01-02 00:00:00	Serious	0	0	0	04.25
2022-01-02 00:00:00	Serious	0	0	0	01.27
2022-01-03 00:00:00	Serious	0	0	0	10.32
2022-01-04 00:00:00	Serious	0	0	0	04.33
2022-01-05 00:00:00	Serious	0	0	0	01.15
2022-01-11 00:00:00	Minor	0	0	0	04.00

2022-01-16 00:00:00	Serious	0	0	0	06.34
2022-01-20 00:00:00	Serious	0	0	0	10.21
2022-01-23 00:00:00	Serious	0	0	0	05.15
2022-01-30 00:00:00	Serious	0	0	0	08.00
2022-02-05 00:00:00	Minor	0	0	0	04.20
2022-02-06 00:00:00	Serious	0	0	0	01.14
2022-02-09 00:00:00	Serious	0	0	0	08.18
2022-02-11 00:00:00	Fatal	0.584795	0	0.584795322	09.12
2022-02-15 00:00:00	Serious	0	0	0	09.00
2022-03-03 00:00:00	Fatal	1.169591	0	1.169590643	08.25
2022-03-11 00:00:00	Serious	0	0	0	07.15
2022-03-11 00:00:00	Fatal	0.584795	0	0.584795322	10.20
2022-03-12 00:00:00	Fatal	0	1.666667	1.666666667	07.55
2022-03-13 00:00:00	Serious	0	0	0	01.25
2022-03-15 00:00:00	Serious	0	0	0	08.10
2022-03-19 00:00:00	Serious	0	0	0	09.04
2022-03-22 00:00:00	Fatal	0.584795	0	0.584795322	02.21
2022-03-23 00:00:00	Fatal	0	1.666667	1.666666667	04.25
2022-03-31 00:00:00	Serious	0	0	0	03.25
2022-04-05 00:00:00	Minor	0	0	0	04.25
2022-04-07 00:00:00	Fatal	0.584795	0	0.584795322	03.35
2022-04-19 00:00:00	Fatal	0.584795	0	0.584795322	02.05
2022-04-23 00:00:00	Serious	0	0	0	01.25
2022-04-29 00:00:00	Serious	0	0	0	02.25

2022-05-01 00:00:00	Serious	0	0	0	05.05
2022-05-02 00:00:00	Serious	0	0	0	05.15
2022-05-07 00:00:00	Serious	0	0	0	02.50
2022-05-19 00:00:00	Minor	0	0	0	06.40
2022-05-19 00:00:00	Serious	0	0	0	06.00
2022-05-29 00:00:00	Serious	0	0	0	08.00
2022-05-30 00:00:00	Fatal	1.754386	0	1.754385965	05.15
2022-06-09 00:00:00	Serious	0	0	0	08.05
2022-06-10 00:00:00	Fatal	0.584795	0	0.584795322	02.30
2022-06-11 00:00:00	Serious	0	0	0	08.00
2022-06-21 00:00:00	Serious	0	0	0	09.35
2022-06-22 00:00:00	Serious	0	0	0	04.25
2022-06-22 00:00:00	Serious	0	0	0	06.30
2022-06-30 00:00:00	Fatal	0	1.666667	1.666666667	03.40
2022-07-08 00:00:00	Serious	0	0	0	01.20
2022-07-13 00:00:00	Serious	0	0	0	11.20
2022-07-14 00:00:00	Serious	0	0	0	12.00
2022-07-16 00:00:00	Serious	0	0	0	09.20
2022-07-24 00:00:00	Minor	0	0	0	07.10
2022-08-05 00:00:00	Fatal	0.584795	0	0.584795322	06.20
2022-08-09 00:00:00	Serious	0	0	0	01.14
2022-08-10 00:00:00	Minor	0	0	0	02.19
2022-08-20 00:00:00	Fatal	0.584795	0	0.584795322	04.00
2022-08-26 00:00:00	Serious	0	0	0	07.35

2022-08-30 00:00:00	Serious	0	0	0	02.10
2022-09-01 00:00:00	Serious	0	0	0	08.12
2022-09-09 00:00:00	Serious	0	0	0	01.37
2022-09-10 00:00:00	Serious	0	0	0	04.45
2022-09-14 00:00:00	Fatal	1.169591	0	1.169590643	06.00
2022-09-19 00:00:00	Fatal	0	0	0	02.30
2022-09-21 00:00:00	Fatal	0	3.333333	3.333333333	15.15
2022-09-26 00:00:00	Minor	0	0	0	21.07
2022-10-02 00:00:00	Serious	0	0	0	12.02
2022-10-02 00:00:00	Serious	0	0	0	00.15
2022-10-03 00:00:00	Minor	0	0	0	08.05
2022-10-09 00:00:00	Serious	0	0	0	03.30
2022-10-19 00:00:00	Serious	0	0	0	05.36
2022-10-26 00:00:00	Serious	0	0	0	01.16
2022-10-28 00:00:00	Serious	0	0	0	07.04
2022-11-02 00:00:00	Fatal	0.584795	0	0.584795322	06.02
2022-11-03 00:00:00	Fatal	0	1.666667	1.666666667	00.26
2022-11-05 00:00:00	Fatal	0.584795	0	0.584795322	01.39
2022-11-06 00:00:00	Fatal	0	5	5	06.30
2022-11-11 00:00:00	Serious	0	0	0	04.15
2022-11-12 00:00:00	Fatal	2.923977	0	2.923976608	08.15
2022-11-15 00:00:00	Serious	0	0	0	12.10
2022-11-16 00:00:00	Serious	0	0	0	02.05
2022-11-19 00:00:00	Fatal	0.584795	0	0.584795322	01.49

2022-11-22 00:00:00	Serious	0	0	0	10.39
2022-11-23 00:00:00	Serious	0	0	0	14.08
2022-11-25 00:00:00	Serious	0	0	0	06.05
2022-11-26 00:00:00	Fatal	0.584795	0	0.584795322	01.05
2022-12-12 00:00:00	Serious	0	0	0	00.10
2022-12-16 00:00:00	Serious	0	0	0	02.50
2022-12-17 00:00:00	Serious	0	0	0	06.50
2022-12-18 00:00:00	Serious	0	0	0	02.45
2022-12-18 00:00:00	Serious	0	0	0	08.00
2022-12-20 00:00:00	Serious	0	0	0	08.15
2022-12-24 00:00:00	Serious	0	0	0	08.45
2022-12-27 00:00:00	Fatal	1.169591	0	1.169590643	01.35
2023-01-18 00:00:00	Serious	0	0	0	04.05
2023-01-20 00:00:00	Fatal	0.584795	0	0.584795322	03.20
2023-01-29 00:00:00	Fatal	0	3.333333	3.333333333	05.20
2023-01-29 00:00:00	Serious	0	0	0	05.00
2023-02-11 00:00:00	Fatal	0.584795	0	0.584795322	03.45
2023-02-12 00:00:00	Minor	0	0	0	04.50
2023-02-17 00:00:00	Serious	0	0	0	09.20
2023-02-22 00:00:00	Serious	0	0	0	03.00
2023-03-05 00:00:00	Serious	0	0	0	06.20
2023-03-05 00:00:00	Fatal	0.584795	0	0.584795322	03.20
2023-03-08 00:00:00	Fatal	0.584795	0	0.584795322	07.30
2023-03-11 00:00:00	Fatal	0.584795	0	0.584795322	05.20

2023-03-22 00:00:00	Fatal	0	1.666667	1.6666666 67	05.15
2023-03-23 00:00:00	Serious	0	0	0	03.25
2023-03-26 00:00:00	Fatal	0.584795	0	0.5847953 22	12.00
2023-04-16 00:00:00	Serious	0	0	0	01.30
2023-04-17 00:00:00	Serious	0	0	0	13.55
2023-04-22 00:00:00	Minor	0	0	0	08.45
2023-04-28 00:00:00	Serious	0	0	0	00.40
2023-04-29 00:00:00	Serious	0	0	0	04.30
2023-05-06 00:00:00	Serious	0	0	0	01.05
2023-05-07 00:00:00	Serious	0	0	0	04.05
2023-05-07 00:00:00	Fatal	0.584795	0	0.5847953 22	01.10
2023-05-16 00:00:00	Serious	0	0	0	03.25
2023-05-20 00:00:00	Minor	0	0	0	04.37
2023-05-28 00:00:00	Serious	0	0	0	01.22
2023-06-07 00:00:00	Fatal	0	0	0	01.00
2023-06-08 00:00:00	Serious	0	0	0	02.08
2023-06-17 00:00:00	Minor	0	0	0	04.08
2023-06-19 00:00:00	Serious	0	0	0	03.00
2023-06-16 00:00:00	Fatal	1.754386	0	1.7543859 65	01.08
2023-06-20 00:00:00	Minor	0	0	0	04.28
2023-06-21 00:00:00	Serious	0	0	0	07.50
2023-07-03 00:00:00	Serious	0	0	0	00.15
2023-07-07 00:00:00	Serious	0	0	0	04.35
2023-07-11 00:00:00	Serious	0	0	0	02.20

2023-07-13 00:00:00	Serious	0	0	0	04.20
2023-07-21 00:00:00	Minor	0	0	0	06.55
2023-07-28 00:00:00	Fatal	0.584795	1.666667	2.2514619 88	06.00
2023-07-31 00:00:00	Fatal	0	1.666667	1.6666666 67	16.55
2023-08-01 00:00:00	Fatal	0	1.666667	1.6666666 67	10.00
2023-08-04 00:00:00	Serious	0	0	0	03.20
2023-08-06 00:00:00	Minor	0	0	0	01.10
2023-08-17 00:00:00	Serious	0	0	0	04.00
2023-08-30 00:00:00	Minor	0	0	0	06.05
2023-09-06 00:00:00	Fatal	0.584795	0	0.5847953 22	11.30
2023-09-11 00:00:00	Serious	0	0	0	06.25
2023-09-15 00:00:00	Serious	0	0	0	07.00
2023-09-20 00:00:00	Fatal	1.169591	1.666667	2.8362573 1	07.20
2023-09-23 00:00:00	Serious	0	0	0	07.30
2023-09-24 00:00:00	Fatal	0.584795	0	0.5847953 22	06.20
2023-09-26 00:00:00	Fatal	0	1.666667	1.6666666 67	06.30
2023-09-29 00:00:00	Serious	0	0	0	04.30
2025-10-01 00:00:00	Minor	0	0	0	03.20
2023-10-06 00:00:00	Serious	0	0	0	06.00
2023-10-07 00:00:00	Serious	0	0	0	09.00
2023-10-09 00:00:00	Fatal	0	1.666667	1.6666666 67	10.00
2023-10-12 00:00:00	Serious	0	0	0	06.00
2023-10-14 00:00:00	Minor	0	0	0	01.15
2023-10-14 00:00:00	Minor	0	0	0	05.45

2023-10-26 00:00:00	Fatal	0.584795	0	0.584795322	06.00
2023-10-29 00:00:00	Fatal	0.584795	0	0.584795322	05.00
2023-11-02 00:00:00	Minor	0	0	0	06.30
2023-11-05 00:00:00	Fatal	0	3.333333	3.333333333	03.00
2023-11-05 00:00:00	Fatal	0	1.666667	1.666666667	04.10
2023-11-14 00:00:00	Serious	0	0	0	02.10
2023-11-19 00:00:00	Fatal	0.584795	0	0.584795322	00.00
2023-11-19 00:00:00	Serious	0	0	0	08.30
2023-11-23 00:00:00	Minor	0	0	0	03.48
2023-11-26 00:00:00	Serious	0	0	0	06.52
2023-11-29 00:00:00	Serious	0	0	0	09.00
2023-11-07 00:00:00	Fatal	2.339181	5	7.339181287	01.20
2023-12-15 00:00:00	Serious	0	0	0	10.00
2023-12-15 00:00:00	Fatal	0.584795	0	0.584795322	11.00
2023-12-17 00:00:00	Serious	0	0	0	00.35
2023-12-17 00:00:00	Serious	0	0	0	06.46
2023-12-24 00:00:00	Serious	0	0	0	03.36
2023-12-25 00:00:00	Serious	0	0	0	01.25
2023-12-27 00:00:00	Minor	0	0	0	02.30
2023-12-29 00:00:00	Fatal	0.584795	1.666667	2.251461988	04.20
2023-12-29 00:00:00	Serious	0	0	0	07.03
2024-02-01	Serious	0	0	0	05.18
2024-04-01	Minor	0	0	0	01.15
2024-06-01	Minor	0	0	0	00.10
2024-10-01	Fatal	0	1.666667	1.666666667	03.10

2024-01-28	Fatal	0.584795	0	0.584795322	02.20
2024-01-29	Minor	0	0	0	01.00
2024-03-01	Minor	0	0	0	06.10
2024-06-01	Serious	0	0	0	02.50
2024-07-01	Fatal	0.584795	0	0.584795322	04.50
2024-08-01	Minor	0	0	0	06.30
2024-01-17	Fatal	0.584795	0	0.584795322	03.00
2024-01-24	Minor	0	0	0	03.35
2024-01-26	Minor	0	0	0	05.05
2024-02-08	Minor	0.584795	0	0.584795322	00.30
2024-02-27	Fatal	0.584795	0	0.584795322	05.25
2024-02-07	Minor	0	0	0	02.25
2024-02-10	Fatal	0.584795	0	0.584795322	07.30
2024-05-03	Minor	0	0	0	03.00
2024-07-03	Minor	0	0	0	07.16
2024-12-03	Minor	0	0	0	04.16
2024-03-26	Minor	0	0	0	08.30
2024-11-03	Minor	0	0	0	06.30
2024-02-03	Fatal	0.584795	0	0.584795322	06.00
2024-08-03	Serious	0	0	0	02.00
2024-03-28	Fatal	0.584795	0	0.584795322	05.30
2024-04-26	Serious	0	0	0	04.20
2024-12-04	Serious	0	0	0	00.40
2024-04-04	Minor	0	0	0	00.20
2024-04-19	Fatal	2.923977	0	2.923976608	04.10
2024-01-04	Fatal	0.584795	1.666667	2.251461988	08.15
2024-05-18	Fatal	1.754386	3.333333	5.087719298	04.15
2024-07-06	Minor	0	0	0	00.00
2024-06-20	Minor	0	0	0	02.20
2024-05-06	Serious	0	0	0	03.50
2024-09-06	Fatal	0.584795	0	0.584795322	07.50
2024-06-30	Minor	0	0	0	06.00
2024-06-21	Fatal	0.584795	1.666667	2.251461988	06.20

2024-02-07	Serious	0	0	0	04.20
2024-04-07	Minor	0	0	0	00.00
2024-08-07	Serious	0	0	0	01.40
2024-07-21	Minor	0	0	0	02.00
2024-07-24	Minor	0	0	0	01.05
2024-07-17	Serious	0	0	0	03.25
2024-07-30	Serious	0	0	0	04.20
				0.5847953	
2024-07-07	Fatal	0.584795	0	22	01.03
2024-08-26	Minor	0	0	0	01.18
2024-08-31	Minor	0	0	0	01.15
2024-08-25	Serious	0	0	0	03.30
2024-08-28	Serious	0	0	0	02.00
2024-08-29	Serious	0	0	0	05.10
2024-09-13	Minor	0	0	0	04.05
				0.5847953	
2024-01-10	Fatal	0.584795	0	22	01.05
2024-10-24	Minor	0	0	0	02.30
2024-10-25	Serious	0	0	0	02.20
2024-10-28	Serious	0	0	0	01.20
2024-10-28	Serious	0	0	0	07.00
2024-03-10	Minor	0	0	0	01.35
				0.5847953	
2024-10-19	Fatal	0.584795	0	22	03.35
				2.8362573	
2024-10-31	Fatal	1.169591	1.666667	1	04.20
2024-12-10	Serious	0	0	0	01.30
2024-11-19	Serious	0	0	0	01.10
2024-11-26	Minor	0	0	0	07.30
				7.3391812	
2024-05-11	Fatal	2.339181	5	87	03.30
2024-11-24	Minor	0	0	0	04.15
				0.5847953	
2024-11-25	Minor	0.584795	0	22	04.30
2024-11-17	Minor	0	0	0	02.05
2024-11-24	Minor	0	0	0	08.15
2024-11-12	Serious	0	0	0	08.15
2024-10-28	Serious	0	0	0	08.28
2024-04-12	Minor	0	0	0	04.13
2024-12-15	Serious	0	0	0	05.10
				1.1695906	
2024-12-30	Fatal	1.169591	0	43	06.10
2024-12-15	Minor	0	0	0	na