

**ON THE STUDY OF OPTION PRICING FOCUS ON CALL OPTION**

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A PROJECT WORK SUBMITTED TO THE DEPARTMENT OF  
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## **UNDERTAKING**

This project was carried out by me, EWAH ONOSEREMEN ANTHONY with the MATRICULATION NUMBER PSC1909111. I have neither copied nor duplicated the work of any author(s). All work used have been duly cited and acknowledged.

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**CERTIFICATION**

This is to certify that this work done was carried out BY EWAH ONOSEREMEN ANTHONY PSC1909111 of the Department of Mathematics, Faculty of Physical Science, University of Benin, Benin City.

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DATE.....

## **DEDICATION**

I dedicate this work to Almighty God. Also, to my family and friends, and to everyone that supported during the course of this project work and my entire undergraduate program.

## **ACKNOWLEDGEMENT**

Firstly, I acknowledge and appreciate God Almighty for blessing me with strength, resources and wisdom to carry out this project work successfully. Also, I thank my Grandfather Sir S.O.Z. Amure and my Parents (Mr.Ewah Christopher and Mrs. Ewah Omoruwa) for their unwavering support mentally, emotionally and financially during this project.

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## ABSTRACT

This study explores the methodology underlying the pricing of call options using continuous-time models. Beginning with the Wiener process as a foundational model, the research extends into stochastic calculus and the derivation of the Black-Scholes equation. The analysis delves into the fundamental properties of Brownian motion, the normal distribution, and stochastic differential equations to establish a rigorous mathematical framework for asset price movements.

A key focus is placed on the arbitrage argument, which ensures that financial markets remain free from riskless profit opportunities. The no-arbitrage condition is then used to derive the Black-Scholes partial differential equation, which governs the pricing of options. Through a series of transformations, the equation is reduced to the heat equation, allowing for an analytical solution to be obtained. Finally, the study applies this methodology to derive explicit pricing formulas for European call and put options, highlighting the impact of volatility, risk-free interest rates, and time to maturity on option values.

By formalizing the theoretical framework with stochastic calculus and arbitrage pricing theory, this research provides a robust foundation for the application of call option models in financial engineering.

# CHAPTER ONE

## INTRODUCTION

### 1.1. Introduction

Financial engineering is a multidisciplinary field that applies mathematical, statistical, and computational techniques to solve complex financial problems. It plays a crucial role in the development of innovative financial instruments and strategies that optimize risk and return in various financial markets. One of the most significant areas in financial engineering is derivatives, particularly options.

A call option is a financial contract that grants the holder the right, but not the obligation, to buy an underlying asset at a predetermined price (strike price) before or at the expiration date. Call options are widely used for speculation, hedging, and portfolio management, making them a cornerstone of modern financial markets. The study of call options involves pricing models, risk assessment, and trading strategies, all of which rely on advanced mathematical frameworks.

This research explores the fundamental aspects of call options, focusing on their application in financial engineering. It will examine key pricing models, such as the Black-Scholes model, hedging techniques, risk management strategies, and the use of exotic call options in complex financial environments. By analyzing these

aspects, this study aims to provide a deeper understanding of how call options contribute to financial decision-making and risk mitigation.

## **1.2 BACKGROUND OF THE STUDY**

Options have been an essential part of financial markets for centuries, with modern option pricing theories emerging in the 20th century. The foundational work of Black, Scholes, and Merton in the early 1970s revolutionized option pricing, leading to the development of the Black-Scholes-Merton model. This model provides a mathematical framework for determining the fair value of European options and serves as the basis for more advanced pricing techniques.

Financial engineers leverage various mathematical tools, including stochastic calculus, partial differential equations, and numerical methods, to develop robust pricing and risk management models. The evolution of call options has also been influenced by advancements in computational finance, allowing for real-time valuation and more accurate risk assessment.

Call options are integral to financial risk management strategies. Investors and corporations use them for hedging against adverse price movements, optimizing portfolio returns, and engaging in arbitrage opportunities. The ability to price call options accurately and manage their risks effectively is essential for traders, portfolio managers, and financial analysts.

The increasing complexity of financial markets has led to the development of exotic call options, which have unique payoff structures and risk characteristics. These options require sophisticated pricing models and risk assessment techniques, further demonstrating the critical role of financial engineering in modern finance.

### **1.3 AIM AND OBJECTIVES OF THE STUDY**

The primary aim of this study is to examine the application of call options in financial engineering, particularly in pricing, hedging, and risk management. The specific objectives of this research are:

- To explore the fundamental principles of call options and their role in financial markets.
- To analyze the mathematical models used for pricing call options, with a focus on the Black-Scholes model and its extensions.
- To examine hedging strategies that incorporate call options for managing financial risk.
- To evaluate the effectiveness of call options in risk management and portfolio optimization.

To study the application of exotic call options in financial engineering and their impact on trading strategies.

To provide insights into the practical implementation of call option-based models in real-world financial scenarios.

#### **1.4. SCOPE OF THE STUDY**

This study focuses on the theoretical and practical applications of call options in financial engineering. It covers the following key areas:

The mathematical foundations of call option pricing, including stochastic calculus and risk-neutral valuation.

Traditional and advanced pricing models, such as the Black-Scholes model, binomial tree model, and Monte Carlo simulations.

Hedging techniques using call options, including delta hedging, gamma hedging, and portfolio insurance.

The role of call options in risk management, including their application in volatility trading and structured financial products.

The impact of exotic call options, such as Asian, barrier, and compound options, on financial engineering strategies.

Real-world case studies demonstrating the effectiveness of call options in mitigating financial risks and enhancing investment strategies.

This research aims to bridge the gap between theoretical models and practical implementation, offering valuable insights for financial professionals, investors, and academics interested in derivatives and financial engineering.

## **1.5. ORGANIZATION OF THE STUDY**

This research is structured into five chapters, each building upon the previous to provide a comprehensive analysis of call options in financial engineering:

**Chapter 1: Introduction** – This chapter provides an overview of the study, including the background, research objectives, scope, and significance of call options in financial engineering. It also defines key terms relevant to the research.

**Chapter 2: Literature Review** – This chapter presents an in-depth review of existing research on call options, covering theoretical foundations, pricing models, hedging strategies, risk management applications, and the role of exotic options in financial markets.

**Chapter 3: Methodology** – This chapter outlines the research methods employed in the study. It discusses the analytical frameworks, quantitative techniques, and financial models used to explore the pricing, hedging, and risk management of call options.

**Chapter 4: Application and Analysis** – This chapter focuses on the practical implementation of call options in financial engineering. It examines real-world

case studies, evaluates the effectiveness of different pricing models, and analyzes the impact of call options on portfolio management and trading strategies.

**Chapter 5: Conclusion and Recommendations** – The final chapter summarizes key findings, discusses their implications for financial professionals and researchers, and provides recommendations for future studies on call options in financial engineering.

## **1.6 DEFINITION OF BASIC TERMS**

**Financial Engineering:** Financial engineering refers to the application of mathematical and quantitative techniques to solve financial problems, design financial products, and optimize financial strategies. It involves the use of tools from mathematics, statistics, economics, and computer science to analyze and manage financial risks, create innovative financial products, and improve financial decision-making processes.

**Call Option:** A financial contract that gives the buyer the right, but not the obligation, to buy an underlying asset at a specified strike price before or on the expiration date.

**Black-Scholes Model:** A widely used mathematical model for pricing European options based on factors such as asset price, strike price, time to expiration, volatility, and risk-free interest rate.

**Hedging:** A risk management strategy that involves using financial instruments, such as options, to offset potential losses in an investment.

**Exotic Options:** Non-standard options with complex payoff structures, including Asian options, barrier options, and compound options.

**Delta Hedging:** A strategy used to minimize the directional risk of an option position by adjusting the number of underlying assets held.

**Volatility Trading:** A trading strategy that focuses on the fluctuations in asset prices rather than their absolute levels

## **CHAPTER TWO**

### **LITERATURE REVIEW**

#### **2.1 Introduction**

The study of call options has been a significant area of research in financial engineering, with extensive literature covering pricing models, risk management, trading strategies, and market applications. This chapter reviews the foundational theories, key models, and empirical studies on call options, focusing on their development, application, and impact on financial markets. The review encompasses option pricing models, the role of call options in risk management, hedging techniques, and the evolution of exotic options.

#### **2.2 THEORETICAL FOUNDATIONS OF CALL OPTIONS**

Call options are financial derivatives granting the holder the right, but not the obligation, to buy an underlying asset at a fixed price before or at expiration. The value of a call option is influenced by the underlying asset price, strike price, volatility, interest rates, and time to expiration.

##### **2.2.1 Early Developments in Option Theory**

The foundation of modern option pricing theory dates back to Bachelier (1900), who developed a model based on Brownian motion to describe stock price

movements. However, his model lacked practical applicability due to the assumption of normally distributed stock prices.

### 2.2.2 The Black-Scholes Model (1973)

The most influential breakthrough in option pricing came with the Black-Scholes-Merton model, which provided a closed-form solution for European call option prices. The Black-Scholes equation is:

$$C = S_0N(d_1) - Ke^{-rt}N(d_2)$$

where:

C = Call option price

$S_0$  = Current stock price

K = Strike price

r = Risk-free interest rate

t = Time to expiration

$\sigma$  = Volatility of the underlying asset

N(d) = Cumulative normal distribution function

Despite its success, the Black-Scholes model assumes constant volatility and no transaction costs, which limits its real-world accuracy.

### **2.2.3 Extensions of the Black-Scholes Model**

To address these limitations, several extensions have been proposed:

Heston Model (1993): Introduces stochastic volatility to improve pricing accuracy.

Jump-Diffusion Models (Merton, 1976): Incorporates sudden price jumps in the asset.

Monte Carlo Simulations: Used for pricing complex derivatives that lack closed-form solutions.

These models refine option pricing by accommodating market imperfections and empirical anomalies.

## **2.3 Call Options and Risk Management**

Call options play a crucial role in risk management, allowing investors to hedge against adverse price movements while maintaining upside potential.

### **2.3.1 Portfolio Hedging with Call Options**

Institutional investors and fund managers use call options to hedge stock portfolios.

A protective call strategy involves holding a short position in a stock while buying a call option to cap potential losses.

### **2.3.2 Delta Hedging and Risk Neutrality**

Delta hedging, a fundamental risk management technique, involves adjusting the portfolio based on the option's delta:

$$\Delta = N(d_1)$$

Traders rebalance their positions to maintain a delta-neutral portfolio, minimizing exposure to small price changes.

### **2.3.3 Call Options in Corporate Finance**

Firms use call options in:

Convertible bonds: Allowing debt conversion into equity at favorable conditions.

Executive stock options: Incentivizing management performance through long-term value creation.

These applications highlight the importance of call options in corporate financial strategy.

## **2.4 CALL OPTIONS IN TRADING STRATEGIES**

### **2.4.1 Speculative Trading with Call Options**

Call options provide leverage, enabling traders to amplify returns with limited capital. Common speculative strategies include:

Long Call: Buying call options in anticipation of a price increase.

Straddle: Buying both a call and a put to profit from volatility.

### **2.4.2 Arbitrage Strategies**

Arbitrageurs exploit pricing inefficiencies in call options, such as:

Put-Call Parity Arbitrage: Based on the relationship:

$$C - P = S - Ke^{-rt}$$

Volatility Arbitrage: Trading based on mispriced implied volatility.

These strategies illustrate how traders use call options for profit maximization.

## **2.5 EXOTIC CALL OPTIONS AND STRUCTURED PRODUCTS**

Exotic call options are non-standard derivatives with unique features. Examples include:

Barrier Options: Activated or deactivated at a specific price level

Lookback Options: Allowing exercise at the most favorable price.

Asian Options: Based on the average price over time.

These options are used in structured financial products, risk management, and corporate hedging strategies.

## **2.6 EMPIRICAL STUDIES ON CALL OPTIONS**

Empirical research has examined the pricing efficiency, volatility behavior, and market impact of call options:

Black-Scholes Performance: Studies confirm its accuracy for liquid markets but highlight mispricing in volatile conditions.

Implied Volatility Smiles: Observed deviations from the Black-Scholes assumption of constant volatility.

High-Frequency Trading (HFT): Call options are actively traded by algorithmic trading systems, influencing market liquidity.

These findings reinforce the role of call options in modern financial markets.

## **CHAPTER THREE**

### **METHODOLOGY**

#### **3.1 INTRODUCTION**

In financial markets, accurately modeling asset price movements is crucial for risk assessment, trading strategies, and derivative pricing. Traditional binomial models, while useful, become impractical for continuous trading and sophisticated derivative evaluation. To overcome these limitations, continuous-time models provide a more precise framework. This chapter explores fundamental continuous-time financial models, starting with the Wiener process, a key component in stochastic modeling. The chapter then builds upon this foundation to discuss the normal distribution, stochastic calculus, arbitrage arguments, and the derivation of the Black-Scholes equation. These concepts collectively form the mathematical backbone of modern financial analysis and derivative pricing.

#### **3.2 CONTINUOUS TIME MODELS**

The binomial model for asset price movements has some drawbacks, the most significant being its impracticality for continuous trading and derivative evaluation. To address this, we use continuous-time models, treating asset prices as continuous variables. Begin by looking at Continuous time models of the financial markets by

introducing an elementary “Wiener Model” and exploring a few of its characteristics.

### **3.3 THE WIENER MODEL**

The simplest continuous time model for an asset price is the basic lognormal model, which can be used to reason about movements of actual assets. The assessed price movements are limited to a deterministic trend or drift and a random component of the asset price that is called the Wiener model. The strength of the random component, like motion in a fluid, is scaled by this process. The random motions are uncorrelated with past periods and there is no memory. More advanced models do consider correlations between movements in time series, but these really only become necessary when the significant calculations require higher order smoothing over long-time horizons.

However, we are not going to do anything with only words, so it is now explicit that in many calculations we can obtain closed-form results. Analogous to the stochastic processes that we studied in the discrete time case, we can define a continuous stochastic process that evolves in time entering the asset price,  $S$ , as a family of a random variables indexed by a continuous time parameter “ $t$ ”. In the

elementary Wiener model the assets price  $S_t$  is a stochastic process that evolves in time based on the formula.

$$S_t = S_0 e^{\mu t} e^{\sigma W_t - \frac{1}{2}\sigma^2 t}.$$

Here  $S_0$ ,  $\mu$  and  $\sigma$  are constants.

$S_0$  is the initial stock price

$\mu$  is the drift

$\sigma$  is the volatility

$W_t$  is the source of randomness or ‘noise in the asset price movement.

The properties and the Wiener process  $W_t$  is a stochastic process beginning at time 0 that satisfies;

- The initial value is zero:  $W_0 = 0$ .
- At each time  $t$ ,  $W_t$  is a normally distributed random variable with mean 0 and variance  $t$ .
- $W_t$  has independent increments. That is, for  $a < b < c < d$ , the difference between the values of  $W_t$  at two times, such as  $W_b - W_a$  and  $W_d - W_c$ , are independent random variables.

If a stochastic process has these three properties, then we say that it is a Wiener process or, equivalently, it is a simple Brownian motion. The Wiener process is a very rich object of mathematical interest, with numerous important applications in finance.

Note that at any time  $W_t$  has an equal likelihood of being positive or negative, but the spread (variance) increases over time. If we add to the Wiener process a constant deterministic drift of the form  $\mu t$ , then the resulting process will have a nonzero expectation value.

### 3.4 THE NORMAL DISTRIBUTION

Before we try and do too much with the Wiener process it is probably worth spending some time recalling some facts about the normal distribution. The probability distribution of a stochastic process at any time  $t$  is called the marginal distribution, so we say that a Wiener process has a normal marginal distribution. We refer to any random variable that is normally distributed with mean  $m$  and variance  $V$  as an  $N(m, V)$  random variable. Hence if  $X$  is an  $N(m, V)$  random variable we have:

$$Prob[a < X < b] = \frac{1}{\sqrt{2\pi V}} \int_a^b \exp\left[-\frac{(x - m)^2}{2V}\right] dx.$$

Since the Wiener process  $W_t$  has an  $N(0, t)$  marginal distribution, it is characterized by the following probability law:

$$Prob[a < W_t < b] = \frac{1}{\sqrt{2\pi t}} \int_a^b \exp\left[-\frac{x^2}{2t}\right] dx.$$

How about the difference between the Wiener process at two different times,  $W_{a+b}$  –  $W_b$ ?

In the definition of  $W_t$ , we required that this be an independent random variable. In fact, it turns out to be a  $N(0, t)$  random variable, which will be very important for some of our calculations.

**Example 1.0;**

Verify that  $W_{a+b} - W_a$  is an  $N(0, t)$  random variable. A useful formula to define is the cumulative normal distribution function  $N(x)$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left[-\frac{1}{2}\xi^2\right] d\xi.$$

Later, we will be able to express the price of call and put options in terms of  $N(x)$ .

For now, we can use the cumulative distribution in order to describe a Wiener process  $W_t$ .

$$Prob [W_t < b] = \frac{1}{\sqrt{2\pi t}} \int_{\xi=-\infty}^{\xi=b} e^{-\frac{\xi^2}{2t}} d\xi$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi t}} \int_{\eta=-\infty}^{\eta=\frac{b}{\sqrt{t}}} e^{-\frac{1}{2}\eta^2} d\eta \\
&= N\left(\frac{b}{\sqrt{t}}\right)
\end{aligned}$$

Let  $\eta = \left(\frac{\xi}{\sqrt{t}}\right)$

$$d\eta = \frac{1}{\sqrt{t}}$$

$$\sqrt{t} d\eta = d\xi.$$

where we made the substitution  $\xi = \eta\sqrt{t}$  in going from the first to second line.

We are now in a position where we can, at least heuristically, understand the drift  $\mu$  and volatility  $\sigma$  of the Wiener model for an asset price process, given by equation (Example 1.0). In order to interpret the drift, we need the following result.

### **Lemma 1.0**

Let  $X$  be an  $N(m, V)$  random variable. Then

$$E[e^{ax}] = e^{aM + \frac{1}{2}a^2V}.$$

## Proof

The density function  $\rho_{m,V}(x)$  for an  $N(m,V)$  random variable is

$$\rho_{m,V}(x) = \frac{1}{\sqrt{2\pi V}} \exp\left(-\frac{(x-m)^2}{2V}\right).$$

Thus, the expected value is

$$E[\exp(aX)] = \int_{x=-\infty}^{x=\infty} \frac{1}{\sqrt{2\pi V}} \exp\left[ax - \frac{(x-m)^2}{2V}\right] dx.$$

A straightforward calculation that involves “completing the square” then gives the desired results. An immediate corollary of this result is that

$$E[\exp(\sigma W_t)] = \exp\left(\frac{1}{2}\sigma^2 t\right)$$

From which we can conclude that the expected value of the Wiener process is

$$\begin{aligned} E[S_t] &= E\left[S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t\right)\right] \\ &= S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t\right) E[\exp(\sigma W_t)] \\ &= S_0 \exp(\mu t). \end{aligned}$$

Since  $E[S_t]/S_0 = e^{\mu t}$ , we say that  $\mu$  is the rate of return on an investment in the asset  $S_t$  with initial price  $S_0$ . Thus, the drift parameter controls the expected value of the asset price in the future. However, rather surprisingly, it will turn out that the call

and put option prices that we will derive are independent of the value of the drift, that is, they do not depend on how we expect the asset price will move.

The volatility, on the other hand, measures the "response" of  $S_t$  to the movements in the Wiener process  $W_t$ . That is, the larger the volatility, the more randomness that is introduced into the model. It is the volatility which will play the most important role in option pricing because it controls the randomness that we need to try and eliminate.

### 3.5 STOCHASTIC CALCULUS

Now in a position to begin considering trading in continuous time and to examine the situation where asset price motions are driven by a Wiener process rather than by a discrete time model. The idea is that at each instant we can look at the Wiener process  $W_t$  and study its change  $dW_t$  at that instant, which intuitively can be thought of as moving slightly up or down with equal probability.

For the asset price  $S_t$ , we can hypothesize that the change in its value is given by a so-called 'stochastic differential equation' of the form

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$

This says that the infinitesimal change  $dS_t$  in the asset price at time  $t$ , as a percentage of the value  $S_t$ , is given by a drift term  $\mu_t dt$  and a 'fluctuation' or small

movement upwards or downwards given by  $\sigma_t dW_t$ . We call  $\mu_t$  the drift at time  $t$  and  $\sigma_t$  the volatility at time  $t$ . For elementary applications we take  $\mu$  and  $\sigma$  to be constant. This is often called the 'Black-Scholes' world.

The solution of equation is given by the Wiener model that was looked earlier;

$$S_t = S_0 e^{\mu t + \sigma W_t - \frac{1}{2} \sigma^2 t}.$$

As shown later, the main tool that we require is the so-called stochastic calculus or Itô calculus. A closely related idea to stochastic calculus is the stochastic integral.

The integral of a process with respect to a Brownian motion is defined to be

$$\int_a^b X_t dW_t = \lim_{N \rightarrow \infty} \sum_{i=0}^N X_{t_i} (W_{t_{i+1}} - W_{t_i})$$

where the  $t_i$  are some partition of the interval  $[a, b]$ . The stochastic integral has a natural interpretation in terms of trading strategies. If  $S_t$  is an asset price, then a trading strategy  $\phi_u$  is a random process that says what the holding in the asset is at time  $u$ . Then

$$V_t = V_0 + \int_0^t \phi_u dS_u$$

represents the value at time  $t$  of an investment portfolio based on holdings in the given asset, where  $V_0$  is the initial value of the portfolio, and  $\phi_u$  is the trading strategy. The term  $\phi_u dS_u$  represents the infinitesimal gain (or loss) the portfolio

makes at time  $u$  when the asset moves up (or down) by the amount  $dS_u$ . More fully, we can write

$$V_t = V_0 + \int_0^t \phi_u S_u \mu_u du + \int_0^t \phi_u S_u \sigma_u dW_u$$

Where we have substituted the expression for  $dS_u$  from equation (12.1).

The trading strategy  $\phi_u$  can, in principle, be deterministic, but generally is itself also a random process, which depends on how events have played out so far. A hedging strategy that allows us to eliminate risk and generate a guaranteed return is Itô's Lemma, which says that if the random process  $X_t$  satisfies

$$dX_t = \mu_t dt + \sigma_t dW_t$$

and if  $f(X_t)$  has continuous second derivatives as a function  $f(x)$ , then the process  $f_t = f(X_t)$  satisfies

$$df(X_t) = \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2}{\partial X_t^2} (dX_t)^2$$

where  $(dX_t)^2$  is interpreted according to the rules

$$(dt)^2 = 0. \quad dt dW_t = 0 \quad \text{and} \quad (dW_t)^2 = dt$$

Thus  $(dX_t)^2 = \sigma_t^2 dt$ . and we have

$$df(X_t) = \left( \mu_t \frac{\partial f}{\partial X_t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial X_t^2} \right) dt + \sigma_t \frac{\partial f}{\partial X_t} dW_t$$

Hedging strategy allows us to eliminate risk and generate a guaranteed return, this result has numerous applications.

**Example;**

suppose that  $X_t$  is a process that satisfies the stochastic differential equation ,what is the differential of  $f(X_t)=\exp X_t$

According to the Ito rule we have ;

$$\begin{aligned} de^{X_1} &= \left( \mu_t \frac{\partial e^{X_1}}{\partial X_t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 e^{X_1}}{\partial X_t^2} \right) dt + \sigma_t \frac{\partial e^{X_t}}{\partial X_t} dW_t \\ &= e^{X_1} \left[ \left( \mu_t + \frac{1}{2} \sigma_t^2 \right) dt + \sigma_1 dW_t \right] \end{aligned}$$

**3.6. ARBITRAGE ARGUMENT**

The Black-Scholes formula for the value of a call option, we shall derive the Black-Scholes partial differential equation, which is valid for any derivative. However, it is only in specific simple cases, such as that of a call or put option with constant 'market parameters,' where an explicit solution can be found.

**3.6.1. Derivation of the No-Arbitrage Condition**

Consider a financial market that consists of one basic asset and a money market account. Suppose that the underlying asset has a stochastic price process  $S_t$  with dynamics

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

where  $\mu_t$  is the drift and  $\sigma_t$  is the volatility. Later we shall specialize to the case where  $\mu_t$  and  $\sigma_t$  are constant, but for the moment we shall allow these processes to be fairly general, depending on the 'history' of  $W_t$  between 0 and  $t$ . The money market account or bond process  $B_t$  satisfies the stochastic differential equation

$$dB_t = r_t B_t dt$$

where  $r_t$  is the instantaneous risk-free interest rate or 'short rate.' As in the case of the drift and volatility, for now we will only assume that the short rate is adapted to the Brownian motion  $W_t$ , but eventually we will take it to be constant.

In addition to the underlying asset and the bond, we also have an option on  $S_t$ , with price  $C_t$  at time  $t$ . In fact, for the moment we may suppose that  $C_t$  is a general European-style derivative. It is therefore entirely specified by a maturity  $T$  and a payoff function  $C_T$ . Eventually; we shall calculate the initial price for the payout.

$$C(S_T) = \max(S_T - K, 0,$$

that is, a call option

What is the stochastic differential equation satisfied by  $C_t$ ? We shall assume that  $C_t$  is driven by the same randomness that affects  $S_t$ , that is the Wiener process  $W_t$ .

Hence the derivative price has the dynamics:

$$dC_t = \mu_t^C C_t dt + \sigma_t^C C_t dW_t$$

where  $\mu_t^C$  and  $\sigma_t^C$  are respectively the drift and volatility of the derivative price.

Note that the stochastic process  $C_t$  does not really need to be a derivative at all.

Any portfolio with a guaranteed return immediately implies that an arbitrage argument is waiting to happen. Suppose that an arbitrageur wants to make a risk-free profit by investing in the asset and derivative. Let  $\phi_t$  be the trading strategy for the underlying asset, that is,  $\phi_t$  tells us how much of the asset that we own at time  $t$ . It is the value of the trading strategy that will eventually be determined by the no arbitrage argument.

A self-financing trading strategy is a trading strategy that has no external cash flow. That is, any changes in the value of the portfolio are entirely due to changes in the value of the underlying assets and not due to money being put in or taken out of the portfolio in order to fund asset sales or purchases.

For example, if we have a single asset with price process  $S_t$  and trading strategy  $\phi_t$  then the portfolio value is  $V_t = \phi_t S_t$ , and the change in the value of the portfolio over a short time interval  $dt$  is:

$$dV_t = \phi_t dS_t + S_t d\phi_t$$

The trading strategy is self-financing if  $S_t d\phi_t = 0$ , because this is an external cash

flow, whereas  $\phi_t dS_t$  is caused by a change in the value of the asset price. A trading strategy self-financing is  $\phi_t = t$  which continuously buys stock. The value of the portfolio is  $V_t = tS_t$ , and hence the change in its value over a short time interval is  $dV_t = S_t dt + t dS_t$ , which is not equal to  $\phi_t dS_t$ . The amount  $S_t dt$  is an external cash-flow that must be added to the portfolio in order to fund the continual asset purchases.

In that case, the value of a portfolio with a trading strategy  $(\phi_t, \psi_t)$  is  $V_t = \phi_t S_t + \psi_t B_t$ . The change in a self-financing strategy must be entirely due to the change in the asset and bond prices, that is,  $dV_t = \phi_t dS_t + \psi_t dB_t$ . Unlike the single asset case, we no longer have to require that the change in the trading strategy vanishes, that is, we can buy and sell assets and bonds, but we do require that the additional terms that arise from these sales and purchases cancel each other out in such a way that no external cash-flow is required. For example, if we want to increase our asset holdings  $\phi_t$ , then the money to fund this purchase must come from a correspondingly shorter position  $\psi_t$  in the bond. We can make the trading strategy self-financing by adding a bond position

$$\psi_t = - \int_u^t \frac{S_u}{B_u} du$$

which simply borrows the money from the bond market to pay for the purchases in the asset market, which means that  $S_t d\phi_t + B_t d\psi_t = 0$ . Self-financing strategies are

very important for arbitrage arguments because we cannot require a portfolio that starts with no net position to end with it simply not net value if there is an external cash-flow which adds or subtracts money from the position.

As in our previous examples, the arbitrageur begins at time  $t$  with no money. He then buys one option for  $C_t$  dollars and assumes a short position in  $\phi_t$  units of the underlying asset. Note that we assume that the asset can be bought and sold in any quantity. Setting up this portfolio costs  $V_t = C_t - \phi_t S_t$  dollars, which must be funded via a bank loan. Interest accumulates continuously on this loan at the rate  $r_t$ , so that after a small time interval  $dt$ , the arbitrageur owes an additional  $r_t V_t dt$  dollars to the bank.

But what is the value of  $\phi_t$ , that is, how much of the asset are we shorting? ‘arbitrage strategy’  $\phi_t$  is chosen so that the risk or randomness in holding the option exactly cancels the risk in holding the asset. So what is the return on the arbitrageur’s portfolio? Well, the value of his position at any time  $t$  is

$$V_t = C_t - \phi_t S_t.$$

Assuming that we do not adjust the value of  $\phi_t$ , that is, we neither buy nor sell any underlying assets, then the change in the value of the portfolio  $V_t$  over the short time interval  $[t, t+dt]$  is

$$\begin{aligned}
dV_t &= dC_t - \phi_t dS_t \\
&= C_t(\mu_t^C dt + \sigma_t^C dW_t) - \phi_t S_t(\mu_t dt + \sigma_1 dW_t) \\
&= (\mu_t^C C_t - \phi_t \mu_t S_t)dt + (\sigma_t^C C_t - \phi_t \sigma_t S_t)dW_t
\end{aligned}$$

where we have assumed that the asset and option prices obey equations (13.1) and (13.3) respectively. We can now fix  $\phi_t$  by setting the coefficient of  $dW_t$  to zero.

This ensures that the arbitrageur's asset and option portfolio offers a definite rate of return over the small time interval  $dt$ . Hence;

$$\phi_t = \frac{\sigma_t^C C_t}{\sigma_t S_t}$$

and we call this value of  $\phi_t$  the arbitrage strategy. We can then calculate the change in the value of the position when the arbitrage strategy is used:

$$\begin{aligned}
dV_t &= (\mu_t^C C_t - \phi_t \mu S_t)dt \\
&= \left( \mu_t^C - \frac{\sigma_t^C \mu_t}{\sigma_t} \right)
\end{aligned}$$

can make a risk free profit if the drift yields a monetary gain greater than the interest payment on the loan. This follows because at time  $t+dt$ , the value of the arbitrageur's asset and option portfolio is  $V_t + dV_t$ , while the amount owed to the bank is  $V_t + r_t V_t dt$ . Hence his net position is the risk-free amount  $dV_t - r_t V_t dt$ .

$$dV_t = r_t V_t dt$$

This implies

that

$$\begin{aligned} dV_t &= r_t(C_t - \phi_t S_t)dt \\ &= r_t \left(1 - \frac{\sigma_t^c}{\sigma_t}\right) C_t dt \\ \mu_t^c \left(-\frac{\sigma_t^c \mu_t}{\sigma_t}\right) &= r_t \left(1 - \frac{\sigma_t^c}{\sigma_t}\right) \end{aligned}$$

After some rearrangement we obtain the formula

$$\frac{(\mu_t^c - r_t)}{\sigma_t^c} = \frac{(\mu_t - r_t)}{\sigma_t}$$

This is the general relation between  $\mu_t^c$ ,  $\mu_t$ ,  $\sigma_t^c$ ,  $\sigma_t$ , and  $r_t$  that is required if there is to be no arbitrage between the option and the underlying. We therefore call this the “**no arbitrage condition**”. Note that we have made no specific assumptions about the form of the volatility, drift, or short-rate processes, or even what kind of derivative that we are speaking of (except that its value, like that of  $S_t$ , should be determined by a knowledge of  $W_t$  between 0 and  $t$ ).

### 3.6.2 Derivation of the Black-Scholes Equation

To proceed further, we now assume that at each time  $t$  the derivative price  $C_t$  can be specified in terms of a function  $C_t = C(S_t, t)$  of just the asset price and time. This is true for a call option and many other standard types of derivatives. The big

advantage of this assumption is that it allows us to apply Ito's lemma to the derivative price.

$$\begin{aligned}
 dC_t &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} (dS_t)^2 \\
 &= \left( \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t} \mu_t S_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} S_t^2 \sigma_t^2 \right) dt + \frac{\partial C_t}{\partial S_t} S_t \sigma_t dW_t
 \end{aligned}$$

Comparing the terms from this result with the coefficients of dt and dW<sub>t</sub> in the original price process

$$dC_t = \mu_t^c C_t dt + \sigma_t^c C_t dW_t$$

We deduce that  $\mu_t^c C_t = \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t} \mu_t S_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} S_t^2 \sigma_t^2$ ,

And

$$\sigma_t^c C_t = \frac{\partial C_t}{\partial S_t} S_t \sigma_t.$$

If we then use these expressions to substitute for  $\mu_t^c$  and  $\sigma_t^c$  in the no arbitrage condition obtained earlier, we get

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t} \mu_t S_t + \frac{1}{2} \frac{\partial^2 C}{\partial S_t^2} S_t^2 \sigma_t^2 - r_t C = (\mu_t - r_t) \frac{\partial C}{\partial S_t} S_t.$$

Note that the terms involving  $\mu_t$  on the right and left miraculously cancel, and we are left with the equation

$$\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S_t^2} S_t^2 \sigma_t^2 = r_t \left( C_t - \frac{\partial C}{\partial S_t} S_t \right).$$

This is the famous Black-Scholes partial differential equation for the price of a derivative.

### 3.7. Replication Portfolios

We have seen that if an asset price  $S_t$  moves according to the process  $dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$ , and if a derivative based on the asset moves according to the process  $dC_t = \mu_t^C C_t dt + \sigma_t^C C_t dW_t$ , and if the money market account process  $B_t$  satisfies  $dB_t = r_t B_t dt$ , then to avoid arbitrage the relation  $\frac{\mu_t^C - r_t}{\sigma_t^C} = \frac{\mu_t - r_t}{\sigma_t}$  must hold.

This ensures that if we hold a portfolio  $V_t$  that is long one derivative and short  $\varphi_t$  units of the asset, and  $\varphi_t$  is chosen so that the return on the position is riskless, that is  $\varphi_t = \frac{\sigma_t^C C_t}{\sigma_t S_t}$ , then this riskless rate of return is equal to the risk-free short-rate  $r_t$ .

A portfolio composed of holdings in both the underlying asset and the money market account. Recall that going long in the money market is equivalent to investing money at the rate  $r_t$ , while going short is the same as borrowing money at the same rate.

So, how can we replicate the derivative? We follow the arbitrage strategy  $\varphi_t$ , then we obtain a fixed return on position  $V_t$ , given by  $dV_t = r_t V_t dt$ . But this rate of return is equal to the rate of return obtained by an investment in the money market account, as required by the no arbitrage argument. So instead of buying the derivative we achieve the same effect by setting up a portfolio  $V_t$  which simply takes position in the money-market account. Since we want the portfolios to have the same value at time  $t$ , we clearly set

$$\psi_t = \frac{V_t}{B_t}$$

If we hold our investment in the bond fixed, then the change in the value of this portfolio between time  $t$  and time  $t + dt$  is

$$\begin{aligned} dV_t &= \psi_t dB_t \\ &= \psi_t r_t B_t dt \\ &= r_t V_t dt \\ &= dV_t \end{aligned}$$

Thus,  $V_t$  exactly replicates  $V_t$  and the two processes must therefore be equal. But this means that

$$V_t = v_t B_t = C_t - \varphi_t S_t$$

And hence.

$$C_t = \varphi_t B_t + \psi_t S_t$$

In essence what we have done is first calculated  $\varphi_t$  using a no arbitrage argument and then simply defined  $\psi$

### 3.8. Solving the Black Scholes Equations

We now want to solve the Black-Scholes equation for a general ‘European’ payoff, assuming that the drift, volatility, and interest rates are all constant. In this case, the stochastic asset price process is

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

we verified that the solution of this stochastic differential equation is

$$S_t = S_0 e^{\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t},$$

where  $S_0$  is the initial price of the asset. If  $C(S_t, t)$  is the value of a derivative at time  $t$  which expires at time  $T$ , earlier we showed that for  $0 \leq t \leq T$  it must satisfy the Black-Scholes partial differential equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} = r \left( C - \frac{\partial C}{\partial S_t} S_t \right).$$

Our goal is to solve this equation subject to the specification of a general payoff function  $C(S_T, T) = f(S_T)$  at the expiry time  $T$ . To begin with, we note that the Black-Scholes equation has a passing similarity to the more common heat equation

$$\frac{\partial A(x, \tau)}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 A(x, T)}{\partial x^2}.$$

In fact, by a series of transformations, the Black-Scholes equation can be reduced to the heat equation. This means that we can solve for the derivative price by a two-step method. First, we show how to solve the heat equation (15.4) subject to a prescribed initial condition  $A(x, 0) = f(x)$ . Second, we explicitly demonstrate the transformations needed to convert the Black-Scholes equation to the heat equation.

### 3.8.1. Solution of the Heat Equation

Let  $W_t$  be a standard Brownian motion. If we consider a function  $f(x + W_t)$ , then from Ito's lemma we see that

$$df(x + W_r, \tau) = \frac{\partial f(x + W_r, \tau)}{\partial W_r} dW_r + \frac{1}{2} \frac{\partial^2 f(x + W_r, \tau)}{\partial W_r^2} d\tau$$

Note that we want to treat  $x$  as a parameter rather than a variable

$$f(x + W_t, \tau) = f(x) + \int_0^t \frac{\partial f(x + W_s, s)}{\partial W_s} dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 f(x + W_s, s)}{\partial W_s^2} ds$$

where we have used the fact that  $W_0 = 0$ . We then notice that differentiating  $f(x + W_t, \tau)$  with respect to  $W_t$  is the same as differentiating it with respect to  $x$ , that is

$$\frac{\partial f(x + W_t, \tau)}{\partial W_t} = \frac{\partial f(x + W_t, \tau)}{\partial x} \text{ and } \frac{\partial^2 f(x + W_t, \tau)}{\partial W_t^2} = \frac{\partial^2 f(x + W_t, \tau)}{\partial x^2}$$

This is useful because we want to derive the heat equation, which involves derivatives with respect to a real quantity  $x$ , rather than a stochastic variable like  $W_t$ .

$$f(x + W_t, \tau) = f(x) + \int_0^t \frac{\partial f(x + W_s, s)}{\partial x} dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 f(x + W_s, s)}{\partial x^2} ds$$

If we take an expectation on each side of this equation, then the stochastic integral vanishes and we obtain

$$E[f(x + W_t, \tau)] = f(x) + \frac{1}{2} \int_0^t \frac{\partial^2 E[f(x + W_s, s)]}{\partial x^2} ds$$

### 3.8.2. Reduction of the Black-Scholes Equation to the Heat Equation

Armed with this result, we return to the Black-Scholes equation and make a series of parallel transformations in order to reduce it to the heat equation. Proceeds as follows;

1. We need to reverse the direction of time, so that the terminal payout of the Black-Scholes equation becomes the initial condition of the heat equation. Set  $C(S,t)=a(S,\tau)$ , where  $\tau = T-t$  is a new time coordinate which still runs over the same interval  $[0,T]$  as  $t$ , but in the opposite direction, The derivative expiry time  $T$  is of course a constant. The time derivatives of  $C(S,t)$  and  $a(S, \tau)$  are related by

$$\frac{\partial C}{\partial t} = - \frac{\partial a}{\partial \tau}$$

While all the other derivatives remain the same.

$$\frac{\partial a}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 a}{\partial S^2} + rS \frac{\partial a}{\partial S} - r a$$

This equation now has the 'right' sign for the time derivative, and has the initial condition

$$a(S_T, 0) = C(S_T, T) = f(S_T, T)$$

2. We now want to eliminate the  $r$  term. We can do this by introducing a discount

factor “e” explicitly into the equation. Set  $a(S, \tau) = \beta(S, \tau)e^{r\tau}$ . The time derivative is then

$$\frac{\partial a}{\partial \tau} = \left( \frac{\partial \beta}{\partial \tau} - r\beta \right) e^{r\tau}$$

And hence the equation can be written as

$$\frac{\partial \beta}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \beta}{\partial S^2} + rS \frac{\partial \beta}{\partial S}$$

3. To proceed further, we want to write the equation in terms of the operator  $S\partial/\partial S$ .

This can be easily accomplished by rearranging the second order term

$$\frac{\partial \beta}{\partial \tau} = \frac{1}{2} \sigma^2 S \frac{\partial}{\partial S} \left( S \frac{\partial \beta}{\partial S} \right) + \left( r - \frac{1}{2} \sigma^2 \right) S \frac{\partial \beta}{\partial S}$$

4. We can simplify the operator  $S\partial/\partial S$  by defining the new variable  $Y = \ln S$ , and noting that

$$S \frac{\partial}{\partial S} = \frac{\partial}{\partial Y}.$$

If we then introduce the new function  $\gamma(Y, \tau) = \beta(S, \tau)$ , we see that the differential equation becomes

$$\frac{\partial r}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 \tau}{\partial Y^2} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial \tau}{\partial Y}$$

5. Finally, we need to get rid of the first order partial derivative. Define  $X = Y +$

$(r - \frac{1}{2} \sigma^2) \tau$  and set  $A(X, \tau) = \gamma(Y, \tau)$ . The partial derivative of  $\gamma$  with respect to  $\tau$  is

given by

$$\begin{aligned} \frac{\partial \gamma}{\partial \tau} &= \frac{\partial A}{\partial \tau} + \frac{\partial A}{\partial X} \frac{\partial X}{\partial \tau} \\ &= \frac{\partial A}{\partial \tau} + \frac{\partial A}{\partial X} \left( r - \frac{1}{2} \sigma^2 \right) \end{aligned}$$

However since

$$\frac{\partial \gamma}{\partial Y} + \frac{\partial A}{\partial X}$$

$$\frac{\partial A}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 A}{\partial X^2}$$

Which is the heat equation

Now that we have shown that the Black Scholes equation can be reduced by a series of transformations to the heat equation we would like to solve for the derivative price  $C(S_t, t)$  subject to the terminal condition

$$C(S_T, T) = F(S_T)$$

Where  $F(S_T)$  is a prescribed function that is the payoff function of the derivative

$$\begin{aligned}
C(S_T, T) &= \alpha(S_t, T - t) \\
&= \beta(S_t, T - t)e^{-r(T-t)} \\
&= \gamma(\log S_t, T - t)e^{-r(T-t)} \\
&= A\left(\log S_t + \left[r - \frac{\sigma^2}{2}\right][T - t], T - t\right)e^{-r(T-t)}
\end{aligned}$$

$$\begin{aligned}
F(S_T) &= C(S_T, T) \\
&= A(\log S_T, 0)
\end{aligned}$$

Hence the initial condition on  $A(x, \tau)$  at  $\tau=0$  is

$$A(x, 0) = F(e^x)$$

In the case of call option we have

$$A(x, 0) = \max(e^x - K, 0)$$

We can now appeal to our earlier formula for the solution of the heat equation with the initial condition  $A(x, 0) = F(e^x)$ ,

$$A(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\left(e^{x+\sigma\sqrt{\tau}\xi}\right) e^{-\frac{1}{2}\xi^2} d\xi.$$

Using this value of  $A(x, \tau)$  and the transformation we can then write the derivative price as;

$$C(S_t, t) = A \left( \log S_t + \left[ r - \frac{\sigma^2}{2} \right] [T - t], T - t \right) e^{-r(T-t)}$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F \left( S_t e^{rT + \sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2 T} \right) e^{-\frac{1}{2}\xi^2} d\xi$$

In particular, if we set  $t=0$ , then we obtain the initial price of the derivative

$$C_0 = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F \left( S_0 e^{rT + \sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2 T} \right) e^{-\frac{1}{2}\xi^2} d\xi.$$

We see that the present value of the derivative depends on the expiry date  $T$ . The initial asset price  $S_0$  the volatility  $\sigma$ , the risk-free interest rate  $r$ , and the final condition of the payoff function  $F(S_T)$ .

Note that, similar to the discrete case, we can write the derivative price in terms of an expectation

$$C_0 = e^{-rT} E[F(S_T)]$$

Where,

$$S_T = S_0 e^{rT + \sigma W_T^* - \frac{1}{2}\sigma^2 T}$$

$W_T^*$  is a random variable that is normally distributed with mean zero and variance  $T$ , with respect to some measure  $P^*$ . This is the standard process in the Black-Scholes case.

Before proceeding, we can begin by pricing the trivial derivative that delivers  $S_T$ , that is,  $F(S_T) = S_T$ .

The price of this derivative is

$$\begin{aligned} C_0 &= e^{-rT} E^*[F(S_T)] \\ &= e^{-rT} E^*[S_T] \\ &= e^{-rT} E^*\left[S_0 e^{rT + \sigma W_T^* - \frac{1}{2}\sigma^2 T}\right] \\ &= S_0 E\left[e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}W}\right] \end{aligned}$$

Using lemma 1.11, we can evaluate the expectation as

$$E^*[e^{\sigma W_T^*}] = e^{\frac{1}{2}\sigma^2 T}$$

And hence see that the derivative price is simply

$$C_0 = S_0$$

which is what we should expect.

### 3.9. Call and Put Option Prices

It was seen that by performing an integration, the Black-Scholes formula for the price of a European call option can be derived. Note that it does not allow for early exercise, meaning that it applies only to European options. However, the formula also holds for American options on non-dividend-paying stocks, since early

exercise is never optimal for such options. We start by calculating the initial prices of the two most widely traded options.

### 3.9.1 Call Option

Recall that the call option payoff is:

$$F(S_T) = \max(S_T - K, 0)$$

Hence, the initial price of the derivative is:

$$C_0 = \frac{rT}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\left(S_0 \exp\left[rT + \sigma\sqrt{T\xi} - \frac{1}{2}\sigma^2 T\right] - K, 0\right) \exp\left[-\frac{1}{2}\xi^2\right] \\ \frac{rT}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max\left(S_0 \exp\left[rT + \sigma\sqrt{T\xi} - \frac{1}{2}\sigma^2 T\right] - K, 0\right) \\ \exp\left[-\frac{1}{2}\xi^2\right] d\xi$$

In order to evaluate the integral, it is useful to remove the max function. This can easily be manipulated into a more convenient form.

$$S_0 \exp\left[rT + \sigma\sqrt{T\xi} - \frac{1}{2}\sigma^2 T\right] - K > 0,$$

Which is equivalent to:

$$\exp\left[rT + \sigma\sqrt{T\xi} - \frac{1}{2}\sigma^2 T\right] > \frac{K}{S_0} \text{ taking logarithms of both sides, we obtain:}$$

$$rT + \sigma \sqrt{T\xi} - \frac{1}{2}\sigma^2T > \log(K/S_0)$$

We then want to isolate the integration variable. This will allow us to discover the integration region where the max function is nonzero. We find that:

$$\xi > \frac{\log\left(\frac{K}{S_0}\right) - \left(rT - \frac{1}{2}\sigma^2T\right)}{\sigma\sqrt{T}}$$

If we define the critical value to be:

$$\xi^* = \frac{\left[\log\left(\frac{K}{S_0}\right) + \left(rT - \frac{1}{2}\sigma^2T\right)\right]}{\sigma\sqrt{T}}$$

Then the max function can be written as:

$$\max\left(S_0 \exp\left[rT + \sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2T\right] - K, 0\right)$$

$$\begin{cases} S_0 e^{rT + \sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2T} - K, & \xi > \xi^* \\ 0, & \xi < \xi^* \end{cases}$$

Since the integral vanishes for  $\xi < \xi^*$ , we only need to integrate over the region where  $\xi > \xi^*$  and the max function takes on a positive value. Hence the derivative price becomes.

$$C_u = \frac{e^{-rt}}{\sqrt{2}} \int_{-\infty}^{\infty} \left(S_0 \exp\left[rT + \sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2T\right]\right)$$

This integral involves two terms, it is easiest to evaluate them separately. Defining:

$$\begin{aligned}
 I_1 &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_0 \exp \left[ rT + \sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2T \right] \exp \left[ -\frac{1}{2}\xi^2 \right] d\xi \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_0 \exp \left[ -\frac{1}{2} + \sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2T \right]
 \end{aligned}$$

$$I_2 = -K \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\xi^*}^{\infty} \exp \left[ -\frac{1}{2}\xi^2 \right] d\xi$$

The derivative price is simply the sum of the two integrals:  $C_0 = I_1 + I_2$

The second integral is easier, so we calculate it first. before we do this, consider the following result:

$$\begin{aligned}
 \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{1}{2}u^2} d\xi &= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{1}{2}u^2} du \\
 &= N(-x)
 \end{aligned}$$

where we made the substitution  $u = -\xi$  in the first line and  $N(x)$  is the standard normal cumulative probability density function, previously defined in equation (11.4). Using this result, we see that

$$I_2 = -K e^{-rT} \left( \frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{-\xi^*} e^{-\frac{1}{2}\xi^2} d\xi$$

$$= -K e^{-rT} N(-\xi^*)$$

However, the standard way of writing the Black-Scholes formula is not in terms of  $\xi$ , but rather in terms of two new constants  $h^+$  and  $h^-$ , defined to be

$$h^\pm = \frac{\left( \log\left(\frac{\tilde{S}}{K}\right) \pm \frac{1}{2}\sigma^2 T \right)}{(\sigma\sqrt{T})}$$

where  $\tilde{S}$  is the forward price  $S_0 e^{rT}$ . If we then rewrite  $-\xi^*$  as

$$\begin{aligned} -\xi^* &= \frac{\left( \log\left(K - S_0\right) - \left(rT - \frac{1}{2}\sigma^2 T\right) \right)}{(\sigma\sqrt{T})} \\ &= \frac{\left( \log\left(\frac{S^0}{K}\right) + rT - \frac{1}{2}\sigma^2 T \right)}{(\sigma\sqrt{T})} \\ &= \frac{\log\left(\frac{S_0 e^{rT}}{K}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \end{aligned}$$

we see that

$$-\xi^* = h^- \quad \text{and} \quad -\xi^* + \sqrt{\sigma}T = h^+$$

Hence, we can write the integral  $I_2$  as

$$I_2 = -e^{-rT} K N(h^-)$$

We now want to calculate the slightly more complicated integral  $I_1$ . We begin by completing the square in the exponential,

$$\begin{aligned} I^1 &= \left( \frac{1}{\sqrt{2\pi}} \right) \int_{\xi^*}^{\infty} S_0 \exp \left[ -\frac{1}{2}\xi^2 + \sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2 T \right] d\xi \\ &= \left( \frac{S_0}{\sqrt{2\pi}} \right) \int_{\xi^*}^{\infty} \exp \left[ -\frac{1}{2}(\xi - \sigma\sqrt{T})^2 \right] d\xi. \end{aligned}$$

We then want to make a change of integration variable to  $\eta = \xi - \sigma\sqrt{T}$ . In this case  $d\xi = d\eta$ , and the lower limit of integration  $\xi^* = -\xi^*$  becomes the new lower limit  $\eta = \xi^* - \sigma\sqrt{T} = -h^*$ . Hence the integral becomes

$$I_1 = \left( \frac{S_0}{\sqrt{2\pi}} \right) \int_{\eta=-h^*}^{\eta=\infty} e^{-\frac{1}{2}y^2} dy$$

We can then use the result (C.237) to write the integral in terms of  $N(x)$ ,

$$I_1 = S_0 N(h^*)$$

If we then sum the values of the integrals  $I_1$  and  $I_2$  we obtain

$$C_0 = e^{-rT} [S_0 e^{rT} N(h^+) - K N(h^-)]$$

which is the famous Black-Scholes formula for the initial value of a call option.

## CHAPTER FOUR

### APPLICATION

#### 4.1. Introduction

This chapter builds upon the methodology discussed and explores the practical application of call options in financial engineering. We focus on pricing models, hedging techniques, risk management strategies, trading strategies, and exotic derivatives, incorporating mathematical examples to illustrate these concepts.

#### 4.2. Call Option Pricing Models in Practice

Call option pricing is crucial for understanding its application in financial markets. The Black-Scholes Model, developed in Chapter 3, is the cornerstone of option pricing. The formula is given by:

$$C = S_0N(d_1) - Ke^{-rt}N(d_2)$$

where:

$C_0$  = Price of the European call option

$S_0$  = Current stock price

$K$  = Strike price

$r$  = Risk-free interest rate

$t$  = Time to expiration

$\sigma$  = Volatility of the underlying asset

$N(d)$  = Cumulative normal distribution function

$$d_1 = \frac{\ln(s_0/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

### **Example : Pricing a Call Option**

Consider a European call option with:

$$S_0 = 50$$

$$K = 55$$

$$r = 5\%$$

$$t = 0.5 \text{ years}$$

$$\sigma = 20\%$$

We compute:

$$d_1 = \frac{\ln(50/55) + (0.05 + 0.5 \times 0.2^2) \times 0.5}{0.2\sqrt{0.5}} = -0.127$$

$$d_2 = d_1 - 0.2\sqrt{0.5} = -0.268$$

Using the normal distribution table:

$$N(d_1) = 0.449, \quad N(d_2) = 0.394$$

The call option price is:

$$C = 50 \times 0.449 - 55 e^{-0.05 \times 0.5} \times 0.394$$

$$C = 22.45 - 20.55 = 1.90$$

Thus, the call option price is #1.90.

### 4.3. Hedging with Call Options

Call options are widely used for portfolio hedging and risk mitigation. The key concept in hedging is Delta Hedging, which involves continuously adjusting a position to maintain a risk-neutral portfolio.

## Delta Hedging Example

Delta ( $\Delta$ ) is the rate of change of the option price with respect to the underlying asset price:

$$\Delta = N(d_1)$$

Using our previous example,

$$\Delta = N(-0.127) = 0.449$$

This means that for every call option sold, we should hold 0.449 shares of the underlying stock to hedge the risk.

## Protective Call Strategy

Investors use protective calls to hedge short stock positions. For instance, a trader shorting #100 shares of a stock at #50 may buy 100 call options with a strike price of #55 to cap potential losses.

#### 4.4. RISK MANAGEMENT WITH CALL OPTIONS

Portfolio Protection;

Institutions hedge against market downturns by purchasing out-of-the-money (OTM) call options, which provide insurance against large price increases.

Interest Rate Hedging;

Financial firms use call options on bonds to mitigate risks from fluctuating interest rates. The Black Model, a variation of Black-Scholes, is used for pricing options on futures and interest rate derivatives:

$$C = e^{-rt}FN(d_1) - KN(d_2)$$

where  $F$  is the forward price.

Example: Hedging Interest Rate Risk

A pension fund anticipates an increase in bond yields and purchases call options on bond futures with:

Strike price = #1000

Current price = \$980

Interest rate = 3%

Volatility = 15%

Time to expiration = 1 year

Using the Black Model, we compute the option value and determine the number of contracts needed to hedge the exposure

$$d_1 = \frac{\ln(980/1000) + \frac{1}{2}(0.15)^2 \times 1}{0.15 \times \sqrt{1}}$$
$$= \frac{\ln(0.98) + 0.0125}{0.15}$$

$$= \frac{-0.0202 + 0.01125}{0.15} = \frac{-0.00895}{0.15} = -0.0597$$

$$d_2 = -0.0597 - (0.15 \times \sqrt{1})$$

$$= -0.0597 - 0.15 = -0.2097$$

Using a standard normal table or a calculator:

$$N(d_1 = -0.0597) \approx 0.4762$$

$$N(d_2 = -0.2097) \approx 0.4170$$

$$C = e^{-0.03 \times 1} [980 \times 0.4762 - 1000 \times 0.4170]$$

$$= e^{-0.03} [466.68 - 417]$$

$$= 0.9704 \times 49.68$$

$$= 48.20$$

The call option price is #48.20.

## 4.5 TRADING STRATEGIES USING CALL OPTIONS

### 4.5.1 Covered Call Strategy

An investor holds a stock and sells a call option to generate income.

Example:

Stock price = \$100

Call option premium = \$5

Strike price = \$110

If the stock remains below \$110, the investor keeps the \$5 premium. If it rises above \$110, they sell at \$110, capping upside gains but benefiting from premium income.

### 4.5.2. Bull Call Spread

Buying a lower-strike call and selling a higher-strike call to limit costs.

Example:

Buy call with strike price = \$50 for \$4

Sell call with strike price = \$55 for \$2

Net cost = \$2, max profit = \$3 if stock closes at or above \$55.

## 4.6 EXOTIC CALL OPTIONS AND STRUCTURED PRODUCTS

### Barrier Options

Call options that activate/deactivate at a certain price level.

#### Example 5: Up-and-Out Call Option

Strike price = #50

Barrier = #70

If the stock reaches #70, the option is worthless, reducing costs for structured products.

### Lookback Call Options

Allow exercising at the lowest stock price during the option's life, maximizing gains.

$$C = e^{-rt}E[\max(S_T - K, 0)]$$

## 4.7 CASE STUDIES AND MARKET EXAMPLES

### 4.7.1 Case Study 1: Portfolio Hedging Using Call Options

A hedge fund manages a \$100 million portfolio consisting of S&P 500 stocks. The fund wants to hedge against potential market surges without selling assets, so it purchases call options on the S&P 500 index.

### Strategy: Buying Call Options on the Index

- The hedge fund buys 1-month call options on the S&P 500 ETF (SPY).
- The SPY ETF is currently trading at \$450 per share.
- Each option contract represents 100 shares of SPY.
- The hedge fund purchases out-of-the-money (OTM) call options with a strike price of \$460 at a premium of \$5 per contract.

Number of contracts needed = Portfolio value / (SPY price × 100)

$$= \frac{100,000,000}{450 \times 100} = 2222 \text{ contracts}$$

Total cost of hedging = 2222 contracts × \$5 per contract × 100 shares per contract

$$= 2222 \times 5 \times 100 = 1,111,000$$

Thus, the hedge costs \$1.11 million, providing the fund with protection against a sharp rally in the S&P 500 above \$460.

## Analysis

- If the S&P 500 rises to \$480, the fund benefits from the portfolio's gain and the call option's appreciation.
- If the market stays below \$460, the fund loses the \$1.11 million premium, but retains its long equity exposure.
- This case shows how call options help hedge against market upside risk while keeping core investments intact.

### 4.7.2 Case Study 2: Foreign Exchange (FX) Risk Management with Call Options

#### **Problem:** A Multinational Company Hedging Currency Appreciation

A European exporter expects to receive \$10 million in payments from U.S. customers in 3 months. If the euro appreciates, the value of these payments in euros will decline.

To protect against this, the company buys EUR/USD call options, which will increase in value if the euro strengthens.

## Strategy: Purchasing EUR/USD Call Options

- The current EUR/USD exchange rate is 1.1000.
- The company buys at-the-money (ATM) call options with a strike price of 1.1000.
- The option premium is 0.0150 per euro.
- Each option contract covers €100,000.

### Mathematical Computation

Number of contracts needed = Payment amount / Contract size

$$= \frac{10,000,000}{100,000} = 100 \text{ contracts}$$

Total cost of options = 100 contracts × €100,000 per contract × 0.0150 per euro

$$= 100 \times 100,000 \times 0.0150 = \text{€}150,000$$

### Analysis

- If EUR/USD rises to 1.1500, the call options are profitable, offsetting losses from FX conversion.

- If EUR/USD stays below 1.1000, the company loses €150,000 (the premium cost) but ensures stable revenue conversion.
- This example highlights how call options protect multinational firms from adverse currency movements.

## CHAPTER FIVE

### SUMMARY AND CONCLUSION

#### 5.1 SUMMARY

This chapter provides a synthesis of the key findings from the study on call options and their application in financial engineering. The research has explored various theoretical models, practical applications, and strategic implementations of call options in financial markets.

##### 5.1.1 Key Findings from the Study

###### Theoretical Foundations (Chapter 2 & 3)

The study began with a detailed literature review on call options, covering their origins, fundamental principles, and pricing models. The methodology (Chapter 3) laid the foundation by introducing continuous-time models, Wiener processes, and stochastic calculus, leading to the derivation of the Black-Scholes equation.

###### Application of Call Options (Chapter 4)

The practical aspect of call options was explored through pricing models, hedging strategies, risk management techniques, and trading applications. The study demonstrated:

- The use of the Black-Scholes model for call option pricing.

- Delta hedging as a risk management tool.
- Various trading strategies such as covered calls and straddle strategies.
- The significance of exotic derivatives, including barrier and lookback options, in financial markets.
- Real-world case studies illustrating the effectiveness of call options in structured financial products.

### **5.1.2 Practical Implications**

- Risk Management: Call options play a crucial role in hedging equity portfolios, managing foreign exchange risk, and mitigating volatility exposure.
- Strategic Trading: Investors and institutions utilize options for speculative trading, arbitrage, and income generation through covered calls.
- Pricing and Valuation: Understanding advanced pricing models enhances market efficiency, enabling fair valuation of derivatives.
- Structured Financial Products: The integration of exotic call options into structured products provides innovative financial solutions for institutional investors.

### **5.1.3 Theoretical Contributions**

- **Refinement of Pricing Models:** The study reaffirms the Black-Scholes model while acknowledging its limitations, paving the way for advanced models like stochastic volatility and jump-diffusion processes.
- **Extension of Hedging Strategies:** By incorporating Itô's Lemma, the research highlights how dynamic hedging can be optimized in volatile markets.
- **New Perspectives on Derivatives:** The analysis of exotic call options contributes to the growing literature on complex financial instruments.

## **5.2 CONCLUSION**

Call options play a crucial role in financial engineering, offering versatile applications in pricing, hedging, and risk management. This study aims to provide a comprehensive understanding of call options, from their fundamental principles to advanced applications in financial markets. By examining key pricing models, hedging techniques, and the role of exotic call options, this research seeks to bridge the gap between theoretical models and real-world financial practices.

The insights gained from this study will be valuable for financial professionals, investors, and academics looking to enhance their understanding of call options and their strategic applications. As financial markets continue to evolve, the study of call options remains essential for managing risk, optimizing investment

strategies, and developing innovative financial instruments. The following chapters will delve deeper into the literature, methodologies, and practical applications that define the significance of call options in financial engineering.

This chapter reviewed the foundational theories, pricing models, risk management applications, and empirical studies on call options. The literature highlights their importance in financial markets, their evolution through advanced pricing models, and their diverse applications in trading and hedging strategies.

The next chapter will discuss the methodology for applying call options in financial engineering, including stochastic models and hedging frameworks.

This chapter has provided a comprehensive exploration of continuous-time financial models, emphasizing their significance in asset pricing and risk management. Beginning with the Wiener process and its stochastic properties, we established a foundation for modeling asset price dynamics. The introduction of stochastic calculus enabled the formulation of differential equations governing asset prices, leading to the derivation of the Black-Scholes equation. The discussion on arbitrage arguments and replication portfolios highlighted the role of financial mathematics in ensuring market efficiency. Ultimately, these mathematical tools and models provide traders, investors, and financial analysts

with a rigorous framework for pricing derivatives and managing financial risks in continuously evolving markets.

Call options provide essential tools for trading, risk management, and structured product development. Financial engineers apply them to create sophisticated hedging and arbitrage strategies, making them indispensable in modern markets.

Further advancements in quantitative finance, algorithmic trading, and AI-driven option pricing will continue to expand the applications of call options in financial engineering.

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