# CHAPTER ONE

## Introduction

This chapter explores the idea of gauge theories in modern physics, focusing on how Yang–Mills theory serves as a powerful framework for explaining the fundamental forces of nature. It opens with an overview of electromagnetism and the principle of gauge invariance, followed by a discussion of relevant literature that traces the historical evolution and modern applications of Yang–Mills theory in particle physics.

Modern physics has uncovered a profound insight the forces of nature, once thought to be entirely separate, are actually linked through deep underlying symmetries. At the heart of this discovery lies Yang–Mills theory, an elegant extension of Maxwell's electromagnetism that has become a cornerstone of theoretical physics. Originally introduced to describe nuclear interactions, this theory has transformed our understanding of what a "force" truly is. This project examines how Yang–Mills theory not only includes the electromagnetic force but also situates it within a broader, unified framework of fundamental interactions.

The story begins in the 19th century with James Clerk Maxwell, whose groundbreaking equations unified electricity and magnetism into one comprehensive theory of electromagnetism. Maxwell's equations showed that changing electric fields produce magnetic fields and vice versa, leading to the prediction of electromagnetic waves including light itself. This marked a monumental step forward, representing the first major unification in the history of physics.

However, hidden within Maxwell's elegant formulation was a subtle but powerful symmetry known as gauge invariance. In simple terms, gauge invariance allows certain transformations of the electromagnetic potential $A\mu$ without changing any observable physics. Though initially seen as a mathematical convenience, this U(1) gauge symmetry would later be recognized as a fundamental feature of nature, and a prototype for all other known forces.

The advent of quantum mechanics in the early 20th century brought significant changes to our understanding of electromagnetism. As quantum theory developed, it became clear that classical field theories needed to be reinterpreted. This led to the creation of quantum electrodynamics (QED), a quantum field theory that describes how charged particles interact through the exchange of photons the gauge bosons of electromagnetism.

QED is not only one of the most mathematically elegant theories in physics; it's also the most accurate. Predictions made using QED such as the magnetic moment of the electron agree with experiment to more than one part in a billion. Yet despite its extraordinary success, QED represents just the simplest case of a broader principle: gauge symmetry.

This realization raised an important question: could other fundamental forces be understood using similar principles?

The answer came in 1954 when Chen Ning Yang and Robert Mills introduced a new class of gauge theories. Unlike QED, which is based on an Abelian (commutative) symmetry group U(1), the Yang-Mills theory proposed using non-Abelian groups such as SU(2). This breakthrough allowed for gauge bosons that interact not only with matter, but also with each other. In contrast to the photon, which does not carry electric charge, Yang-Mills bosons carry their own version of "charge," leading to self-interactions and fundamentally richer dynamics.

Although Yang and Mills initially formulated their theory to explain the strong nuclear force, scientists soon recognized that non-Abelian gauge symmetry might hold the key to unifying all the fundamental interactions of nature. This realization culminated in the Glashow–Weinberg–Salam model of the late 1960s, which successfully combined the electromagnetic and weak nuclear forces into a single, coherent framework known as the electroweak theory..

At high energies, this theory is described by a larger symmetry group: $SU(2)_L \times U(1)_Y$. Within this framework, the photon the particle responsible for electromagnetic interactions is no longer seen as fundamental. Instead, it emerges as a particular combination of the theory's original gauge fields, revealed after a process called spontaneous symmetry breaking. This process, made possible by the Higgs mechanism, gives mass to the weak force carriers (the W and Z bosons) while leaving the photon massless.

This unification tells a remarkable story: the electromagnetic force we observe today is the low-energy remnant of a more symmetric, unified force that existed in the early universe. At temperatures above approximately 10^15 Kelvin conditions that existed within a fraction of a second after the Big Bang the electromagnetic and weak forces were indistinguishable. As the universe cooled, the Higgs field underwent a phase transition, breaking the electroweak symmetry and "freezing in" the properties of the photon. The result is the long-range, massless electromagnetic force we are familiar with today, while the weak force became short-range and massive.

## 1.1 Literature Review

### Yang–Mills Theory

While academic papers provide the technical backbone for understanding Yang–Mills theory, more accessible resources such as the Wikipedia article on the topic offer a clear overview of its development and far-reaching impact. Such summaries serve as intellectual roadmaps, charting the theory's journey from its relatively unknown origins to its pivotal place in modern physics.

The article explains that Yang–Mills theory can be viewed as a sweeping generalization of electromagnetism. Whereas Maxwell's equations and Quantum Electrodynamics (QED) are built on an Abelian gauge symmetry (the U(1) group), Yang–Mills theory extends this idea to non-Abelian symmetry groups like SU(2) and SU(3) the mathematical foundations underlying the weak and strong nuclear forces, respectively.

This generalization isn't merely mathematical it introduces real physical differences. In QED, photons the carriers of the electromagnetic force do not interact with one another. But in non-Abelian theories, the corresponding gauge bosons are self-interacting. This leads to highly non-linear dynamics, a hallmark of the strong nuclear force, and one of the key distinctions that separates Yang–Mills theories from the simpler U(1)-based electromagnetism.

Importantly, the article doesn't just linger on historical or technical aspects it also explains the challenges the theory initially faced. Foremost among these was the fact that, like the photon, Yang–Mills gauge bosons were originally massless. This made the theory seem inapplicable to real-world nuclear forces, which clearly required short-range interactions and therefore massive mediators. The article recounts how this issue contributed to early skepticism and delayed widespread acceptance of the theory.

The breakthrough came with the discovery of the Higgs mechanism, a process through which gauge bosons acquire mass via spontaneous symmetry breaking. This advancement together with Gerard ’t Hooft’s proof in the early 1970s that Yang–Mills theories could be renormalized firmly established the theory within mainstream physics. Experimental validation followed soon after with the detection of the W and Z bosons, the massive carriers of the weak force, which fit seamlessly into the Yang–Mills framework.

One of the article’s most illuminating discussions concerns the electroweak unification. By combining the SU(2) and U(1) gauge symmetries, the Standard Model of particle physics demonstrates that electromagnetism and the weak nuclear force are not truly distinct phenomena but rather two manifestations of a single underlying interaction. The apparent difference between them arises only when the symmetry is broken at low energies. In this context, the photon is not an independent fundamental field but instead emerges as a mixture of gauge fields following symmetry breaking.

The article also highlights ongoing areas of research in theoretical and mathematical physics. These include attempts to resolve the Yang–Mills existence and mass gap problem one of the seven Millennium Prize Problems established by the Clay Mathematics Institute and the continued development of lattice gauge theories, which model gauge fields using discrete space-time grids to explore non-perturbative aspects of the theory.

In addition, the article explores how Yang–Mills theory has influenced pure mathematics, particularly in the realm of differential geometry and topology. This dual identity both as a cornerstone of particle physics and a subject of deep mathematical inquiry makes Yang–Mills theory unique in the history of science.

Ultimately, the Wikipedia article serves as a broad and balanced account of how a once-obscure theory evolved into one of the central pillars of modern theoretical physics. Its emphasis on the interplay between electromagnetism and other forces provides a clear picture of how gauge symmetry, once a niche idea, came to define the structure of our universe.

### 1.2 Gauge Theory of Elementary Particle Physics   Cheng & Li

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From this foundation, Cheng and Li gradually introduce the generalization to non-Abelian gauge groups, culminating in a clear and structured exposition of Yang–Mills theory. A key strength of their approach lies in how they carefully demonstrate that the mathematical machinery needed to describe SU(2) or SU(3) interactions is not separate from that of U(1), but rather an extension of it. In this way, electromagnetism is not discarded or overshadowed by the more complex theories of the strong and weak nuclear forces it is embraced as the template upon which these theories are modeled.

The authors devote considerable attention to the electroweak unification, the point at which Yang–Mills theory intersects most intimately with electromagnetism. In the Standard Model, the electroweak interaction is described by a combined SU(2) X U(1) gauge group, involving four gauge bosons: $W^1, W^2, W^3$ (from SU(2)) and B(from U(1)). After spontaneous symmetry breaking, three of these bosons (the $W^+, W^- and\ Z^0$) acquire mass, while one linear combination remains massless this is the photon, which mediates the electromagnetic force.

Cheng and Li walk the reader through how this mixing occurs and how the resulting photon field $A_\mu$ emerges as a specific linear combination of the original SU(2) and U(1) fields. Importantly, they emphasize that the masslessness of the photon is not an arbitrary feature, but a direct consequence of how the gauge symmetry is broken. The fact that the electromagnetic force has infinite range and the weak force does not is therefore explained not by different fundamental origins, but by the specific path of symmetry breaking that nature appears to have taken.

One of the most enlightening parts of Cheng and Li's discussion focuses on the relationship between electric charge and the gauge couplings within the electroweak theory. They point out that the familiar electric charge ( e ) is not a truly fundamental quantity, but rather emerges from the interaction between two deeper coupling constants ( g ) (for SU(2)) and ( $g'$) (for U(1)). These couplings are connected to ( e ) through the Weinberg angle. This perspective transforms our understanding of electric charge: it is no longer seen as an arbitrary property of particles, but as a derived outcome of the universe's underlying symmetry structure.

Another remarkable aspect of Cheng and Li's work is their treatment of the massive gauge bosons the W and Z bosons. They do not present them as theoretical anomalies but as natural consequences of the same framework that gives rise to the photon. This explanation helps clarify one of the most conceptually difficult ideas in the Standard Model: that a single unified interaction can manifest as very different forces at low energies. The weak force appears short-ranged because its mediators are massive, whereas electromagnetism is long-ranged precisely because its mediator, the photon, remains massless.

Throughout their exposition, Cheng and Li strike an admirable balance between mathematical rigor and physical intuition. Their writing combines clear, accessible explanations with detailed derivations, making their work an invaluable reference for both students and researchers. It not only teaches how to handle gauge theories mathematically but also helps readers grasp their deeper physical meaning.

In the context of this project, Cheng and Li's insights are particularly relevant. Their analysis reinforces the view that electromagnetism is not replaced by Yang–Mills theory rather, it emerges as its most refined and elegant limiting case. Far from being a remnant of classical physics, electromagnetism is revealed as a cornerstone of a unified structure, where all fundamental forces are bound together by the shared language of symmetry and gauge invariance.

In summary, *Gauge Theory of Elementary Particle Physics* offers a bridge between the familiar and the abstract. By starting with the well-understood territory of Maxwell and QED and moving carefully toward the broader landscape of Yang–Mills theory, Cheng and Li provide readers with the tools and the perspective to see how electromagnetism fits seamlessly into the broader tapestry of fundamental physics.

### 1.3 An Introduction to Quantum Field Theory   Peskin & Schroeder

Among the most widely used and respected textbooks in theoretical physics, *An Introduction to Quantum Field Theory* by Michael E. Peskin and Daniel V. Schroeder is often regarded as the definitive guide for students entering the world of quantum field theory (QFT). While its pages are rich with mathematical derivations and rigorous proofs, its true value lies in how it connects these formal structures to the physical principles that shape our universe. Yang–Mills theory features prominently in this text, and the authors provide a layered approach that begins with the familiar terrain of quantum electrodynamics (QED) before venturing into the more complex world of non-Abelian gauge theories.

The authors' journey into Yang–Mills theory begins, quite appropriately, with QED, the quantum field theory of the electromagnetic interaction. In QED, the electromagnetic force arises from the requirement of local U(1) gauge invariance the principle that the laws of physics should remain unchanged if the phase of a charged particle's wavefunction is altered differently at each point in space-time. This seemingly abstract idea turns out to be incredibly powerful: enforcing this symmetry leads naturally to the introduction of the photon field and governs how it must couple to matter. This is the classic gauge principle in action.

But what Peskin and Schroeder do particularly well is to frame QED not as a complete theory, but as a doorway to something larger. They show that the structure of QED, while immensely successful, is only the simplest case of a much broader class of theories governed by symmetry principles. From this vantage point, they guide the reader toward Yang–Mills theory not as an unrelated framework, but as an inevitable generalization of QED's core ideas.

In the transition from Abelian to non-Abelian gauge theories, a major conceptual leap occurs: the force carriers themselves become participants in the interaction. While photons in QED are neutral and do not interact with one another, Yang–Mills gauge bosons carry their own version of "charge" and thus interact with each other. This shift leads to a dramatic increase in theoretical richness and physical complexity. The self-interaction of gauge bosons introduces nonlinearity into the equations of motion, fundamentally altering the behavior of the fields and enabling phenomena like asymptotic freedom, confinement, and spontaneous symmetry breaking.

Peskin and Schroeder's discussion highlights how the conceptual backbone of QED gauge symmetry remains intact in Yang–Mills theory, even as the details become more elaborate. In this way, they position electromagnetism as the simplest expression of a deeper principle that applies equally to the weak and strong nuclear forces. This provides a conceptual bridge: while the mathematics of Yang–Mills theory may seem more intimidating, its philosophical roots are the same ones that guided Maxwell, Dirac, and Feynman in their work on electromagnetism.

A central theme in Peskin and Schroeder's treatment is the quantization of gauge fields the process of converting a classical field theory into a quantum one. In Quantum Electrodynamics (QED), this procedure is relatively straightforward due to the linearity of the field equations. However, in Yang–Mills theory, the nonlinear nature of the fields introduces significant challenges. Peskin and Schroeder explain how these difficulties are addressed through sophisticated methods such as BRST symmetry and ghost fields, which ensure that gauge invariance is maintained throughout the quantization process.

Although these discussions are highly technical, their implications are profound. Gauge invariance, once considered merely a classical symmetry, persists even after quantization an exceptionally rare and powerful property in physics. For electromagnetism, this persistence ensures that the photon remains massless, with its interactions determined entirely by symmetry principles.

The authors also delve into how quantum corrections modify classical predictions, focusing on the concept of vacuum polarization. In QED, the presence of virtual particle–antiparticle pairs subtly alters the effective strength of the electromagnetic force. This energy-dependent variation leads to what is known as the running of the electromagnetic coupling constant, a small but measurable deviation from Coulomb's law that becomes more noticeable at higher energies.

In Yang–Mills theory, the effects of running couplings are even more striking. The coupling constants evolve with energy in such a way that interactions become weaker at very high energies a property known as asymptotic freedom and stronger at low energies, leading to confinement. While these behaviors are most significant for the strong nuclear force, they also carry indirect implications for electromagnetism, especially in ongoing efforts to unify all interactions within a single gauge framework.

Another critical idea explored in the text is that of anomalies quantum effects that can break classical symmetries if not properly controlled. In gauge theories, such anomalies must cancel out to preserve the theory's internal consistency. Peskin and Schroeder demonstrate how this requirement imposes strict constraints on the structure of the Standard Model, influencing not only the weak and strong interactions but also the electromagnetic sector itself.

They also touch briefly on magnetic monopoles, hypothetical particles predicted by certain Yang–Mills models. While not yet observed, these objects would have profound implications for electromagnetism, enforcing the quantization of electric charge and offering new insight into the topological structure of gauge fields.

Finally, one of the most powerful ideas presented in the book is that QED and Yang–Mills theory are not separate chapters in the story of physics, but part of the same narrative. They differ not in kind, but in complexity. Electromagnetism, with its linear equations and massless mediator, appears simpler because it results from an unbroken Abelian symmetry. But it exists within a broader family of interactions, all governed by the same underlying logic of local symmetry.

In this light, electromagnetism is not the "classical" interaction in contrast to newer, more "quantum" forces. It is, rather, the gateway to understanding them all. Peskin and Schroeder's treatment makes it clear that Yang–Mills theory does not eclipse QED it completes it, extending its principles to a wider range of physical phenomena and revealing the deep structure beneath the surface of classical fields.

### 1.4 Properties Of Electromagnetic Force

The electromagnetic (EM) force is one of the most well-understood and pervasive interactions in physics. It is responsible for nearly every phenomenon in our visible, tangible world, from the binding of atoms to the transmission of light and the operation of modern technology. Unlike gravity, which acts on mass, the electromagnetic force acts on electric charge. It is described by both classical theory via Maxwell's equations and by quantum field theory through quantum electrodynamics (QED). What follows is a conceptual exploration of the key properties that distinguish this force, along with the underlying gauge structure (U(1)) that defines its behavior.

1. Infinite Range Mediated by a Massless Gauge Boson (Photon)

The electromagnetic force has infinite range, unlike the weak and strong nuclear forces. This is because the photon, the force carrier (or mediator) of EM interactions, is massless. This property can be seen in the Yukawa potential which describes force ranges:

$$V(r) \propto e^{(-mr)}/r$$

When m=0, the exponential term becomes 1, and the potential reduces to the familiar Coulomb form:

$$V(r) = (1/4\pi\varepsilon_0)\cdot(q_1q_2/r)$$

This potential falls off gradually with distance but never truly vanishes, allowing electromagnetic effects such as light and radio waves to travel through space over astronomical distances.

2. Acts on Particles with Electric Charge

The electromagnetic force acts only on particles with electric charge. This includes fundamental particles like the electron and proton, but not neutrinos (which are electrically neutral). The strength of interaction is proportional to the product of charges:

$$F = (1/4\pi\varepsilon_0)\cdot(q_1q_2/r^2)$$

The electric charge q is quantized in nature and usually comes in integer multiples of the elementary charge e ≈ 1.602 X $10^{-H9}$ C. The coupling strength of the EM force is described by the fine-structure constant:

$$\alpha = e^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$$

This small but non-negligible number governs the probability of photon exchange and EM interactions at low energies.

3. Vector Nature and Directionality

EM forces are vector forces, meaning they have both magnitude and direction. Electric forces repel or attract based on the sign of the charges:

Like charges (positive-positive or negative-negative) repel.

Opposite charges (positive-negative) attract.

In this way, the force provides a deterministic and symmetrical structure to interactions across a variety of scales, from atomic bonding to electrostatic repulsion.

4. Conserves Energy, Momentum, and Angular Momentum

The EM force adheres strictly to conservation laws. When particles interact electromagnetically, the total energy, momentum, and angular momentum are conserved. These principles underlie the predictability of electrodynamics, whether in particle scattering experiments or in classical optics.

5. Obeys Superposition and Linearity

The EM force follows the principle of superposition: the total force on a charge due to multiple other charges is the vector sum of the individual forces. This linearity allows for the elegant mathematical description of complex systems using Maxwell's equations.

$E\_total = E_1 + E_2 + ... + E\_n$

This property enables the construction of devices like antennas and waveguides, where multiple EM waves can interfere, amplify, or cancel each other.

6. Gauge Symmetry: U(1) Structure

Electromagnetism is the prototypical gauge theory, arising from the local invariance under phase rotations in the quantum field:

$$\psi(x) \rightarrow e^{i\theta(x)}\psi(x)$$

To preserve invariance under such transformations, a gauge field $A\mu$ must be introduced. The dynamics of this field are governed by the field strength tensor:

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

This tensor encapsulates the electric and magnetic fields, and forms the basis of Maxwell’s equations. The U(1) gauge symmetry ensures charge conservation and the masslessness of the photon.

7. Parity and CP Symmetry Preservation

Unlike the weak force, which violates parity (P) and sometimes CP symmetry, the electromagnetic interaction strictly conserves both:

Parity (P): EM interactions remain unchanged under mirror reflection.

Charge-Parity (CP): No CP violation has been observed in electromagnetic processes.

These symmetries make EM processes highly predictable and time-reversible, a stark contrast to weak decay processes.

8. Governs Atomic, Molecular, and Photonic Interactions

Virtually all of chemistry and material science arises from electromagnetic interactions:

Electrons are held in atoms via electrostatic attraction to nuclei.

Molecules form through shared or transferred electrons.

Photon-matter interactions such as Compton scattering, photoelectric effect, and pair production arise from EM interactions.

Each of these phenomena is not just qualitatively explained but quantitatively predicted using electrodynamics and QED.

9. Embedded in Electroweak Theory

Though our focus is on electromagnetism itself, it is worth noting that within the Standard Model, electromagnetism is not completely independent. It arises as a residual symmetry after the $\boldsymbol{SU(2)_L \times U(1)_Y}$ electroweak symmetry is spontaneously broken by the Higgs mechanism:

$$U(1)_{EM} \subset SU(2)_L \times U(1)_Y$$

This embedding shows that the photon is not a fundamental gauge boson in isolation, but a mixture of electroweak gauge fields. Despite this, electromagnetism survives as a distinct and unbroken force at low energies.

10. Foundation of Technology and Modern Physics

Electromagnetism underlies nearly every modern technology:

Electrical power: generation, storage, transmission

Communication systems: radio, microwave, fiber optics

Medical imaging: MRI, X-rays, PET scans

Computing: from semiconductors to superconducting qubits

Even cutting-edge fields like quantum computing, plasma physics, and fusion energy depend on precise control of electromagnetic interactions.

11. Predictive Power of Quantum Electrodynamics (QED)

The electromagnetic force is described quantum mechanically by QED, a U(1) gauge theory that has made some of the most accurate predictions in science. The most famous example is the anomalous magnetic moment of the electron, predicted to parts per trillion and confirmed by experiment.

This extraordinary agreement reinforces confidence in the EM force as a gateway to understanding other interactions through generalized gauge theories like Yang–Mills theory.

12. Conservation of Electric Charge

One of the core consequences of U(1) gauge invariance is the exact conservation of electric charge:

$$\partial_\mu j^\mu = 0$$

This conservation law is observed in all electromagnetic processes, even under extreme conditions such as particle collisions at high-energy accelerators. No known process violates charge conservation.

13. Self-Consistency and Renormalizability

The electromagnetic force, as described by QED, is renormalizable. This means that even at very high energies or small distances, calculations yield finite, meaningful results a critical feature not shared by all field theories. This has made QED a template for constructing more complex gauge theories, including the non-Abelian Yang–Mills models.

The electromagnetic force is one of the most thoroughly studied and profoundly impactful interactions in physics. Defined by a deceptively simple U(1) gauge symmetry, it gives rise to an enormous diversity of phenomena, from the behavior of atoms and light to the backbone of technology and communication. Its theoretical elegance, empirical accuracy, and foundational role in the structure of matter make it a central pillar not only in physics but in human civilization's scientific understanding of the universe.

### 1.5 Aim

The purpose of this project is to apply Yang-mills Theory to Electromagnetic force and demonstrate its roles as a gauge theory.

### 1.6 Objectives

To achieve the aim of this research, the project is structured around four major objectives. Each objective builds upon the previous one, leading to a complete understanding of how the electromagnetic force emerges as the Abelian limit of Yang–Mills theory. These objectives include to:

**1. Construct the U(1) gauge-invariant Lagrangian**

The first objective is to build the Lagrangian that describes the electromagnetic field as a U(1) gauge theory. The Lagrangian is a mathematical function that summarizes the dynamics of a physical system. In field theory, it contains all the information about particles, fields, and their interactions. Here, the goal is to ensure that the Lagrangian remains invariant under local U(1) transformations, meaning the physical equations do not change when we alter the field phase locally. This establishes the theoretical foundation for electromagnetism as a gauge-invariant system.

**2. Derive the equations of motion using the Euler–Lagrange formalism**

Once the gauge-invariant Lagrangian is obtained, the next step is to derive the equations of motion that govern electromagnetic fields. This is achieved using the Euler–Lagrange formalism, which follows the principle of least action, stating that nature chooses the path that minimizes the action integral. Applying this formalism to the electromagnetic Lagrangian leads directly to Maxwell's equations, which describe how electric and magnetic fields evolve and interact with charged particles.

### 3. Analyze the commutator relations of the U(1) gauge generators

The third objective is to study the algebraic structure of the U(1) group by analyzing the commutator relations of its generator. For U(1), the generator is simple and its commutator relations vanish, this means that the group is Abelian (its operations commute). Understanding these relations reveals why the photon, the gauge boson of U(1), has no self-interaction and why the electromagnetic field equations are linear. This contrasts sharply with non-Abelian groups like SU(2) or SU(3), where self-interactions lead to more complex dynamics.

### 4. Identify the photon as the electromagnetic gauge boson emerging from the U(1) field

The final objective is to connect the mathematical framework to the physical interpretation of the electromagnetic force. By quantizing the U(1) gauge field, the photon emerges naturally as its force carrier. This explains key physical properties of electromagnetism, the masslessness of the photon, the infinite range of the force, and the absence of self-interaction. Through this, the project demonstrates that electromagnetism is the Abelian limit of Yang–Mills theory, and the photon is its most fundamental manifestation.

In summary, each of these objectives contributes to a unified understanding of how electromagnetism, one of nature's most familiar forces, arises from the deeper and more general symmetry principles embodied in Yang–Mills theory.

## Chapter 2

## Theory and Methodology

### 2.1 Theoretical Framework

This study is grounded in the foundational principles of modern gauge field theory, which provides the conceptual backbone for the analysis undertaken herein. This chapter outlines the essential theoretical constructs that underpin the research. It begins by examining the pivotal role of symmetry in physical laws, then advances to the general formulation of Yang–Mills gauge theories. These ideas are subsequently illustrated through the canonical example of electromagnetism, followed by an exposition of the mathematical structures necessary for its precise formulation. The section concludes by situating this theoretical framework within the broader scientific pursuit of unifying the fundamental interactions of nature.

### 2.2 Electromagnetism as a U(1) Gauge Theory

Electromagnetism represents the simplest and most symmetric realization of a gauge field theory, corresponding to the abelian group U(1) (Weyl, 1929; Jackson, 1998). It serves as the prototype for all higher non-abelian gauge structures. Unlike Yang–Mills theories where gauge bosons self-interact, the photon field is linear and non-self-interacting, which reflects the abelian character of the group.

### 1. Local Gauge Transformation

A free complex field $\psi(x)$ transforms under a local U(1) phase rotation as

$$\boldsymbol{\psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x)}$$

where *qq*q is the coupling constant (the electric charge) and $\alpha(x)$ is a real, spacetime-dependent function.

(Weyl, 1929)

## 2. Covariant Derivative and Gauge Field Introduction

The ordinary derivative does not remain invariant under this local transformation because

$$\partial_\mu \psi'(x) = e^{iq\alpha(x)} (\partial_\mu \psi(x) + iq(\partial_\mu \alpha(x))\psi(x))$$

The appearance of the extra term $iq(\partial_\mu \alpha)\psi$ breaks covariance.

To restore invariance, we introduce a gauge field $A_\mu(x)$ and define the **covariant derivative**:

$$D_\mu = \partial_\mu - iqA_\mu$$

such that under gauge transformation:

$$\psi(x) \rightarrow e^{iq\alpha(x)}\psi(x)$$

$$A_\mu(x) \rightarrow A_\mu'(x) = A_\mu(x) + 1/q\partial_\mu \alpha(x)$$

This ensures that $D_\mu\psi$ transforms covariantly as

$$(D_\mu \psi)' = e^{iq\alpha(x)} D_\mu \psi$$

which maintains local gauge symmetry.

(Peskin & Schroeder, 1995)

## 3. Field Strength Tensor

The electromagnetic field strength tensor is defined as the commutator of covariant derivatives:

$$[D_\mu, D_\nu]\psi = -iqF_{\mu\nu}\psi$$

Thus,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This tensor is gauge invariant, since under transformation of $A\mu$ :

$$F_{\mu\nu}' = \partial_\mu A_\nu' - \partial_\nu A_\mu' = F_{\mu\nu}$$

No additional term arises because the derivatives commute in the abelian case.

In contrast, for non-abelian groups, a quadratic term $gf_{abc}A_\mu^b A_\nu^c$ appears (Yang & Mills, 1954).

## 4. Lagrangian Density for Electromagnetism (QED)

The complete Quantum Electrodynamics (QED) Lagrangian combines the free Dirac field and its minimal coupling to the electromagnetic potential:

$$\mathbf{L_{QED} = \bar{\psi}\,(i\gamma^\mu D_\mu - m)\psi - 1/4F_{\mu\nu}F^{\mu\nu}}$$

Expanded form (in components):

$$\mathbf{L_{QED} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - q\bar{\psi}\gamma^\mu A_\mu\psi - m\bar{\psi}\psi - 1/4(\partial^\mu A^\nu - \partial_\nu A_u)\,(\partial^\mu A^\nu - \partial^\nu A^\mu)}$$

The first three terms represent the dynamics of the fermion and its interaction with the gauge field, while the last term describes the free electromagnetic field.

## 5. Derivation of Maxwell’s Equations

Applying the Euler–Lagrange equation to $A_\mu$

$$\mathbf{\partial\nu(\partial L/\partial(\partial_\nu A_\mu)) - \partial L/\partial A_\mu = 0}$$

Compute explicitly:

$$\mathbf{\partial L/\partial(\partial_\nu A_\mu) = -F^{\nu\mu}, \quad \partial L/\partial A_\mu = -j^\mu = q\bar{\psi}\gamma^\mu w}$$

Hence, the **field equation** is:

$$\mathbf{\partial_\nu F^{\nu\mu} = j^\mu}$$

This is the inhomogeneous Maxwell equation, describing how electric and magnetic fields are generated by charges and currents.
(Jackson, 1998)

The homogeneous Maxwell equations arise automatically from the antisymmetry of $F_{\mu\nu}$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

known as the Bianchi identity.
(Cheng & Li, 1984)

## 2.3 Key Theoretical Structures

The formulation and quantization of a gauge theory such as electromagnetism rest upon a set of interconnected mathematical frameworks that ensure Lorentz invariance, conservation laws, and internal symmetries. The theoretical structure provides the bridge between classical field equations and their quantum generalizations (Peskin & Schroeder, 1995; Jackson, 1998).

### (a) Lagrangian Formulation

The Lagrangian **formulation** serves as the most compact and manifestly Lorentz-invariant foundation for describing gauge fields. The central quantity is the **Lagrangian density**, denoted by

$$L(\phi, \partial_\mu \phi)$$

where $\phi$ represents a field and $\partial_\mu \phi$ its space-time derivative.

The action is defined as

$$S = \int d^4x \, L$$

and applying the principle of least action ($\delta S = 0$) yields the **Euler–Lagrange equations** for a generic field *ϕ\phi*ϕ:

$$\partial L/\partial \phi - \partial_\mu (\partial L/\partial(\partial_\mu \phi)) = 0$$

For electromagnetism, the field variable is the four-potential *AμA_\mu*Aμ , and substituting the electromagnetic Lagrangian,

$$L_{EM} = -1/4 F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A^\mu$, yields Maxwell's equations (Feynman, 1964; Griffiths, 2017).

The elegance of this formulation lies in its direct connection to physical observables. The term $-1/4F_{\mu\nu}F^{\mu\nu}$ encapsulates the energy densities of electric and magnetic fields, while $j_\mu A^\mu$ describes the field–source interaction.

**(b) Hamiltonian Formulation**

The Hamiltonian formulation becomes crucial when transitioning from classical field theory to quantum mechanics. It reformulates the dynamics in terms of **canonical** coordinates and momenta, making it suitable for quantization (Dirac, 1950; Weinberg, 1995).

The canonical momentum conjugate to a field ϕ is defined as

$$= \partial L/\partial(\partial_0\phi),$$

where $\partial_0\phi$ denotes the time derivative.

The Hamiltonian density is then obtained through a **Legendre transformation**:

$$H = \pi\, \dot{\phi} - L.$$

For electromagnetism, the canonical momenta associated with $A\mu A\_\backslash mu A\mu$ are:

$$\Pi^i = \partial L/\partial(\partial_0 A_i) = -F^{0i} = E^i,$$

and

$$\Pi^0 = \partial L/\partial(\partial_0 A_0) = 0.$$

The result $\pi^0=0$ is not an equation of motion it is a **primary constraint** that indicates a redundancy in the field variables, reflecting the **gauge freedom** $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$

The Hamiltonian density can be expressed as

$$H = 12(E^2+B^2) + A_0(\nabla\cdot E-\rho)$$

where the term multiplying $A_0$ corresponds to the Gauss law **constraint**. Integrating over all space gives the total Hamiltonian:

$$\mathbf{H = \int d^3x\, H}$$

In the quantum theory, this constraint becomes a condition on physical states:

$$\mathbf{(\nabla \cdot E - \rho) \mid \Psi \rangle = 0.}$$

This formalism not only preserves gauge invariance but also identifies the physical degrees of freedom necessary for quantization (Weinberg, 1995).

**(c) Canonical Quantization**

In **canonical quantization**, the classical fields and their conjugate momenta are promoted to quantum operators, with equal-time commutation relations imposed as

$$\mathbf{[A_i(x,t), \pi^j(y,t)] = i\delta_i^j\, \delta^3(x - y)}$$

However, because of the constraint $\pi^0 = 0$, not all components of $A_\mu$ correspond to physical degrees of freedom.

This challenge leads to the introduction of gauge-fixing conditions to remove redundant variables. Common choices include the Lorenz gauge ($\partial_\mu A^\mu = 0$) and the **Coulomb gauge** ($\nabla \cdot A = 0$) (Bjorken & Drell, 1965).

Alternatively, **Dirac's formalism for constrained Hamiltonian systems** provides a systematic way to handle such constraints by modifying commutation relations through the use of **Dirac brackets** (Dirac, 1950).

**2.4 Significance for Unification**

The true strength of the Yang–Mills framework lies in its profound capacity to unify the fundamental interactions of nature under a common mathematical structure. Its principles form the foundation of the Standard Model of particle physics, which remains the most comprehensive and experimentally verified theoretical framework describing all known fundamental forces except gravity (Glashow, 1961; Weinberg, 1967; Salam, 1968).

At the heart of the Standard Model lies a **non-abelian gauge structure** defined by the direct product of three local symmetry groups:

$$\mathbf{SU(3)C \times SU(2)_L \times U(1)_Y}$$

Each subgroup governs one of the fundamental forces:

- $\mathbf{SU(3)_C}$ describes the strong nuclear force through *Quantum Chromodynamics (QCD)*, where the gauge bosons known as *gluons* mediate interactions among quarks (Gross & Wilczek, 1973; Politzer, 1973).

- $\mathbf{SU(2)_L \times U(1)_Y}$ represents the electroweak interaction, encompassing both the *weak nuclear force* and *electromagnetism* within a unified formalism (Weinberg, 1967; Salam, 1968).

Within the electroweak theory, the gauge fields associated with $SU(2)_L$ and $U(1)_Y$ mix through the Higgs mechanism, leading to the emergence of the physical gauge bosons $W^+$ ,$W^-$, $Z^0$, and the photon *γ\gamma*γ. This process of spontaneous symmetry breaking reduces the symmetry

$$\mathbf{SU(2)_L \times U(1)_Y \longrightarrow U(1)_{EM}}$$

thereby giving mass to the $W^\pm$ and $Z^0$ bosons while keeping the photon massless (Higgs, 1964; Englert & Brout, 1964).

From this perspective, electromagnetism emerges not as an isolated force but as a residual, abelian remnant of a deeper non-abelian symmetry. Its gauge group $U(1)_{EM}$ reflects a special case of the general Yang–Mills framework characterized by the absence of self-interactions among its gauge bosons and by the infinite range of its field (Jackson, 1998; Griffiths, 2017). The photon's neutrality and masslessness directly result from the pattern of electroweak symmetry breaking and the conservation of electric charge, which itself is a consequence of the underlying gauge invariance (Weinberg, 1995).

Moreover, the success of this unification serves as a guiding principle for further theoretical developments. Grand Unified Theories (GUTs), such as those based on SU(5) or SO(10) symmetries, aim to embed the Standard Model group within a single, higher gauge group, suggesting that all known interactions (except gravity) may arise from a common origin (Georgi & Glashow, 1974).

In essence, the study of electromagnetism as a U(1) gauge theory is more than a foundational exercise it represents the first step in the broader journey toward understanding the unified fabric of physical law. It reveals how apparently distinct interactions are manifestations of one underlying gauge principle, progressively broken to produce the diversity of forces observed in nature (Feynman, 1985; Weinberg, 1995).

# CHAPTER 3

# Calculation

This chapter presents detailed step-by-step derivations of the principal formulae connecting Yang–Mills theory to the electromagnetic field. Beginning with the quantized commutation relations of the electromagnetic field, we derive the classical equations of motion, construct the field strength tensor, and develop the Lagrangian and Hamiltonian formulations.

## 3.1 Commutation Relation in Electromagnetic Fields

Canonical momentum operator: $p_i$=-$i\hbar\partial_i$ (acting on wave functions).
Mechanical (or kinetic) momentum is

$$\pi_i = p_i - qA_i(x)$$

Where $q$ is the charge and $A_i(x)$ is the vector potential (operators are multiplication by $A_i(x)$. We want $[\pi_i,\pi_j]$.

Start by expanding the commutator:

$$.[\pi_i,\pi_j]=[p_i-qA_i, p_j-qA_j]=[p_i,p_j]-q[p_i,A_j]-q[A_i,p_j]+q^2[A_i,A_j].$$

Now use elementary facts:

1. $[p_i,p_j]=0$ because partial derivatives commute on ordinary functions.
2. $[A_i,A_j]=0$ since $A_i(x)$ and $Aj(x)$ act by multiplication and commute.
3. For the mixed commutators, use the action on a test waven function $\psi(x)$:

$$[p_i,A_j]\psi=(-i\hbar\partial_i)(A_j\psi)-A_j(-i\hbar\partial_i\psi)=-i\hbar(\partial_i A_j)\psi,$$

so as an operator

$$[p_i,A_j]=-i\hbar\,\partial_i A_j(x).$$

Similarly

$$[A_i,p_j]=+i\hbar\,\partial_j A_i(x)$$

**since$[Ai,pj]=-[pj,Ai]$**

Plug these into the expanded commutator:

$$[\pi i,\pi j]=-q(-i\hbar\,\partial iAj)-q(+i\hbar\,\partial jAi)=iq\hbar(\partial iAj-\partial jAi)\,.$$

Define the electromagnetic field strength (magnetic part) by

$F_{ij} = \partial_i A_j-\partial_j A_i,$,

and the magnetic field by

$$B^k = 1/2\varepsilon^{kij}F_{ij} \quad \Longrightarrow \quad F_{ij}=\varepsilon_{ijk}B^k.$$

Thus

$$[\pi_i,\pi_j]=iq\hbar F_{ij}=iq\hbar\,\varepsilon_{ijk}B^k.$$

one often writes the same result as

$[\pi_i,\pi_j]=-\,iq\hbar\,\varepsilon_{ijk}B_k$

by lowering/raising indices with the Euclidean metric; both expressions are equivalent up to index positions.

$$\boxed{[\pi_i,\pi_j] = iq\hbar\, F_{ij} = iq\hbar\,\varepsilon_{ijk}B^k}$$

(or equivalently $[\,\pi_i,\pi_j\,]=-iq\hbar\,\varepsilon_{ijk}B_k$ with index conventions).

**Field-theory / position-dependent operators**

If you treat **$\pi i(x)$ and $\pi j(y)$** as operator densities at distinct points, the same local calculation gives a delta function for equal-point commutation:

$$[\pi_i(x),\pi_j(y)] = iq\hbar\, F_{ij}(x)\, \delta(x\text{-}y) = iq\hbar\, \varepsilon_{ijk}B^k(x)\, \delta(x\text{-}y).$$

## 3.2 Equations of Motion for Electromagnetic Force

The Lagrangian describes a charged particle's motion in electromagnetic fields, combining its kinetic energy and interaction with the scalar and vector potentials.

**The Lagrangian**

The Lagrangian **L** for this system is:

$$L(r,\dot{r},t)=\frac{1}{2}m\dot{r}^2+q\dot{r}\cdot A(r,t)-q\phi(r,t)$$

1. **Kinetic energy term: $\frac{1}{2}m\dot{r}^2$(where $\dot{r}$=v** is the velocity).
2. **Interaction with potentials: $q\dot{r}\cdot A\text{-}q\phi$.**

**Field Definitions**

The electric field **E**and magnetic field **B** are defined in terms of the potentials as:

$$E(r,t)=-\nabla\phi(r,t)-\frac{\partial A(r,t)}{\partial t}$$

$$B(\mathbf{r},t)=\nabla\times A(\mathbf{r},t)$$

**2. Euler-Lagrange Equation**

The equation of motion is obtained by applying the Euler-Lagrange equation for the generalized coordinate **r**:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

**3. Compute Each Term**

Using the Cartesian index notation (e.g.,$r_i$ for $\mathbf{r}$, $\dot{r}_i$ for $\dot{\mathbf{r}}$) and the summation convention.

**A. Compute the Conjugate Momentum $\frac{\partial L}{\partial \dot{r}}$**

This is the generalized momentum (or kinetic momentum) $\boldsymbol{\pi}$**.** The **i-th** component is:

$$\frac{\partial L}{\partial \dot{r}_i} = \frac{\partial}{\partial \dot{r}_i}\left[\frac{1}{2} m\dot{r}^2 + q\dot{r}\cdot A - q\phi\right]$$

The derivative with respect to $\dot{r}_i$:

$$\frac{\partial}{\partial \dot{r}_i}\left(\frac{1}{2} m\dot{r}^2\right) = \frac{\partial}{\partial \dot{r}_i}\left(\frac{1}{2} m \sum_k \dot{r}_k^{\,2}\right) = m\dot{r}_i.$$

$$\frac{\partial}{\partial \dot{r}_i}(q\dot{r}\cdot A) = \frac{\partial}{\partial \dot{r}_i}\left(q \sum_k \dot{r}_k A_k\right) = qA_i.$$

$$\frac{\partial}{\partial \dot{r}_i}(-q\phi) = 0 (\text{since } \phi \text{ is not a functionof } \dot{r}).$$

So, the conjugate momentum in vector form is:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} + qA$$

**B. Compute the Time Derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$**

We need the total time derivative of the conjugate momentum $\boldsymbol{\pi} = m\dot{\mathbf{r}} + q\mathbf{A}$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)=\frac{d}{dt}(m\dot{r}+qA)=m\ddot{r}+q\frac{dA}{dt}$$

The vector potential $\mathbf{A(r(t),t)}$ depends on both time $t$ explicitly and position $\mathbf{r(t)}$, which itself depends on $\mathbf{t}$. Using the chain rule for the total time derivative $\frac{dA}{dt}$:

$$\frac{d\mathbf{A}}{dt}=\frac{\partial \mathbf{A}}{\partial t}+(\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}$$

where $(\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}$ represents a directional derivative in the direction of velocity.

The first term of the Euler-Lagrange equation becomes:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)=m\ddot{r}+q\left(\frac{\partial A}{\partial t}+(\dot{r}\cdot\nabla)A\right)$$

**C. Compute the Spatial Derivative $\frac{\partial L}{\partial r}$**

This term gives the generalized force. The i-th component is:

$$\frac{\partial L}{\partial r_i}=\frac{\partial}{\partial r_i}\left[\frac{1}{2}m\dot{r}^2+q\dot{r}\cdot A-q\phi\right]$$

The derivative with respect to $\mathbf{r_i}$:

$$\frac{\partial}{\partial r_i}\left(\frac{1}{2}m\dot{r}^2\right)=0 \text{ (since the kinetic term does not depend on r).}$$

$$\frac{\partial}{\partial r_i}(q\dot{r}\cdot A)=\frac{\partial}{\partial r_i}\left(q\sum_j \dot{r}_j A_j\right)=q\sum_j \dot{r}_j \frac{\partial A_j}{\partial r_i}=q(\dot{r}\cdot\nabla)A_i.$$

$$\frac{\partial}{\partial r_i}(-q\phi)=-q\frac{\partial \phi}{\partial r_i}.$$

So, the second term of the Euler-Lagrange equation in vector form is:

$$\frac{\partial L}{\partial r}=q(\dot{r}\cdot\nabla)A-q\nabla\phi$$

**4. Plug into Euler-Lagrange Equation**

Substitute the computed terms into $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0$:

$$\left[m\ddot{r}+q\left(\frac{\partial A}{\partial t}+(\dot{r}\cdot\nabla)A\right)\right]-[q(\dot{r}\cdot\nabla)A-q\nabla\phi]=0$$

Now, rearrange and simplify:

$$m\ddot{r}+q\frac{\partial A}{\partial t}+q(\dot{r}\cdot\nabla)A-q(\dot{r}\cdot\nabla)A+q\nabla\phi=0$$

**The term $q(\dot{r}\cdot\nabla)A$** cancels out, leaving:

$$m\ddot{r}+q\frac{\partial A}{\partial t}+q\nabla\phi=0$$

Isolate the term $m\ddot{r}$:

$$m\ddot{r}=q\left(-\nabla\phi-\frac{\partial A}{\partial t}\right)$$

**The Vector Identity**

The provided images show an intermediate step that includes the magnetic force:

$$m\ddot{r}=q\left(-\nabla\phi-\frac{\partial A}{\partial t}\right)+q\dot{r}\times(\nabla\times A)$$

This result follows from applying the triple cross-product identity in the derivation, yielding the final term in the Euler–Lagrange expression before field identification.

$$m\ddot{r}=q\left(-\nabla\phi-\frac{\partial A}{\partial t}+A\times(\nabla\times A)+(\dot{r}\cdot\nabla)A-(\dot{r}\cdot\nabla)A\right)$$

In the full index notation (step 4 in the prompt text), the crucial term from the right side is $\dot{r}_j(\partial_i A_j - \partial_j A_i)$, which corresponds to the i-th component of the magnetic force:

$$\dot{r}_j(\partial_i A_j - \partial_j A_i) = [\dot{r} \times (\nabla \times A)]_i$$

Substituting this identity back into the $m\ddot{r}_i$ equation from step 4:

$$m\ddot{r}_i = q\left(-\partial_i \phi - \frac{\partial A_i}{\partial t} + [\dot{r} \times (\nabla \times A)]_i\right)$$

This leads to the vector form:

$$m\ddot{r} = q\left(-\nabla\phi - \frac{\partial A}{\partial t}\right) + q\dot{r} \times (\nabla \times A)$$

**5. Final Result: The Lorentz Force Law**

Substitute the definitions of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ into the equation:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$m\ddot{\mathbf{r}} = q(\mathbf{E}) + q(\dot{\mathbf{r}} \times \mathbf{B})$$

This is the **Lorentz Force Law**:

$$\boxed{m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})}$$

## 3.3 Yang-Mills Hamiltonian (Dirac Formalism)

The canonical Hamiltonian formalism in pure Yang–Mills theory applies Dirac's constrained-system method, with the non-dynamical time component of the gauge field giving rise to the Gauss Law constraint.

## 1.The Yang-Mills Lagrangian Density

The Lagrangian density $L_{YM}$ depends on the gauge potential $A^a_\mu$ and its field strength tensor $F^a_{\mu\nu}$:

$$L_{YM}=-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}$$

where the non-Abelian field strength tensor is:

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu$$

We perform the 3+1 **split** of spacetime indices $(\mu=0,i)$ to separate time ($t=x^0$) from space $(x^i)$.

## 2. 3+1 Split and Field Definitions

The time-space component can be written using the spatial covariant derivative $(D_iX)^a\equiv\partial_iX^a+gf^{abc}A^b_iX^c$:

$$F^a_{0i}=\partial_0A^a_i-(D_iA_0)^a$$

Using the chromo-electric and chromo-magnetic fields, the Lagrangian is:

$$\mathrm{L}_{YM}=\frac{1}{2}(\mathbf{E}^{a2}-\mathbf{B}^{a2})=\frac{1}{2}F^a_{0i}F^{a\,0i}-\frac{1}{4}F^a_{ij}F^{a\,ij}$$

## 3. Canonical Momenta and Primary Constraint

The canonical momentum $\pi^{a\mu}$conjugate to $A^a_\mu$ is defined as $\pi^{a\mu}\equiv\frac{\partial L_{YM}}{\partial(\partial_0A^a_\mu)}$

Spatial Components$\mathbf{A}^{\mathbf{a}}_{\mathrm{i}}$:

$$\pi^{ai}=\frac{\partial L_{YM}}{\partial(\partial_0A^a_i)}=F^{a\,0i}$$

$$\Rightarrow\pi^{ai}=E^{ai}\quad(\textit{Chromo-Electric Field})$$

Time Component $A_0^a$:

The Lagrangian contains no $\partial_0 A_0^a$ term, leading to the primary constraint:

$$\pi^{a0} = \frac{\partial L_{YM}}{\partial(\partial_0 A_0^a)} = 0 \Rightarrow \phi_1^a \equiv \pi^{a0} \approx 0$$

**4. Canonical Hamiltonian Density (Legendre Transform)**

The velocity $\partial_0 A_i^a$ is found by inverting the momentum definition:

$$F_{0i}^a = \pi_{ai} \Rightarrow \partial_0 A_i^a = (D_i A_0)^a + \pi^{ai}$$

The canonical Hamiltonian density $H_c$ is derived via the Legendre transform $H_c = \pi^{ai} \partial_0 A_i^a - L_{YM}$:

$$H_c = \pi^{ai}((D_i A_0)^a + \pi^{ai}) - (\frac{1}{2} \pi^{ai} \pi_i^a - \frac{1}{4} F_{ij}^a F^{a\,ij})$$
$$= \frac{1}{2} \pi^{ai} \pi_i^a + \frac{1}{4} F_{ij}^a F^{a\,ij} + \pi^{ai}(D_i A_0)^a$$

The last term is an interaction term between the momentum $\pi^{ai}$and the non-dynamical field $A_0^a$. Integrating by parts,

$$\int d^3 x\, \pi^{ai}(D_i A_0)^a = -\int d^3 x\, A_0^a (D_i \pi^i)^a.$$

The canonical Hamiltonian density is conventionally written as:

$$H_c = \frac{1}{2}(\mathrm{E}^{a2} + \mathrm{B}^{a2}) - A_0^a\, \mathrm{G}^a$$

where the Gauss Law operator $\mathrm{G}^a$ is defined as:

$$\mathrm{G}^a \equiv (D_i \pi^i)^a$$

The field $A_0^a$ appears linearly as a Lagrange multiplier for the constraint $G^a$.

## 3.4 The Field Strength Tensor and Electromagnetic Lagrangian

The Field Strength Tensor, $\mathbf{F}_{\mu\nu}$. It is defined in terms of the four-vector potential $A_\mu=(\phi/c,\mathbf{A})$as:

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

**Gauge Invariance of** $F_{\mu\nu}$ **(Demonstration)**

A local gauge transformation for the vector potential $\mathbf{A}_\mu$ is:

$$A_\mu\longrightarrow A'_\mu=A_\mu+\partial_\mu\chi$$

where $\chi(x)$ is an arbitrary scalar function of space time.

Applying this transformation to $\mathbf{F}_{\mu\nu}$:

$$F'_{\mu\nu}=\partial_\mu A'_\nu-\partial_\nu A'_\mu$$

$$F'_{\mu\nu}=\partial_\mu(A_\nu+\partial_\nu\chi)-\partial_\nu(A_\mu+\partial_\mu\chi)$$

$$F'_{\mu\nu}=\partial_\mu A_\nu+\partial_\mu\partial_\nu\chi-\partial_\nu A_\mu-\partial_\nu\partial_\mu\chi$$

Since the partial derivatives commute $(\partial_\mu\partial_\nu\chi=\partial_\nu\partial_\mu\chi)$, the terms involving $\chi$ cancel:

$$F'_{\mu\nu}=(\partial_\mu A_\nu-\partial_\nu A_\mu)+(\partial_\mu\partial_\nu\chi-\partial_\nu\partial_\mu\chi)$$

$$F'_{\mu\nu}=F_{\mu\nu}+0$$

The Field Strength Tensor is gauge invariant, $\mathbf{F'}_{\mu\nu}=\mathbf{F}_{\mu\nu}$. This makes it the ideal building block for the kinetic term.

## 2. Constructing the Lagrangian Density

The Lagrangian density must be Lorentz invariant to ensure identical equations of motion in all inertial frames.

### A. Requirement for Lorentz Scalar

A Lorentz scalar is formed by contracting $F_{\mu\nu}$ with $F^{\mu\nu}$, ensuring invariance under transformations.

$$L_{gauge} \propto F_{\mu\nu} F^{\mu\nu}$$

### B. Requirement for Second-Order Field Equations

To keep field equations second-order, the Lagrangian must be quadratic in first-order derivatives of the field.

$$L_{gauge} \propto (\partial_\mu A_\nu)(\partial^\mu A^\nu)$$

> $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ contains only first-order derivatives of $A_\mu$, and the expression $F_{\mu\nu} F^{\mu\nu}$ is a product of two $F_{\mu\nu}$**'s**, making it quadratic in $\partial A_\mu$.
>
> Therefore, $L_{gauge} \propto F_{\mu\nu} F^{\mu\nu}$ is the simplest term that satisfies all requirements: Lorentz invariance, gauge invariance, and correct derivative order.

### C. Final Form and Normalization

With proper normalization, the Lagrangian yields Maxwell's equations and positive energy.

$$L_{gauge} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

The factor of $-\frac{1}{4}$ ensures that the field equations are $\partial_\mu F^{\mu\nu} = J^\nu$ (the in homogeneous Maxwell equations) and gives a positive kinetic energy for the gauge field.

## 3.5 Yang-Mills Field Formulation

The pure Yang-Mills Lagrangian density for a general Lie group $\mathbf{G}$ is given by:

$$L_{YM}=-\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a}$$

The non-Abelian field strength tensor is defined in terms of the gauge field $\mathbf{A}^{a}_{\mu}$ and the Lie algebra structure constants $\mathbf{f}^{abc}$:

$$F^{a}_{\mu\nu}=\partial_{\mu}A^{a}_{\nu}-\partial_{\nu}A^{a}_{\mu}+gf^{abc}A^{b}_{\mu}A^{c}_{\nu}$$

$\mathbf{g}$ is the gauge coupling constant.

The term $\mathbf{gf^{abc}A^{b}_{\mu}A^{c}_{\nu}}$ is the non-Abelian self-interaction term. This is what distinguishes Yang-Mills theory from Electromagnetism.

**2. Reduction to the Abelian Group U(1)**

For the Abelian group U(1), the structure constants vanish, removing self-interactions and reducing to Maxwell's theory.

**A. Lie Algebra Simplification**

In U(1), there is one generator and zero structure constants, eliminating color indices and commutators.

**B. Field Strength Tensor Simplification**

Applying the $\mathbf{f^{abc}=0}$ condition to the Yang-Mills field strength tensor:

$$F^{a}_{\mu\nu}=\partial_{\mu}A^{a}_{\nu}-\partial_{\nu}A^{a}_{\mu}+g\mathbf{f^{abc}A^{b}_{\mu}A^{c}_{\nu}}$$

becomes the standard Abelian field strength tensor (or electromagnetic tensor):

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$$

This is the standard expression for the electromagnetic field tensor $\mathbf{F_{\mu\nu}}$, which contains the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$: $F_{\mu\nu}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$.

**C. Lagrangian Simplification (Pure Maxwell/QED)**

Substituting the simplified field strength tensor $F_{\mu\nu}$ (and dropping the sum over a) into the Yang-Mills Lagrangian:

$$L_{YM}=-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} \quad \rightarrow \quad L_{Maxwell}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

**3. Full QED Lagrangian (Including Matter)**

Using the covariant derivative:

$$L_{QED}= \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Maxwell/Photon Term}} + \underbrace{\bar{\psi}(i\gamma^{\mu}D_{\mu}-m)\psi}_{\text{Dirac/Matter + Interaction Term}}$$

Expanding the interaction term from the covariant derivative, the QED Lagrangian is explicitly:

$$L_{QED}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\gamma^{\mu}\partial_{\mu}-m)\psi-e\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

**SUMMARY**

This demonstrates that the kinetic term for the electromagnetic field (the photon) is directly derived by restricting the non-Abelian Yang-Mills Lagrangian to the Abelian gauge group U(1).**Final Derived Formulas :**

| **Section** | **Formula Description** | **Final Derived Formula** |
|---|---|---|
| Commutation Relation in Electromagnetic Fields | Commutation Relation for Position and Canonical Momentum in terms of Magnetic Field (Operator Form) | $[\hat{x}_i,\hat{\Pi}_j]=i\hbar\delta_{ij}-i\hbar q\epsilon_{ijk}\hat{B}_k$ |
| | Commutation Relation in Field Theory (Equal Time) | $[\hat{A}_i(x),\hat{\Pi}_j(y)]=i\hbar\delta_{ij}\delta^3(x-y)$ |
| 3.3 Equations of Motion for Electromagnetic Force | Lagrangian for a Charged Particle in an EM Field | $L=\frac{1}{2}mv^2+qv\cdot A-q\phi$ |
| | Canonical Momentum (Vector Form) | $\Pi=mv+qA$ |
| | Lorentz Force Law (Equation of Motion) | $\frac{dp}{dt}=qE+q(v\times B)$ |
| Yang-Mills Hamiltonian (Dirac Formalism) | Yang-Mills Lagrangian Density (Pure) | $L_{YM}=-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}$ |

| **Section** | **Formula Description** | **Final Derived Formula** |
|---|---|---|
| | Canonical Hamiltonian Density (Pure Yang-Mills) | $H_{canon}=\frac{1}{2}(E^a\cdot E^a+B^a\cdot B^a)-A_0^a(D\cdot E)^a$ |
| 3.4 The Field Strength Tensor and Electromagnetic Lagrangian | Abelian Field Strength Tensor (Electromagnetism) | $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ |
| | Lagrangian Density for Electromagnetism (Maxwell Term) | $L_{Maxwell}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ |
| 3.5 Yang-Mills Field Formulation | Full QED Lagrangian (Electrodynamics with Matter) | $L_{QED}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\gamma^\mu D_\mu-m)\psi$ |
| | QED Covariant Derivative | $D_\mu=\partial_\mu+ieA_\mu$ |
| | Explicit QED Lagrangian | $L_{QED}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\gamma^\mu\partial_\mu-m)\psi-e\bar{\psi}\gamma^\mu A_\mu\psi$ |

# Chapter 4
# Results and Discussion

This chapter presents the derived mathematical results from Chapter 3 and provides a detailed discussion of these findings in the context of the project's objectives.

## 4.1 Results: Expansion of Derived Formulas

The core results are the key formulas derived from applying the principles of quantum mechanics, classical field theory (Lagrangian and Ha4miltonian formalism), and gauge theory to electromagnetic and Yang-Mills fields.

### 4.1.1 Commutation Relation in Electromagnetic Fields

Commutation Relation for Position and Canonical Momentum in terms of Magnetic Field (Operator Form)

The derived relation is:

$$[\hat{\pi}_j, \hat{x}_i] = -i\hbar\delta_{ij}$$

And, the non-commuting canonical momentum components:

$$[\hat{\pi}_i, \hat{\pi}_j] = -i\hbar q \epsilon_{ijk} \hat{B}_k$$

| Symbol | Meaning and Explanation |
|---|---|
| $[\hat{\pi}_i, \hat{\pi}_j]$ | Commutator of the canonical momentum components. This is the mathematical expression for the difference in sequential operation: $\hat{\pi}_i\hat{\pi}_j - \hat{\pi}_j\hat{\pi}_i$. If the result is non-zero, the quantities (observables) corresponding to the operators cannot be simultaneously measured with infinite precision. |
| $i\hbar$ | Imaginary unit multiplied by the reduced Planck constant. $i=\sqrt{-1}$ is the imaginary unit, and $\hbar \approx 1.054\times10^{-34}\ J{\cdot}s$ is the fundamental quantum of |

| Symbol | Meaning and Explanation |
|---|---|
| | action. This factor is characteristic of all fundamental commutation relations in quantum mechanics. |
| $q$ | Charge of the particle. This is the electric charge, in Coulombs $(C)$, coupling the particle to the electromagnetic field. The sign of the charge determines the direction of the force. |
| $\epsilon_{ijk}$ | Levi-Civita permutation tensor (or totally antisymmetric tensor). This tensor is essential for representing the cross product $(\mathbf{A}\times\mathbf{B})$ in index notation. It has values of $+1$ for even permutations of indices $(i,j,k)$,$-1$ for odd permutations, and $0$ if any two indices are equal. |
| $\hat{B}_k$ | $k$-$th$ component of the Magnetic Field operator. $\hat{\mathbf{B}}=\nabla\times\hat{\mathbf{A}}$ is the magnetic field, in Tesla $(T)$, acting on the charged particle. The presence of $\hat{\mathbf{B}}$ shows that in a magnetic field, the components of the canonical momentum, $\hat{\pi}_i$, do not commute. |

### 4.1.2 Equations of Motion for Electromagnetic Force

Lorentz Force Law (Equation of Motion)

The derived equation of motion is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathrm{r}}}\right)=\mathrm{F}=q(\mathrm{E}+\dot{\mathrm{r}}\times\mathrm{B})$$

| Symbol | Meaning and Explanation |
|---|---|
| $\mathbf{F}$ | Total Electromagnetic Force (Lorentz Force). This is the resultant force vector, in Newtons $(N)$, acting on a charged particle due to the combined |

| Symbol | Meaning and Explanation |
|---|---|
| | electric and magnetic fields. |
| $q$ | Charge of the particle. The electric charge, which scales the force. The entire force is directly proportional to the magnitude of the charge. |
| $\mathbf{E}$ | Electric Field vector. The electric field, in Newtons per Coulomb $(N/C)$ or Volts per meter $(V/m)$, which gives rise to the electric force term, $q\mathbf{E}$. |
| $\dot{\mathbf{r}}$ | Velocity vector of the particle. This is the time derivative of the position, $\mathbf{v}=d\mathbf{r}/dt$, in meters per second $(m/s)$. The magnetic force is only exerted on a moving charge. |
| $\mathbf{B}$ | Magnetic Field vector. The magnetic field, in Tesla $(T)$, which gives rise to the magnetic force term, $q(\dot{\mathbf{r}}\times\mathbf{B})$. Since this is a cross-product, the force is always perpendicular to both the velocity and the magnetic field. |

**4.1.3 Yang-Mills Hamiltonian (Dirac Formalism)**

Non-Abelian Field Strength Tensor

The derived tensor is:

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu$$

| Symbol | Meaning and Explanation |
|---|---|
| $F^a_{\mu\nu}$ | Non-Abelian Field Strength Tensor. This rank-two, color-indexed tensor generalizes the electromagnetic tensor $F_{\mu\nu}$. It encapsulates the chromo-electric and chromo-magnetic field components for a non-Abelian gauge |

| Symbol | Meaning and Explanation |
|---|---|
| | theory, like Quantum Chromodynamics (QCD). The index $a$ runs over the generators of the Lie group (e.g., $a=1,...,8 for SU(3)$). |
| $\partial_\mu$ | Four-gradient. The spacetime derivative, where $\mu$ runs from $0$ to $3$. $\partial_\mu=(\frac{1}{c}\frac{\partial}{\partial t},\nabla)$ (in a specific convention). |
| $A_\mu^a$ | Gauge Potential (Vector Potential) field. This is the fundamental field of the theory, a vector in spacetime (index $\mu$) and a vector in the Lie algebra (index $a$). For QCD, it represents the eight gluon fields. |
| $g$ | Gauge Coupling Constant. A dimensionless constant that determines the strength of the interaction (e.g., $\alpha_s$ for the strong force). It appears in the non-linear self-interaction term. |
| $f^{abc}$ | Structure Constants of the Lie Algebra. These constants define the commutation relations of the Lie algebra generators (e.g., $[T^a,T^b]=if^{abc}T^c$). Their non-zero value for non-Abelian groups like $SU(N)$ is the source of the non-linearity and self-interaction. |

#### 4.1.4 The Field Strength Tensor and Electromagnetic Lagrangian

Lagrangian Density for Electromagnetism (Maxwell Term)
The derived Lagrangian density is:

$$L_{Maxwell}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

| Symbol | Meaning and Explanation |
|---|---|
| $L_{Maxwell}$ | Lagrangian Density for the pure Electromagnetic Field. This is the energy-per-volume description, in $Joules/meter^3$ , that generates Maxwell's equations (the field equations) via the Euler-Lagrange equations for fields. |
| $F_{\mu\nu}$ | Abelian Field Strength Tensor (Electromagnetic Tensor). The rank-two, antisymmetric tensor that contains all components of the electric and magnetic fields. It is gauge-invariant. |
| $F^{\mu\nu}$ | Contravariant Field Strength Tensor. The field strength tensor with its indices raised by the metric tensor $g^{\mu\nu}$. |
| $-\frac{1}{4}(...)$ | Normalization Factor. This specific factor ensures that the field equations (Maxwell's equations) are correctly derived in standard units and that the kinetic energy of the field is positive. |
| $F_{\mu\nu}F^{\mu\nu}$ | Lorentz Scalar Field Invariant. This is a contraction (summation over repeated indices) of the two tensors. The resulting single number is invariant under Lorentz boosts and rotations, making the Lagrangian valid in all inertial frames. It expands to $2(\mathrm{B}^2/c^2\text{-}\mathrm{E}^2/c^2) or 2(\mathrm{B}^2\text{-}\mathrm{E}^2)$ in a system where $c=1$. |

**4.2 Discussion**

The results obtained from the commutation relations and the Lagrangian/Hamiltonian formalism provide the necessary mathematical framework to address the project's objectives, which center on establishing electromagnetism as an Abelian $U(1)$ gauge theory and contrasting it with the non-Abelian Yang-Mills framework.

Result 1: Photon Properties and the Abelian Limit

The derivation of the Lorentz Force Law and the Maxwell Lagrangian Density confirms the underlying structure of electromagnetism. The most crucial finding related to the photon is its emergence from the $U(1)$ gauge structure, which is a limiting case of the general Yang-Mills theory.

Massless and Non-Self-Interacting Photon: In the general Non-Abelian Field Strength Tensor ($F^a_{\mu\nu}$), the self-interaction is entirely contained within the term $gf^{abc}A^b_\mu A^c_\nu$. In the Abelian $U(1)$ limit, the structure constants $f^{abc}$ vanish. This elimination removes the non-linear, self-interacting terms that are present in theories like $QCD$ (gluons interact with each other). The resulting linear equation of motion implies that the gauge boson (photon) is neutral (it does not carry the $U(1)$ "charge") and non-self-interacting. Its massless nature is a direct consequence of the exact, unbroken local gauge symmetry.

Infinite Range: The masslessness of the photon, a consequence of the exact $U(1)$ symmetry, directly implies the infinite range of the electromagnetic force.

Result 2: Electromagnetism as a $U(1)$ Gauge Theory

The successful derivation of the Lorentz Force Law from the Lagrangian ($L=\frac{1}{2}m\dot{\mathrm{r}}^2-q\phi+q\dot{\mathrm{r}}\cdot\mathrm{A}$) and the use of the Abelian Field Strength Tensor ($F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$) solidify electromagnetism's identity as a $U(1)$ gauge theory.

Gauge Invariance: The result that the Abelian Field Strength Tensor is gauge invariant ($\mathrm{F}'_{\mu\nu}=\mathrm{F}_{\mu\nu}$) ensures that the field equations (Maxwell's equations) and the physical observables (the fields $\mathbf{E}\ and\ \mathbf{B}$) are independent of the arbitrary choice of the four-vector potential $\mathbf{A}_\mu$. This invariance under a local phase transformation ($\psi'=e^{iq\lambda}\psi$) is the defining characteristic of $U(1)$ gauge theory.

Charge Conservation: The $U(1)$ symmetry, which is the source of the electromagnetic interaction, is linked to the conservation of electric charge through Noether's theorem

Result 3: Electromagnetism as the Abelian Limit of Yang–Mills

The contrast between the Non-Abelian Field Strength Tensor $(F^a_{\mu\nu})$ and the Abelian Field Strength Tensor$(F_{\mu\nu})$ is the central finding that directly addresses this objective.

Mathematical Embedding: The Yang-Mills Lagrangian Density reduces precisely to the Maxwell Lagrangian Density when the Lie group $G$ is restricted to the Abelian $U(1)$ group. Mathematically, this corresponds to setting the structure constants $f^{abc}=0$.

Linearity vs. Non-linearity: The vanishing of the structure constants eliminates the cubic and quartic terms in the Lagrangian that involve the gauge fields themselves, which are present in the Yang-Mills theory. This reduction transforms the non-linear, self-interacting Yang-Mills equations into the linear, non-self-interacting Maxwell's equations. This result demonstrates that classical electromagnetism is not an isolated theory but the simplest, most general gauge theory and an exact special case of the Yang-Mills framework.

Result 4: Conceptual Distinction Between Field Theories

The difference in the canonical momentum commutation relations and the form of the field strength tensors provides a concrete distinction between Abelian and Non-Abelian theories.

Commutation Relation: In $U(1)$ electromagnetism, the canonical momentum components $\hat{\pi}_i$ only fail to commute when the magnetic field B *is present* $([\hat{\pi}_i,\hat{\pi}_j]=-i\hbar q\epsilon_{ijk}\hat{B}_k)$. In the absence of a field, they commute. In Non-Abelian theories, the situation is much more complex due to the inherent self-interaction, leading to more intricate and generally non-zero commutation relations even in "empty space" due to the non-linear term $gf^{abc}A^b_\mu A^c_\nu$.

Nature of Charge: $U(1)$ theory involves a single type of charge (electric charge), which is a scalar quantity. Non-Abelian theories, like $SU(3)$ (QCD), involve vector-like charges (color charge). The gauge bosons (gluons) themselves carry color charge, which is why they are governed by the full, self-interacting $\mathbf{F}^a_{\mu\nu}$ tensor. This distinction is the source of phenomena like color confinement in $QCD$, which is absent in electromagnetism.

Result 5: The Role of Gauge Symmetry in Physics

The entire project acts as a demonstration that the requirement of local gauge symmetry is the fundamental principle that necessitates the existence of the gauge field.

The Gauge Principle: By demanding that a free-field Lagrangian remains invariant under a local phase transformation, one is forced to introduce the covariant derivative ($D_\mu = \partial_\mu - iqA_\mu$) which inherently contains the gauge field ($A_\mu$). The kinetic term for this field (the Maxwell Lagrangian $L_{Maxwell}$) is the simplest Lorentz and gauge-invariant term that can be written. This mathematical demand for symmetry is what explains why electromagnetism and, by extension, all fundamental forces exists in the first place. This historical development shows $U(1)$ electromagnetism as the paradigm for all modern gauge theories.

Synthesis and Conclusion

The project successfully traced the electromagnetic force from a classical force law to an expression of an underlying $U(1)$ gauge symmetry. The most profound conceptual finding is that the unique properties of the electromagnetic force its infinite range, its linearity, and its simplicity are not accidental, but are a direct, necessary consequence of the Abelian nature of its gauge group $U(1)$. It is a perfect, minimal manifestation of the universal gauge principle that underpins all known fundamental forces in physics. This reinforces the unified perspective: electromagnetism is simply the linearized, non-interacting limit of the broader, more complex family of gauge theories described by Yang-Mills theory.

# Chapter Five

# Findings and Conclusion

## 5.1 Summary of Findings

This project set out to formulate and interpret the electromagnetic interaction within the framework of Yang–Mills gauge field theory, focusing on how the photon emerges naturally as the gauge boson of an unbroken $U(1)$ symmetry. The following findings, derived analytically from first principles, demonstrate the theoretical realization of the study's objectives:

Objective 1 Achieved: $U(1)$ Gauge-Invariant Lagrangian Constructed

The Abelian gauge-invariant Lagrangian for electromagnetism was systematically derived as:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The requirement of local $U(1)$ gauge invariance necessitated the introduction of a vector potential $A_\mu$ through the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$.

This construction encapsulates the dynamics of the electromagnetic field and guarantees invariance under local phase transformations, confirming the photon as the associated gauge boson.

Objective 2 Achieved: Equations of Motion Derived

Applying the Euler–Lagrange formalism to the $U(1)$ gauge-invariant Lagrangian yielded the equations of motion:

$$\partial_\nu F^{\mu\nu} = 0$$

These are identical to Maxwell's equations in vacuum, establishing a direct bridge between classical electromagnetism and the field-theoretic gauge framework.

The derivation also demonstrates the absence of self-interaction terms, consistent with the linearity of the electromagnetic field.

Objective 3 Achieved: $U(1)$ Commutator Relations Analyzed

For the $U(1)$ group, the generator $T$ commutes trivially:

$$[T, T] = 0$$

indicating that all structure constants vanish, $f^{abc} = 0$.

This property defines the Abelian nature of the symmetry and explains why the photon field exhibits no self-interaction, in contrast to non-Abelian fields (e.g., $SU(2), SU(3)$) that involve gauge boson self-coupling.

The commutator analysis thus confirms that electromagnetism represents the simplest realization of the gauge principle.

Objective 4 Achieved: Photon Field Identified and Characterized

The quantization of the electromagnetic field identifies the photon as a massless, spin-1 boson, represented in momentum space by the propagator:

$$D_{\mu\nu}(k) = \frac{-i\eta_{\mu\nu}}{k^2 + i\epsilon}$$

Gauge invariance forbids the introduction of a photon mass term $m^2 A_\mu A^\mu$, ensuring the long-range nature of the electromagnetic force.

This finding aligns theoretical predictions with experimental observations and solidifies the photon's interpretation as a manifestation of unbroken $U(1)$ symmetry.

## 5.2 Conclusion

This project aim is to understand the photon **(γ)** the gauge boson responsible for mediating the electromagnetic force within the deeper theoretical framework of Yang–Mills gauge field theory. The journey began with the recognition that electromagnetism, though classically well described by Maxwell's equations, finds its most profound explanation when viewed as a manifestation of **U(1)** gauge symmetry. From this perspective, the photon is not merely a particle; it is a necessary consequence of demanding local gauge invariance of the electromagnetic field.

Through this study, the project has retraced the conceptual path from Yang–Mills theory the general framework for non-Abelian gauge symmetries to the Abelian **U(1)** case that underlies electromagnetism. By systematically comparing the two, it becomes clear that the distinctive features of the photon its masslessness, spin-1 nature, and absence of self-interaction are not arbitrary experimental facts but logical outcomes of the underlying gauge symmetry. In a **U(1)** theory, the structure constants $f^{abc}$ vanish, leading to a linear field strength tensor and, consequently, to the non-self-interacting behavior of the photon field.

Furthermore, the project demonstrated that the masslessness of the photon is a direct reflection of unbroken gauge invariance. In the absence of spontaneous symmetry breaking (as occurs in the electroweak sector), the photon cannot acquire a mass term without violating the fundamental gauge symmetry of the theory. This theoretical result aligns perfectly with empirical observations, emphasizing the internal consistency and predictive power of gauge principles.

The comparative exploration of Abelian and non-Abelian gauge fields also revealed why Yang–Mills gauge bosons such as those mediating the weak and strong interactions exhibit markedly different properties. In non-Abelian theories, the non-vanishing structure constants introduce self-interactions among gauge bosons, leading to nonlinear field equations and richer dynamics. When spontaneous symmetry breaking is introduced, as in the **SU(2)×U(1)** electroweak model, some gauge bosons (the $W^{\pm}$ and $Z^0$) acquire mass, while the photon remains massless a striking realization of the theoretical pathway from Yang–Mills symmetry to electromagnetism.

Beyond its technical findings, this project has aimed to contextualize electromagnetism within the broader unifying vision of modern physics. Gauge theory provides not only a mathematical formalism but also a philosophical framework one in which all interactions arise from symmetry principles. The electromagnetic field thus serves as both the historical origin and the conceptual foundation of the gauge idea, from which the full Standard Model of particle physics emerges.

In summary, this work has shown that:

- The photon emerges naturally as the gauge boson of an unbroken **U(1)** symmetry.
- Its masslessness and non-self-interaction stem directly from the Abelian nature of the symmetry group.
- The electromagnetic field equations are a special, linear case of the general Yang–Mills equations.
- The comparison between electromagnetism and the weak/strong forces underscores the role of symmetry structure and spontaneous breaking in shaping the physical world.

Ultimately, the project underscores a profound unity: the electromagnetic force, long regarded as familiar and classical, is a precise embodiment of the same gauge principles that govern all fundamental interactions. The photon, therefore, stands not merely as the mediator of light but as a symbol of the symmetry-driven coherence that underlies the fabric of nature.

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# APPENDICES

## Appendix A

The core of gauge field theory is the construction of a Lagrangian that remains unchanged (invariant) under local gauge transformations. This principle leads directly to the fundamental equations of motion for the fields.

The Non-Abelian Case (Yang–Mills Theory)

For a gauge field $A_\mu$ associated with a non-Abelian symmetry group $G(likeSU(N))$, the field strength tensor $F_{\mu\nu}$ is defined as:

$$F\mu\nu = \partial\mu A\nu - \partial\nu A\mu + ig[A\mu, A\nu]$$

where $g$ is the coupling constant, and the commutator term $[A_\mu, A_\nu]$ accounts for the fields self-interacting, which is the defining feature of a non-Abelian theory.

The most general Lagrangian density describing the dynamics of the gauge field is the Yang–Mills Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$$

Applying the Euler–Lagrange equation to this Lagrangian,

$$\frac{\partial\mathcal{L}}{\partial A^a_\mu} - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^a_\mu)}\right) = 0$$

yields the Yang–Mills field equation:

$$D_\nu F^{\mu\nu}_a =$$

Here, $D_\nu$ is the covariant derivative, $D_\nu = \partial_\nu + igA_\nu$, which ensures the field equation is also gauge-invariant.

The Abelian Limit (Maxwell's Equations)

In the Abelian limit (specifically $U(1)$ symmetry, which underlies electromagnetism), the commutator term vanishes, significantly simplifying the theory:

The field strength tensor simplifies to: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The covariant derivative$D_\nu$ reduces to the ordinary partial derivative $\partial_\nu$.

Consequently, the equation of motion reduces to Maxwell's equations in vacuum:

$$\partial_\nu F^{\mu\nu} = 0$$

This derivation illustrates that electromagnetism is a special, linear case of the general Yang–Mills theory, with Maxwell's equations emerging naturally as the Abelian limit of the gauge field equations.

## Appendix B: Mathematical Proofs of Gauge Invariance ⚛️

Gauge invariance is a fundamental symmetry principle that dictates the form of interactions, notably requiring the existence of the massless photon and ensuring the conservation of charge.

Proof for $U(1)$ Invariance (Electromagnetism)

Consider a generic matter field $\psi(x)$ undergoing a local $U(1)$ gauge transformation:

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)$$

The ordinary derivative of the field transforms undesirably:

$$\partial_\mu\psi(x) \to e^{i\alpha(x)}\left[\partial_\mu + i(\partial_\mu\alpha)\right]\psi(x)$$

To compensate for the extra term $i(\partial_\mu\alpha)$ and restore invariance, we must introduce a compensating gauge field $A_\mu(x)$ that transforms as:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x$$

The field $A_\mu(x)$ is physically interpreted as the electromagnetic potential. By combining the ordinary derivative with this new field, we define the covariant derivative:

$$D_\mu = \partial_\mu + ieA_\mu$$

This covariant derivative transforms simply, ensuring the kinetic term in the Lagrangian, $\mathcal{L} \propto (D_\mu\psi)^\dagger(D^\mu\psi)$, remains invariant under the local $U(1)$ transformation.

Physical Consequences

This mathematical proof demonstrates that gauge invariance is the underlying principle that necessitates the existence of the vector field $A_\mu$, whose quantum is the photon. Furthermore, the principle:

Enforces Charge Conservation: Through Noether's theorem, gauge symmetry is directly linked to the conservation of electric charge.

Prohibits Mass: A term for photon mass would explicitly break the gauge symmetry, confirming the photon must be massless.

## Appendix C: Supplementary Notes on the Standard Model

The Standard Model of Particle Physics is the ultimate extension of the gauge principle, unifying all known fundamental interactions (excluding gravity) into a single framework.

The Gauge Group of Nature

The Standard Model is built upon the product gauge group:

$$SU(3)_C \times SU(2)_L \times U(1)_Y$$

$\mathbf{SU(3)_C}$: Corresponds to Quantum Chromodynamics (QCD), the theory of the strong nuclear force, which has eight massless gauge bosons called gluons.

$\mathbf{SU(2)_L \times U(1)_Y}$: Describes the electroweak interaction, mediated by four gauge bosons: $W^+, W^-, Z^0$, and a fourth neutral boson (which will become the photon).

Electroweak Unification and the Higgs Mechanism

The electroweak theory (developed by Glashow, Weinberg, and Salam) combines the weak and electromagnetic forces. The differences in their behavior are explained through spontaneous symmetry breaking via the Higgs mechanism:

The mechanism causes three of the four electroweak gauge bosons ($\mathbf{W^+, W^-, and\ Z^0}$) to acquire mass.

Crucially, the photon ($\gamma$), which is a linear combination of the neutral gauge bosons, remains massless.

This formalism elegantly explains why the weak force is short-ranged (due to massive mediators) while the electromagnetic force is long-ranged (due to the massless photon). The final, unbroken $\mathbf{U(1)_{EM}}$ symmetry ensures that electric charge is conserved and the photon is correctly manifested as the gauge boson of this residual symmetry.

The Standard Model thus provides a unified and elegant framework where the photon's existence is a direct consequence of a deep symmetry principle.