LINEARIZED WATER WAVE THEORY

BY

MERCY NWANSIMDI SOMEZE

MAT.NO. PSC1909096

DEPARTMENT OF MATHEMATICS

FACULTY OF PHYSICAL SCIENCES

UNIVERSITY OF BENIN, BENIN CITY, EDO STATE, NIGERIA.

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UNDERTAKING

This project work was carried out by me, **SOMEZE NWANSIMDI MERCY** with the Matriculation Number **PSC1909096.**

I have not copied the work of any author, all works have been duly cited and acknowledged.

Someze Nwansimdi Mercy Date

CERTIFICATION

This is to certify that this project work titled **'Linearized Water Wave Theory'** was carried out by SOMEZE MERCY NWANSIMDI MERCY in the Department of Mathematics, University of Benin, Benin City.

Prof. S. E. Uwamusi **Date**

(Project Supervisor)

Prof. Robert .I. Okuonghae **Date**

(Head of Department)

DEDICATION

This project is first dedicated to my God Almighty, the source of all wisdom and guidance, and inspiration, whose divine presence has been my steadfast companion guiding me through all this journey. Thank you for blessing me with the strength, perseverance and clarity of mind to complete this work. May this project be a testament to your love and a reflection of your grace. Secondly, to my family, your steadfast affection, backing, and motivation have illuminated my path during this endeavour. Your faith in my abilities has empowered me to pursue my aspirations and surmount obstacles. This project is dedicated to you, with sincere appreciation for your constant presence and support. I am deeply grateful for your encouragement to aim high and for serving as the cornerstone of my achievements.

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ABSTRACT

Linearized water wave theory is a fundamental concept in fluid dynamics that has been extensively used to study wave propagation in various aquatic environments. Water waves play a crucial role in many engineering and scientific applications, including ocean and coastal engineering, ship hydrodynamics, and offshore engineering. However, the complexity of nonlinear wave dynamics has limited the accuracy of traditional numerical models, emphasizing the need for a simplified yet robust approach. Linearized water wave theory offers a promising solution by assuming small-amplitude waves, enabling the simplification of the governing equations and providing an efficient tool for wave analysis.

This project explores the mathematical and physical principles underlying linearized water wave theory and its application in various fields such as oceanography, coastal engineering and naval architecture. The study begins with an overview of the basic equations governing water wave motion including the linearized Euler equation and boundary conditions. The dispersion equation which relates the wave frequency to its wavenumber is derived and analysed to properly understand wave propagation characteristics. In this study, we developed and applied linearized water wave theory to investigate wave propagation in a simplified fluid domain. We also discretized the linearized Navier-Stokes equations and then introduced a wave-like solution to represent the small-amplitude waves. By substituting this solution into the linearized equations, we obtained a set of ordinary differential equations that describe the wave propagation characteristics. Through mathematical analysis and numerical simulations, this study aims to provide a comprehensive understanding of linearized water wave theory and its applications in fluid dynamics.

The applications of this study are diverse and far-reaching. Our results can be used to improve the design and optimization of various aquatic structures, such as seawalls, breakwaters, and offshore platforms, by providing a better understanding of wave-structure interactions. Additionally, our findings can be applied to enhance the accuracy of wave forecasting models, which are crucial for coastal erosion prediction, ship navigation, and offshore operations. Furthermore, the linearized water wave theory can be extended to study more complex wave phenomena, such as wave-current interactions and wave-induced sediment transport, offering a promising avenue for future research.

CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF STUDY:

Waves are universal concept found in diverse scientific fields, originating from the comprehension of oscillations and disturbances. In the realm of physics, waves are fundamentally linked to the recurring movements of particles or fields. The study of water waves dates back to ancient roots where observable occurrences like water ripples laid the foundation for the development of wave theory. Water waves have been researched for generations with substantial improvements in both comprehension and modelling using mathematics. The early stages of linearized water wave theory can be linked back to Sir George Stokes ground breaking work in the nineteenth century. Stokes fundamental work “On the Theory of Oscillatory Waves” was first released in 1847 and it developed the equations overseeing the motion of tiny amplitude waves on a surface of an inviscid and incompressible fluids. The linearized potential flow equations are the basis of linearized water wave theory. Stokes’ work developed the base for follow up studies on linearized water wave theory which was further intended to nurture mathematical models and analyse the basic physical concept of water.

One significant improvement occurred in the early twentieth century, when the linearized potential flow equations were broadened to encompass the effects of viscosity in fluids. This stimulated the discovery of Linearized Navier-Stoke equation, leading to a better representation of actual water wave propagation. This linearized equation takes fluid viscosity into account and therefore can be used for researching wave damping and as well as dissipation.

Having a comprehensive understanding of the intricate dynamics of ocean wave is essential for a myriad of applications extending from coastal engineering to offshore structural designs. Water waves plays a significant role in various natural processes, including the regular rise and fall of tides and the transport of waves over enormous oceans.

The study of water wave dynamics has been for years a key focus in a variety scientific areas, from oceanography to engineering. Water, an omnipresent force on our globe, creates a dynamic interaction of waves that not only shape coast but influence marine activities and adds to the vital equilibrium of the ecosystem. Understanding the behaviour of water waves is crucial in fields such as coastal engineering, which requires a thorough analysis of their impact on structures.

This chapter begins an exploration into the theory's origins, delving into the underlying concepts of linearized water wave theory, and giving a simpler yet securely built basis for comprehending the dynamics water waves.

1.2 SIGNIFICANCE OF STUDY:

The significance of studying linearized water wave theory stems from its practical applications. This theory is an important topic in fluid mechanics and coastal engineering. This theory is critical in anticipating the consequences of large wave phenomena like tsunamis and storm surges. Scientists and engineers can build early warning systems and structures that can survive the devastating pressures connected to with such occurrences if they precisely model them. This theory underlying predictive capabilities enable researchers and engineers to forecast how wave will behave under variety of scenarios. This anticipation is useful in developing and executing actions to reduce potential dangers associated with wave impacts. Understanding wave dynamics such as erosions, sediment transport, shoreline displacements, and changes in coastal morphology, allows for the management of environmental changes caused by waves. The study of this theory not only solves present difficulties but also provides the groundwork for future researches in fluid mechanics and coastal engineering.

1.3 SCOPE OF STUDY:

The study of linearized water wave theory covers a wide range of topics including mathematical modelling of wave propagation, analysis of wave interaction with barriers and prediction of wave characteristics such as amplitude, wavelength and frequency. This theory is notably relevant in domains such as coastal engineering, nautical science, and naval constructions where accurate wave behavioural pattern assumptions are vital for constructing offshore infrastructures, coastal safety measures and vessel stability. Waves have a role in atmospheric and oceanic interactions and researching water wave theory helps meteorologists improve climates and weather predictions by taking wave effects into consideration. Exploring linearized water wave theory unfolds a broad scope, encompassing the complicated dynamics wave behaviour in fluid mechanics. This endeavour has practical applications in coastal and offshore engineering, directing the construction of wave-resistant structures. In essence, the scope goes beyond the theoretical world, reaching into actual applications and cultivating a complete understanding of wave dynamics.

1.4 LIMITATIONS OF STUDY:

While linearized water wave theory provides useful insights into the behaviour of water wave, it has some drawbacks. One of the limitations is its applicability to small-amplitude waves. In reality, wave can have large amplitudes, particularly during extreme events. As a result of this, the linearized water wave theory may fail to effectively predict the behaviour of large-amplitude waves and their interactions with structures. We encounter limitations such as; small-amplitude condition and linear relationships may oversimplify the complexities of real-world wave dynamics.

In addition, linearized water wave theory presumes idealised assumptions such as an evenly distributed and unbounded fluid domain. In actuality, water bodies may have irregular boundaries, varying depths and the existence of obstructions and other structures. These elements have a considerable impact on wave propagation and may not be accurately reflected by linearized theory alone.

Furthermore, challenges occur in fully depicting scenarios with stronger waves or deep-water conditions. The theory may not properly account for wind and coastline asymmetry, leading to inaccurate predictions of extreme events.

Again, the lack of emphasis on wave breaking, as well as the reduced coastal geometry assumptions, will make it difficult to apply complex natural coastlines.

1.5 IMPORTANCE OF STUDY:

Water wave theory helps in risk assessment in terms of enabling researchers to predict wave behaviours, assessing the probable impact of natural events like tsunami or storms and also provision of alleviation strategies in those coastal areas. This study also aids in understanding and managing environmental changes and also help provide insight for coastal planners and policymakers to make informed decisions regarding environmental conservation and infrastructural developments**.** In summary, studying water wave theory is essential for physical applications in engineering, navigations and environmental protection, as well as or advancing scientific understanding in various disciplines.

1.6 DEFINITION OF SOME TERMS:

The following are some terms associated with water wave;

1.6.1 FLUID MECHANICS: This is a branch of physics that deals with the studies of the behaviour of fluids, both liquids and gases and their interactions with forces. This disciplines involves the fundamental principles that governs the movement of fluids, including elements like viscosity, pressure, fluid flow. These principles plays a vital role in a wide range of scientific and engineering applications.

1.6.2 COASTAL ENGINEERING: This involves the designs and management of coastal areas to tackle problems such as erosions and storm surges. Coastal engineering integrates the principles of fluid mechanics and sediment transport to create solutions for coastal protection.

1.6.3 SMALL-AMPLITUDE CONDITION: In wave theory, this refers to the assumption that wave amplitudes are sufficiently small compared to wavelengths.

1.6.4 COASTAL ASYMMETRY: Is the uneven distribution of land and water along a coast or shoreline.

1.6.5 COASTAL ASYMPTOTE: This is a line that a curve approaches but never reaches. Coastal asymptote and asymmetry behaviours are taken into consideration in coastal engineering to comprehend the transformation of coastlines and the enduring processes occurring along the coast over an extended period.

1.6.6 NATURAL COASTLINES: These are stretches of land where the landmass meets a body of water such as ocean or sea, without artificial modifications. These are characterized by diverse features like estuaries, beaches, dunes etc...

1.7 FLUID MECHANICS

Fluid mechanics is a field of engineering science that studies the behaviour of fluids such as liquids and gases either in motion or at rest. Fluid mechanics plays an important role in many fields, including civil, chemical, mechanical, and aerospace engineering. Knowledge of fluid properties behaviour is very essential for creating and evaluating a wide range of structures including turbines, pipes and heat transfer.

1.7.1 DEFINITION OF FLUIDS:

A fluid is described as anything that regularly becomes deformed when subjected to either shear or tangential stress, irrespective of its size. As compared to solids which are immune to deformations, fluids are capable of moving and takes the shape of their containers.

1.7.2 TYPES OF FLIUDS:

Fluids are classified into two types, namely:

1.7.2.1 REAL FLUIDS:

These are viscous fluids, meaning they have an internal resistance to motion. Real fluids exist both as gases and liquids and their viscosities differs. Real fluid encompasses all of the fluids we experience in the real world. They may appear to be varied in thickness. Viscosity is a characteristics of fluid that act as barrier for multiple layers of fluids flowing past each other. Examples of real fluid include; water, milk, honey, oil, air, oxygen, carbon dioxide etc.

1.7.2.2 IDEAL FLUIDS:

An ideal fluid also known as an inviscid fluid, is an abstract idea that can be used to explain fluid mechanics problems. It presupposes that the fluid is incompressible, frictionless and has no viscosity. This presumptions streamline the analysis of mathematics while offering an elementary comprehension of fluid dynamics. Fortunately, idea fluids do not exist in real world; they merely represent an idea that is applied in theoretical term.

1.8 EULER-STOKES EQUATION:

People often conflate the Euler-Stokes equation and the Navier-Stokes equations since they both pertain to fluids.

The Euler-Stokes equation named after Leonhard Euler and George Gabriel Stokes is an essential equation in fluid dynamics, it describes the flow of an incompressible fluid and it combines the concept of mass, momentum conservation to offer understanding of fluid motion in a wide range of engineering applications.

The Euler-Stokes equation is built on Navier-Stokes equation (which outline the motion of viscous fluids). But in the case of an incompressible fluids, the viscous elements are ignored leading to the reduced EULER-STOKES equations.

1.8.1 DIFFERENCES BETWEEN EULER-STOKES AND NAVIER-STOKES EQUATION:

1.8.1.1 Euler-Stokes equation describes inviscid flow thus indicating that they do not account for viscosity. They are suitable for discussing ideal fluid which possess no internal resistance to shearing. While, Navier-Stoke equation include viscous terms, which allows for a more real picture of fluid behaviour by simply taking into consideration viscosity. It is suitable for depicting real-world having internal friction.

1.8.1.2 Euler-Stoke equation rely on mass, momentum, and energy conservation laws but they lack consideration for viscous effects. Euler-Stoke equation is an accumulation of partial differential equations. While Navier-Stoke equation is a continuation of Euler-Stoke equation that takes into consideration viscosity. Navier-Stoke equation is a more advanced set partial differential equation that consist of both convective and diffusive terms.

1.8.1.3 The Euler-Stoke equation is given as;

$\frac{\rho d\vec{V}}{dt} = -\nabla p + \rho g$ -- (1.1)

which can be rewritten as;

$\rho(\frac{\partial \vec{v}}{\partial t} + \vec{v}.\nabla\vec{v}) = -\nabla P + \rho g$ -------------------------------- (1.2)

where;

ρ = the fluid density

$\frac{\partial \vec{v}}{\partial t}$ = the rate of change of velocity with respect to time

$\vec{v}.\nabla\vec{v}$ = the convective term where $\vec{v}$ represent the velocity vector and $\nabla\vec{v}$ denotes the gradient of the velocity field.

P = the pressure

g = the acceleration due to gravity.

While the Navier-Stoke equation is given as;

$\frac{\rho d\vec{V}}{dt} = -\nabla P + \mu\nabla^2\vec{v} + \rho g$

which can also be rewritten as;

$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}.\nabla\vec{v}\right) = -\nabla P + \mu\nabla^2\vec{v} + \rho g$ ---------------------------------- (2)

where;

ρ =the mass density of the fluid

$\frac{\partial \vec{v}}{\partial t}$ =the rate of change of velocity with respect to time

$\vec{v}.\nabla\vec{v}$ = the convective term where $\vec{v}$ represents the velocity vector and $\nabla\vec{v}$ denotes the gradient of the velocity field.

$\mu\nabla^2\vec{v}$ =the viscous diffusion term which is neglected in the Euler-Stoke equation due to the assumption of inviscid flow.

μ = the kinematic viscosity

P = the pressure of the fluid

1.9 BERNOULLI'S EQUATION OF FLUID:

The Bernoulli equation is based on the concept of energy conservation as applied to fluid motion. It maintains that in a steady flow of an incompressible and inviscid fluid the total energy per unit mass along a streamline is constant. This equation is founded on three fundamental principles, namely;

1.9.1.1 Pressure Energy: Is defined as the energy associated with a fluid's pressure. It is defined as the work performed by a fluid to maintain a specific pressure at a given place. According to the Bernoulli equation, increasing pressure energy causes a decrease in kinetic and potential energy, and vice versa.

1.9.1.2 Kinetic Energy: is defined as the energy associated with fluid motion. It relies on the fluid mass and velocity. According to the Bernoulli equation, when kinetic energy increases, pressure energy and potential energy decrease, and vice versa. This means that when fluid velocity increases, pressure falls.

1.9.1.3 Potential Energy: is the energy involved with elevating a fluid. It is determined by the fluids height above a reference point as well as its gravitational acceleration. The Bernoulli equation states that when potential energy increases, pressure energy and kinetic energy decreases and vice versa.

The Bernoulli equation can be derived either by the direct application of Newton second law or the Euler equation of motion. The Euler equation is a generalized form of Newton's second law that applies to flow problem while maintaining the restricted scope of zero viscosity, resulting in the same outcome.

1.9.2 Derivation of Bernoulli's Equation using Rectangular coordinates of Euler equation

The vector form of the Euler equation can be integrated along a streamline. We will limit the derivation to constant flow.

In a steady flow, the Euler equation in rectangular coordinates can be stated as:

$$\frac{d\vec{v}}{dt} = \left(\frac{u\partial\vec{v}}{\partial x} + \frac{v\partial\vec{v}}{\partial y} + \frac{w\partial\vec{v}}{\partial z}\right) = \vec{v}.\nabla\vec{v} = -\frac{1}{\rho}\nabla p - g\hat{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

For a steady flow, the velocity field is given by $\vec{v} = \vec{v}(x, y, z)$, the streamlines are lines in the flow field that are perpendicular to the velocity vector at each point. Remember that streamlines, pathlines, and streak lines all intersect for constant flow. The movement of a particle along a streamline is regulated by equation (3). At the course of time interval dt, the particle experiences vector displacement $d\vec{s}$ over the streamline.

Taking the dot product of the terms in (3) with displacement $d\vec{s}$ over the streamline will produces a scaler equation that connects pressure, speed, and elevation. Taking the dot product

$$\overrightarrow{(v.\nabla\vec{v})}.\, d\vec{s} = -\frac{1}{\rho}\nabla p.\, d\vec{s} - g\hat{k}.\, d\vec{s} \quad \ldots\ldots (3.1)$$

where $d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Now evaluating each of the three terms in (3)

$-\frac{1}{\rho}\nabla p.\,\vec{ds} = -\frac{1}{\rho}(\frac{\hat{i}\partial p}{\partial x} + \frac{\hat{j}\partial p}{\partial y} + \frac{\hat{k}\partial p}{\partial z}).\,(\,dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= -\frac{1}{\rho}(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}\,dz)$

$= -\frac{1}{\rho}\nabla p.\,\vec{ds} = -\frac{1}{\rho}dp$ ………………(3.2)

$-g\hat{k}.\,\vec{ds} = -g\hat{k}.\,(\,dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= -g\,dz$ ……………………………… (3.3)

Using the Helmholtz decomposition theorem, we can rewrite

$(\vec{v}.\,\nabla\vec{v}).\,\vec{ds} = \frac{1}{2}\nabla(\vec{v}.\vec{v}) - \vec{v}\times(\nabla\times\vec{v}).\,\vec{ds}$

$= \frac{1}{2}\nabla(\vec{v}.\vec{v}).\vec{ds} - \vec{v}\times(\nabla\times\vec{v}).\,\vec{ds}$………..……… (3.4)

The last term on the RHS of this equation (3.4) is zero since $\vec{v}$ is parallel to $\vec{ds}$.

$\therefore\ (\vec{v}.\,\nabla\vec{v}).\,\vec{ds} = \frac{1}{2}\nabla(\vec{v}.\vec{v}).\,\vec{ds}$

$= \frac{1}{2}(\frac{i\partial v^2}{\partial x^2} + \frac{\hat{j}\partial v^2}{\partial y^2} + \frac{\hat{k}\partial v^2}{\partial z^2}).\,(\,dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= (\frac{\partial v^2}{\partial x^2}\,dx + \frac{\partial v^2}{\partial y^2}dy + \frac{\partial v^2}{\partial z^2}dz)$

$= \frac{1}{2}d(v^2)$ ………………………………….. (3.5)

Substituting equation (3.2), (3.3), and (3.5) into equation (3.1); we will arrive at

$\frac{1}{2}d(v^2) = -\frac{1}{\rho}\partial p - gdz = \frac{1}{2}d(v^2) + \frac{1}{\rho}\partial p + gdz = 0$

We then take the integral and we have this;

$\int\frac{1}{2}d(v^2) = -\frac{1}{\rho}\partial p - gdz$ ……………………… (3.6)

$= \frac{p}{\rho} + \frac{v^2}{2} + gz = \text{constant}$

If the density is constant, we obtain Bernoulli equation

$$= \frac{p}{\rho} + \frac{v^2}{2} + gz = \text{constant} \quad \text{.............................} \quad (3.7)$$

where p is the pressure of the fluid, v is the velocity of the fluid, ρ is the fluid density, g is acceleration due to gravity, z is the height.

1.9.3 Restrictions of the Bernoulli's Equation: There are several restrictions of Bernoulli's equation, namely;

1.9.3.1 Incompressible fluids: the Bernoulli equation assumes that the fluid density remains constant throughout the flow. This assumption holds for numerous industrial uses involving liquids but it might be applicable to compressed fluids like gases.

1.9.3.2 Steady flow: when the equation indicates that the fluid movement is steady, it simply, means that the velocity and the pressure assigned to any location is constant. In the real world many fluid systems experience unstable flows like start-ups and shutdowns which could affect the accuracy of the Bernoulli equation.

1.9.3.3 Negligible viscosity: the Bernoulli equation reveals that the fluid is non-viscous, which simply imply that there is no internal friction. The assumption is fair for several engineering applications but it may not hold for highly vicious fluids involving boundary layers.

1.9.3.4 Inviscid flow: the equation presupposes that the fluid flow is inviscid, this means that the no shearing stress or tangential stress acting on it. For many engineering uses, the aforementioned assumption is true, nevertheless, it might not be useful in cases where viscosity is important.

1.9.3.5 Irrotational flow: this equation implies that the flow of fluid is irrational, which implies that there are no vortices or rotational motion amongst the fluid. While this theory is appropriate in many cases, it may not hold in scenarios where turbulence exists.

1.10 Laplace Equation of Fluid Mechanics:

One essential ideal in fluid mechanics is the Laplace equation, which defines a fluid's behaviour in terms of pressure distribution. The Laplace equation also known as potential equation, named after Pierre-Simon Laplace, a French mathematician and astronomer, is a second-order partial differential equation that appears in various fields of physics, including fluid dynamics, electromagnetism, and heat transfer. This equation serve as the foundation for potential flow theory which is frequently applied in aerodynamics and hydrodynamics. According to the potential flow theory, a fluid can be termed ideal, inviscid and incompressible. This allows engineers to study and determine fluids in real life scenarios.

To derive Laplace equation, we consider an incompressible fluid, in scenarios like this, the fluid is characterized as an irrotational flow.

Taking the divergence of the velocity vector, we get

$$\nabla.\vec{v} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \quad \ldots\ldots (3.8)$$

where u, v, and w represent the velocity components in the x, y, and z directions respectively. This equation reflects the condition of incompressibility which states that velocity vector divergence is zero.

Now letting $\vec{v} = \nabla\varphi$……………………………………….. (3.9)

where φ is a scaler function known as the velocity potential, it is given as

$$\varphi = \varphi(x, y, z), \nabla\varphi = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial z} \quad \text{……………………………… (4.0)}$$

We can take the divergence equation in terms of velocity potential:

$$\nabla.\nabla\varphi = \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \text{-------------------- (4.1)}$$

Equation (4.1) is the Laplace equation, where ∇^2 is the Laplacian operator.

CHAPTER TWO

WAVE

2.1 ORIGIN OF WAVE:

From centuries of man's desire for knowledge and scientific investigations, our discoveries of the world around us have laid the foundation of our understanding of waves. Waves in their multiple forms have fascinated scientists all over the past years leading to remarkable studies of the basis of the real life and core concepts that oversees the universe as a whole.

According to research, one of the very first interactions with wave is attributed to the study of water waves. Ancient civilizations like the Egyptians were intrigued by the periodic ripples of river Nile and the captivating movement of waves in water. The early discoveries set the framework for our comprehension of waves, but wave theory was not established till a while later in the evolution of mankind.

In time immemorial, philosophers from Greek like Pythagoras and Aristotle studied the nature of sound wave which was eventually identified as a means of propagating waves. Years later, researchers improved and built on Aristotle's understanding of wave as a disturbance that travels through air. The study of wave made significant progress during the era of cultural and intellectual rebirth. The complexities of fluid dynamics were studied by intellectuals like Leonardo da Vinci, who shed light on the principles that regulate the movements of water waves and their behaviours.

Also, researchers like Johannes Kepler and Galileo Galilei made revolutionary advances in our knowledge of light and sound which formed the basis for existing wave theory. Prominent figures like Isaac Newton and Christian Huygens played significant roles in the advancement of mathematical accuracy within the study of waves during the 17^{th} and 18^{th} centuries. When it came to analysing the dynamics of waves, Huygens' wave theory of light offered a convincing clarifications, while Newton's law of motion and universal gravitation provided theoretical foundations. Wave study expanded across diverse fields encompassing engineering, mathematics, physics, and beyond. The development of Maxwell equations and electromagnetic theory of light completely changed our knowledge of wave phenomena.

Waves still fascinate scientists and engineers in the present day era, driving progress in numerous sectors like quantum physics, communications, acoustics and seismic researches.

2.2 DEFINITION OF WAVE AND WAVE MOTION:

Wave is a disturbance or an oscillation that travels through a transmitting medium transferring energy from one location to another without permanently altering the medium. It is a vibration that moves from one point to another transferring energy without causing the transfer of matter.

Wave motion refers to the movement of oscillations through a medium. It involves the transfer of energy from one point to another without physical displacement of matter over long distances. Wave motion can occur in various forms.

2.3 PROPERTIES OF WAVE:

2.3.1 Amplitude: Denoted by A is the maximum displacement of particles from their equilibrium position**.** It measures the energy of a wave.

2.3.2 Wavelength: The wavelength of a wave is the distance between two successive crest and troughs of a wave. It is usually denoted by the Greek letter lambda (λ). Wavelength is calculated using $\lambda = \frac{2\pi}{m}$, where m is the wave number. The S I unit of wavelength is in meter.

2.3.3 Frequency: This is the number of full oscillations that takes place in a given amount of time. It is represented by the symbol (f), and it is measured in Hertz (Hz).

2.3.4 Period: Is the time it takes for one complete oscillation to occur. Period denoted by P is the reciprocal of frequency. It is measured in seconds.

2.3.5 Wave Velocity/Speed: This represents the speed at which wave travels in a medium. It is denoted by either a (v) or (c). Wave speed $C = f\lambda$

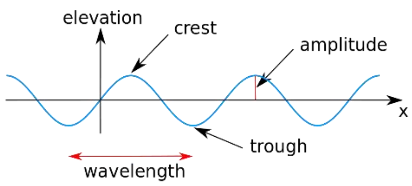
2.3.6 Crest and Trough: Crest is the highest point of elevation in a wave while Trough is the highest point of depression in a wave.

2.3.7 Reflection: This occurs when a wave bounces back after being in contact with a surface, resulting to a change in direction.

2.3.8 Refraction: This occurs when a wave bends as it enters a medium with different density leading to a change in the speed and direction.

2.3.9 Diffraction: This simply means the bending and spreading out of waves when passing through openings.

The figure below shows the properties of a wave:



(Fig. 2.1)

2.4 TYPES OF WAVES:

Basically, there are two types of waves namely:

2.4.1 Mechanical Waves:

These are waves that require material media such as water, air or solid material for their propagation. This waves occur from the vibration of particles within the medium and transfer energy through those media as they travel. These waves are sub-classified into two;

2.4.1.1 Transverse waves:

In transverse waves, the particle oscillations are perpendicular to the wave movements. Examples include wave on a string or rope, water waves.

2.4.1.2 Longitudinal waves:

In this wave, particles of the medium oscillate parallel to the direction of wave propagation. Compression of the medium occur as the wave passes through. An example of this kind of wave is the sound wave.

2.4.2 Electromagnetic Waves:

They are a form of transverse wave that do not require material medium for its propagation. These waves consist of electric and magnetic fields perpendicular to each other and to the direction of wave movements. Examples of electromagnetic waves are;

radio waves, microwaves, infrared waves, visible light waves, ultraviolet waves, x-rays and gamma rays, surface waves etc.

2.5 WATER WAVES:

Waves on water are a common natural occurrence in lakes, rivers, seas, and even little puddles. Water waves are the end result of the collision between seas and other external forces likes gravitational forces, winds, tsunamis. Offshore designing, coastal engineering and naval construction are the very few engineering applications that require an understanding of the properties and patterns of water waves.

Water waves are disturbances that move through water bodies, carrying energy from one point to another devoid of actual material transfer. The rising and falling motion of the water surface is how they are usually seen. It is important to study water waves since they have different wavelengths, amplitudes and frequencies and they can travel over great distances.

2.6 TYPES OF WATER WAVES:

Water waves are classified based on their sources and generations mechanisms, they include;

2.6.1 Wind-generated water waves:

The most prevalent kind of water waves are the wind-generated waves, sometimes referred to as the surface waves or wind waves. This waves is formed simply by the interaction of winds with water. The water receives energy from the winds in form of ripples which eventually grow into waves with much bigger size and intensity. Wind waves can be further classified based on their wavelengths and periods.

2.6.1.1 Capillary waves:

These waves are also called ripples, they are the smallest wind-generated waves. Surface tension is the main factor influencing them and their wavelengths are shorter than 1.74cm. These waves are responsible for the texture of water surfaces and they are usually seen in calm conditions.

2.6.1.2 Gravity waves:

These are large-wind generated waves that occur when gravity is the restorative force, this wind create pressure differences over the water surface which drive the water surface into circular orbits. These waves comes in various shapes and sizes ranging from little swells and ripples to massive storm-generated waves. Gravity waves moves great distances across seas and oceans while transmitting energy and momentum over great distances. Gravity waves are further classified into; swells and seas.

Swells waves are longer period waves with longer wavelength greater than 20meters. They are caused by distant storms and also carry energy to large areas in the oceans without any loss. While sea waves are waves with shorter wavelengths between 1 and 20 meters. They originate from a local wind field and are influenced by the distance the wind blows across.

2.6.2 Seismic or Tsunamis:

Unlike wind-generated waves, tsunamis are large-scale waves characterized by long wavelength and their high velocities. When they arrive at the coastlines, they have the ability to spread across vast ocean basins and wreak havoc. Tsunami are caused by underwater disturbances such as volcanic eruptions, earthquake or landslides. They move vertically through the water column.

(Fig. 2.2)

2.6.3 Tidal waves:

These are long-period waves that are generated by the gravity interactions of earth, sun, and moon. They result in the rise and fall of sea levels along coastlines. Tidal waves are divided into diurnal and semidiurnal tides based on their frequency of occurrence.

2.6.4 Human-generated waves: These are divided into two classes, namely;

2.6.4.1 Ship waves: These are also called bow waves, they are caused by the passage of ships through the water bodies, as the ship move through water, it create disturbances in the form of a wave.

2.6.4.2 Wake waves: These are waves created by the wake of boats and ships. They can effect navigations and shoreline erosions.

2.6.5: Internal waves: These are waves that are generated by densities differences between the water columns such as differences in temperatures. These waves can move along density interfaces in the ocean.

2.6.6 Seiches (Long waves): These waves are caused by disturbances such as wind, seismic activities, and atmospheric changes. Seiches occur in half-closed or closed bodies of water like bays, lakes etc. and can lead to a significant water level rise and fall.

2.7 WHAT GENERATES WATER WAVES

Understanding the generations of water waves is essential in various engineering applications such as ship designs and offshore constructions, disaster preparedness. The factors which generates water waves include;

2.7.1 Wind-induced waves: The wind is one of the forces that generates water waves. These waves are created when the wind carries energy and momentum to the water molecules on the surface of a water body. Wind speed duration and wind fetch affect the height and wavelength of wind-induced waves. There are three main stages involved in the wind generation process, firstly, there is capillary waves which creates ripples on the surface of the water as wind blows on it. Then as the speed of the wind increases, capillary waves merges and forms large waves called gravity waves, these waves have longer wavelengths and amplitudes. Eventually as the wind strength increases, the gravity waves changes into fully formed long-distances wind waves

2.7.2 Seismic activity: Seismic activity can produce water waves in addition to waves caused by wind. Waves can be created in water bodies by earthquakes, volcanic eruptions and undersea landslides. When these waves reach coastal locations they can travel great distances and cause havocs. Large volumes of water are vertically displaced as a result of volcanic activity and this is the primary cause of tsunamis. The destructive power of these tsunami are caused by these wave's sharp rise in height and fall in speed as they approach shallow waters close to the coast.

2.7.3 Gravity forces: water waves are also produced by gravitational forces that arises from the interactions of the sun, earth, and moon. These interactions results in periodic rise and fall of water levels in coastal regions. The gravitational force that the celestial bodies have on the earth's oceans causes tidal waves. The combined effects of the sun and moon is the main source of tides. A high tides is caused by the moon gravitational pull creating a bulge on the surface of the earth. Due to the centrifugal force brought on by the rotation of the earth-moon system, another budge also form on the opposite side of the planet, a low tide is formed from this, thou it has less impact than that of the moons, the suns gravity pull also play aids in the formation of tides. The gravitational forces on the moon and sun also influence the timing and amplitudes of tides, resulting in the phenomena called spring tides and neap tides. Gravity waves plays important role in the complex dynamics of tidal variations, affecting marine ecosystems, and shaping coastal environments.

2.8 SHALLOW WATER:

A region of water body that are smaller than the waves' wavelengths are referred to as the shallow water. Waves in shallow water interacts with the oceans floor as a result of bottom topography. In shallow water the wavelength and period and also the water depths affect how quickly the wave moves. These waves behave differently from their deep water counterparts as a result of the frictions and their interactions with seabed; which causes waves to alter in height and form. Shallow water are commonly seen near coastlines where water depths gradually increases from shorelines.

2.8.1 Characteristics of Shallow Water

2.8.1.1 Wave Breaking: As waves are approaching shallow regions their energies are concentrated causing wave crests to become steeper, this eventually makes the wave crest to become unstable and it collapses resulting in wave breaking. Wave breaking plays a role in offshore dynamics and sediment transports.

2.8.1.2 Wave Dispersion: In shallow water, waves experiences a phenomena called the wave dispersion where waves of different wavelengths and depths travels at different speeds. This dispersion occurs a result of the interactions between water depths and wavelengths. As a result, water waves travel faster than shorter waves leading to wave separations and changings of wave shapes.

2.8.1.3 Seabed Interactions: the interaction of seabed and waves can induce sediment transport and erosion. Wave energy can be dissipated by frictional forces exerted by the seabed leasing to wave reductions.

There are certain challenges encountered in shallow waters, these includes;

2.8.2 Coastal Erosions: Shallow water areas are to coastal erosions which can threaten infrastructures and ecosystems. Engineers can develop ways to alleviate erosions and protect shorelines through specific measures like seawalls and groynes.

2.8.3 Wave-induced forces: the volatile nature of shallow water wave produces forces that affect offshore structures, in other to avert these, engineers need to precisely evaluate these forces by designing structures that can sustain loads caused by these waves.

2.8.4 Sediment Transportation: sediment transfer is more common in shallow water areas and it can have impact on infrastructures and harbour entrances. To lessen the negative consequences of sediment movements, they must carefully consider sediment dynamics and implement right measures in place.

2.9 DEEP WATER WAVES:

A region of water body where the depth is much greater than the wavelength passing through it is known as deep water waves. In deep water, the speed is determined by the wavelength alone; the water depths has no impact on it. Waves in deep water has properties like diffractions, refractions, interference but they do not really interacts with ocean floor. Deep water waves are commonly seen in open sea, away from coastlines where the water depth is considerable.

2.9.1 Characteristics of Deep Water

2.9.1.1 Wave Propagation: waves moves without influence on the water bottom in deep water. These waves maintain their shapes and travel in constant speeds irrespective of their wavelengths, this aids the transmission of long period waves across large distances in open oceans.

2.9.1.2 Energy Dissipation: deep water experiences less energy dissipations unlike the shallow water due to seabed interactions, because of this waves in deep water can travel long distances without loss of energy.

2.9.1.3 Oceanic Current: deep water regions are associated with oceanic currents which can influence the propagation and behaviour of waves. These oceanic currents can come from different factors such as the earth's rotations, wind patterns, and temperature gradients.

There are certain engineering implications in deep water, deep water provides opportunities for subsea explorations and installations of offshore structures like oil rigs, wind turbines, underwater communication cables. Engineers designs these structures to withstand harsh marine environments. Also Engineers develop innovative solutions such as underwater acoustic communication and optical fibre network to establish reliable communication links in deep water.

2.10 ROGUE WAVES

Rogue waves also known as freak waves or monster waves are rare but incredibly strong and huge waves that can show up out of the blue in a clam or temperate sea conditions. These waves are distinguished by their exceptional height and they frequently loom over neighbouring waves and their steep faces which sometimes seem almost vertical. Scientists, coastal communities around the world are always fascinated and concerned by rogue waves because they causes serious hazards to ships, offshore buildings and coastal settlements. Since rogue waves are unpredictable and challenging, it is therefore difficult to get specific figures of their frequencies; however, research indicate that they appear more often than before. Although, rogue waves can occur in any ocean or sea, they are frequently linked to areas with strong ocean currents, including the Gulf Stream in the North Atlantic and the Agulhas Current in South Atlantic.

Below is a diagram of a rogue wave;

(Fig 2.3)

2.10.1 CAUSES AND FORMATION OF ROGUE WAVES

Although, the exact processes underlying the development of rogue waves remain incompletely understood, but there are several components that are believed to be involved in the formation of these waves.

Firstly; constructive interferences of smaller waves where energy level is concentrated and amplified to form a single, massive wave is one of the causes of this wave. Secondly, the nonlinear factor in wave dynamics like the instabilities of wave trains caused by modulations can also cause the sudden appearance of rogue waves. Also, nonlinear wave-currents can also cause rogue waves.

2.10.2 Impacts of Rogue Wave:

Rogue waves appear suddenly with little or no warnings thereby posing a threat to ships, offshore platforms and coastal infrastructures. These waves causes vessels to capsize thereby leading to injuries or loss of valuable cargos. Rogue waves endanger coastal communities, causing beach erosions, infrastructure damages and flooding.

Continued researches and studies in ocean sciences are important for enhancing our ability to predict, respond to and alleviate risks presented by rogue waves.

2.11 CONCEPT OF AMBIENT CURRENT IN THE OCEAN:

The term 'ambient current' refers to the background flow of water in the ocean and it is different from other kinds of currents like the geostrophic or tidal currents that are solely driven by the wind.

2.11.1 CHARACTERISTICS OF AMBIENT CURRENTS:

When compared to other currents, ambient currents are typically characterised by their generally stable and slow flow. Large-scale elements like the regional patterns of circulations usually have impacts on them. Since ambient currents do not show noticeable temporal changes, unlike the tidal currents, they are frequently referred to as the average water flow in a given area.

Ambient currents can be categorized into two main types: surface currents and subsurface ambient current.

Surface Ambient Current: this currents are strong in the ocean topmost layer and are mostly caused by the wind stress. These currents are important for dispersing nutrients and heats, which in turn affects the weather patterns and distribution of marine life.

Subsurface Ambient Current: These currents are driven by density differences caused by the temperatures and salinity. These currents are found below the surface layer, it helps move water masses across different oceans region.

2.11.2 SIGNIFICANCE OF AMBIENT CURRENT

Understanding ambient current is essential for comprehending the overall circulation pattern and dynamics of the ocean. By studying this current, researchers can gain insights in the transport and dispersion of pollutants marine ecosystems and climate patterns. Also ambient currents play important role in the distribution of heat and nutrient which are vital for the sustentation of marine life. Knowledge of ambient currents helps in the optimization of fishing routes, predicting the spread of pollutants in the case of accidents and planning offshore buildings i.e. having a good knowledge on this currents can help predict good locations for the constructions of oil rigs and wind farms.

2.11.3 MEASUREMENTS AND MODELLING OF AMBIENT CURRENTS:

Many methods are used in examining ambient currents, including remote sensing technologies like the satellite altimetry and ocean surface drifters as well as the direct observations with devices like current meters and acoustic Doppler current profilers. The speed, direction, and fluctuation of ambient currents at various depths and places are all well-documented by these measurements.

Numerical modes are often used to stimulate and predict ambient currents in addition to the direct measurements .to duplicate the flow of water in the ocean, these models take into consideration a number of variables like salinity, wind forcing, temperature, bathymetry and other. Researches may validate and enhance the accuracy of these models, allowing for more accurate predictions of ambient currents in various settings and locations by comparing models and data.

2.12 WAVE BREAKING IN THE OCEAN:

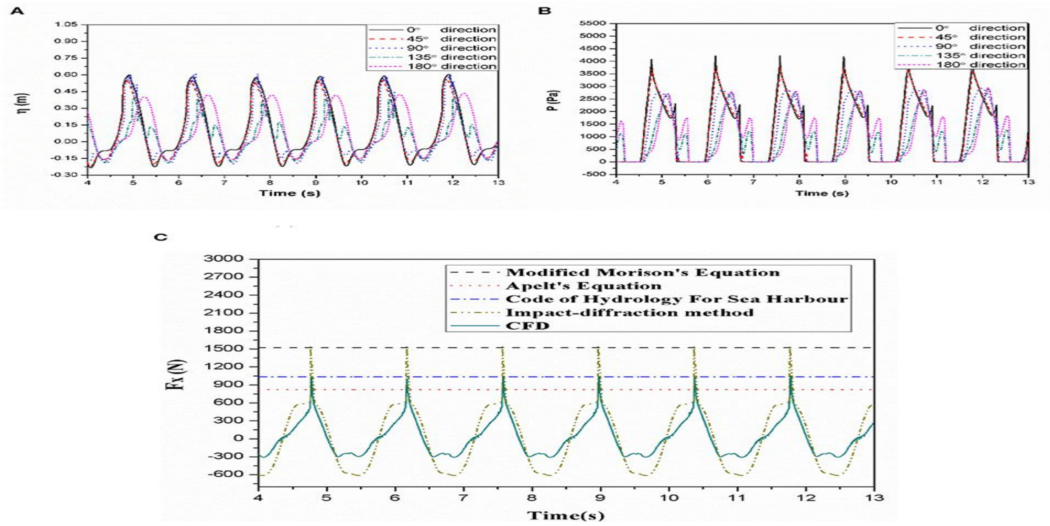
Wave breaking is a dynamic and complex phenomenon that occurs when waves reach a critical point of instability and transition from an overly waveform to turbulent motion. It is a fundamental process in coastal dynamics and plays an important role in shorelines shaping, sediment transportations, and wave energy dissipations.

2.12.1 TYPES OF WAVE BREAKING

Wave breaking are classified into several types based on their characteristics and behaviours. The types of wave breaking include;

2.12.1.1 SPILLING BREAKERS: This occur when waves gradually lose energy as they approach shorelines, resulting in gradual and gentle descent of the wave crest. These waves typically break over gently sloping beaches and are characterized by a rolling motion as the wave crest spills forward. Spilling breakers are common in moderate wave conditions and are less powerful compared to plunging or surging breakers.

(*Fig 2.*4 below shows an illustration of a spilling breaker)*



2.12.1.2 PLUNGING BREAKERS: This breaking occur when waves encounter a sudden change in bottom topography or a steep shoreline, causing the wave crest to become unstable and curl over. These waves form a hollow tube or barrels as they break, with the wave crest pitching forward and crashing into trough below. Plunging breakers are often associated with more energetic wave conditions and are favoured by surfers fro their challenging and exhilarating rides.

(*Fig 2.5 below shows the graph of a plunging breaker)*

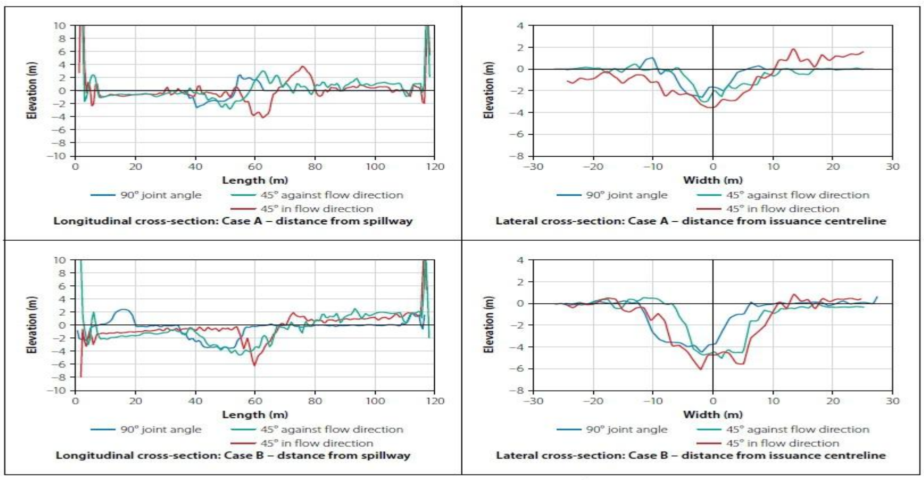


Figure 12 Observed equilibrium bed profile indicating effect of the rock joint angle (Q = 80 m³/s/m, H = 100 m, TWL = 20 m)

2.12.1.3 SURGING BREAKERS: Surging breakers occur when waves approach a steep or rough shoreline or coastal structures, causing the wave energy to be directed towards a powerful surge. This wave rise up to the shorelines, with significant forces and cause hazards to coastal buildings and swimmers.

2.12.1.4 COLLAPSING BREAKERS: When waves lose their coherence and collapse onto themselves, collapsing breakers happen. Chaotic and unpredictable breaking patterns may arise from the complicated wave interactions or abnormalities in wave movements. Collapsing breakers can be dangerous to coastal buildings and designs and shoreline stability, and they can be difficult for boats to manage.

2.12.1.5 OVERTOPPING BREAKERS: Waves that break over coastal barriers like seawalls, ditches or dikes and pour onto the protected region are known as overtopping breakers. These waves especially, especially during storms or high wave events can result in over flooding, erosions, and damage to coastal buildings. Coastal engineers and planners who develop and manage coastal protection systems are always concerned about overtopping breakers.

2.12.1.6 DUMPING BREAKERS: When waves break suddenly and strongly, a lot of water is dumped onto the coast, creating dumping breakers. Strong surges and undertows produced by these waves can endanger swimmers and properties along the coast. Dumping breakers can be dangerous at times of strong wave energy or during storm. Dumping breakers are often associated with steeps and narrow beaches.

CHAPTER THREE

PROGRESSIVE AND STATIONARY WAVES

3.1 EQUATION OF MOTION:

The motion we are analysing is two-dimensional and the vertical component of the velocity is not negligible, it is small but since we are concerned with linear motion the square of it can be disregarded. We further assume that the motion is irrotational, this applies for the waves started in an inviscid fluid by natural forces. Since the motion is irrotational, there is existence of a velocity potential ∅ such that $\vec{q}=-\nabla\varphi$. Therefore the continuity equations can be written as;

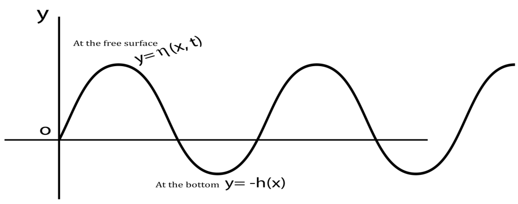
$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 \quad \dots\dots\dots (1)$$

3.2 BOUNDARY CONDITIONS

3.2.1 KINEMATIC BOUNDARY CONDITION:

This boundary condition is also known as the free surface boundary condition, it describes the behaviour of the water surface, and it usually specify the kinematic and dynamic conditions at the surface.

Let us consider the water of depth ‘h’



(fig 3.1; wave at the surface and bottom)

The propagating wave height is given as $\eta(x, t)$, this implies that the equation of surface is given as $y = \eta(x, t)$ ………………………………………………….. (2.1)

Since the surface is moving with the fluid, we have;

$\frac{d(y-\eta)}{dt} = 0$ ………………………………………………………………… (2.2)

= $\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(y - \eta) = 0$ …………………………………………… (2.3)

= $-\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + v = 0$……………………………………………………….. (2.4)

$= v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}$ ………………………………………………………………. (2.5)

By linearity, we have;

$\frac{\partial \eta}{\partial t} = v =- \frac{\partial \varphi}{\partial y}$ …… ………………………………………………… (2.6)

$\therefore \frac{\partial \eta}{\partial t} =- \frac{\partial \varphi}{\partial y}$ ………………………………………………………… (2.7)

where η(x,t) is the free surface and it depends on x and t.

Equation 2.7 is the kinematic boundary condition.

3.2.2 PRESSURE BOUNDARY CONDITION:

The pressure distribution at the fluid domain boundaries is usually described by the pressure boundary condition. It takes into account the forces that the fluid applies to the boundaries. In water waves, it controls the way pressure varies throughout the fluid domain, affecting the wave behaviour close to the border.

To derive the pressure boundary condition; Let P be the pressure inside the fluid and let P' be the pressure outside the fluid. Since we are dealing with irrotational fluid, we use the Bernoulli equation as the equation of motion, and it is given as

$$P = \rho\left(\frac{\partial\varphi}{\partial t} - gy - \frac{q^2}{2}\right) + f(t) \quad \text{...} (3.1)$$

We assume f(t) to be in $\frac{\partial\varphi}{\partial t}$, and by linearity, $q^2 = 0$, therefore, we have;

$$P = \rho\left(\frac{\partial\varphi}{\partial t} - gy\right) \quad \text{..} (3.2)$$

But we know that in a free surface there is equal pressure, i.e. P=P'.

Therefore, $\frac{dP}{dt} = 0$ and this implies that;

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\rho\left(\frac{\partial\varphi}{\partial t} - gy\right) = 0 \quad \text{..} (3.3)$$

$$= \frac{\partial^2\varphi}{\partial t^2} + u\frac{\partial^2\varphi}{\partial x\partial t} + v\frac{\partial^2\varphi}{\partial y\partial t} - vg = 0 \quad \text{..} (3.4)$$

If we are to ignore the non-linear terms, we will have;

$$= \frac{\partial^2\varphi}{\partial t^2} - vg = 0 \quad \text{..} (3.5),$$

but recall $v = -\frac{\partial\varphi}{\partial y}$

Therefore equation 3.5 becomes;

$= \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial y} g = 0$ ………………………………………………………... (3.6)

Equation 3.6 is the pressure boundary condition.

3.2.3 PHYSICAL BOUNDARY CONDITION:

This refers to a range of circumstances pertaining to the physical attributes of the water wave. These conditions include material conditions (like permeability for porous interface), thermal conditions (like temperature distribution at boundaries), and boundary motion constraints (like no-slip conditions for solid boundaries).

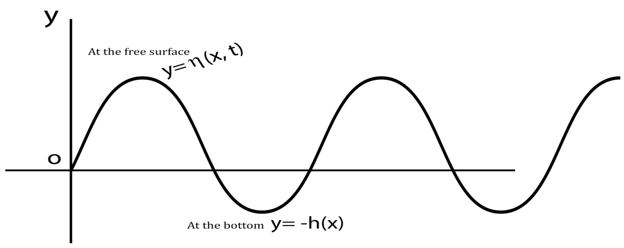
The normal velocity of the fluid and the velocity of the boundary are equal at the boundary $y =- h$

Therefore, $v =- \frac{\partial \varphi}{\partial y} = 0$, $y =- h$

$\frac{\partial \varphi}{\partial y} = 0, \;\; y =- h$ …………………………………………………………. (4)

3.3.1 DERIVATION OF POTENTIAL VELOCITY FOR PROGRESSIVE WAVE:

Consider a wave of depth ‘h’



Let the equation of the free surface be $\eta = a\sin(mx - nt)$ .... (5)

We assume that; $\varphi = f(y)\cos(mx - nt)$ .... (6)

Now, we show that equation (6) satisfy the Laplace equation by substituting equation (6) into the Laplace equation ($\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$) .... (7.1)

$\frac{\partial\varphi}{\partial x} =- f(y)msin(mx - nt)$ .... (7.2)

$\frac{\partial^2\varphi}{\partial x^2} =- f(y)m^2\cos(mx - nt)$ .... (7.3)

$\frac{\partial\varphi}{\partial y} = \frac{dF}{dy}\cos(mx - nt)$ .... (7.4)

$\frac{\partial^2\varphi}{\partial y^2} = \frac{d^2F}{dy^2}\cos(mx - nt)$ .... (7.5)

Inserting equation (7.3) and (7.5) into equation (7.1), we get;

$=- f(y)m^2\cos(mx - nt) + \frac{d^2F}{dy^2}\cos(mx - nt) = 0$

$= f''\cos(mx - nt) - f(y)m^2\cos(mx - nt) = 0$

$= f'' - f(y)m^2 = 0$

We get the indicial roots by solving the auxiliary equation using the operator-D method

$= D^2 - m^2 = 0$

$D^2 = m^2$

$D = \pm \sqrt{m^2}$

$D = \pm m$

This gives us the general solution; $f(y) = Ae^{my} + Be^{-my}$………………………….. (7.6)

Substituting equation (7.6) into equation (6), we have;

$\varphi = (Ae^{my} + Be^{-my})\cos(mx - nt)$……………………………………………….. (8.1)

From the physical boundary condition (equation (4), $\frac{\partial\varphi}{\partial y} = 0,\ \ y = -h$), we have;

$\frac{\partial\varphi}{\partial y} = (mAe^{my} - mBe^{-my})\cos(mx - nt) = 0$

At $y = -h$, we have

$\left(Ae^{-mh} - Be^{mh}\right)mcos(mx - nt) = 0$

$= \left(Ae^{-mh} - Be^{mh}\right) = 0$

$= Ae^{-mh} = Be^{mh}$

$A = \frac{Be^{mh}}{e^{-mh}}$

$A = Be^{2mh}$……………………………………………………………….. (8.2)

Put equation 8.2 into equation 8.1;

$\varphi = (Be^{2mh}.e^{my} + Be^{-my})\cos(mx - nt)$=0

Next, we factor out Be^{mh}

$\varphi = Be^{mh}(e^{m(y+h)} + e^{-m(y+h)})$cos (mx-nt) =0

$= 2Be^{mh}[\frac{e^{m(y+h)}+e^{-m(y+h)}}{2}]\cos(mx - nt)$

$= 2Be^{mh}coshm(y + h)\cos(mx - nt)$

$\varphi = Gcoshm(y + h)\cos(mx - nt)$ ………………………………….. (9.1)

where $G = 2Be^{mh}$.

Next we use the kinematic boundary to get the value of G; $[\frac{\partial\eta}{\partial t} =- \frac{\partial\varphi}{\partial y}]$……………….. (2.7)

$\eta = \text{a}\sin(mx - nt)$

$\frac{\partial\eta}{\partial t} =- ancos(mx - nt)$……… (9.2)

$-\frac{\partial\varphi}{\partial y} =- mGsinhm(y + h)\cos(mx - nt)$ ………. (9.3)

Put equation 9.2 and 9.3 into equation 2.7

$-ancos(mx - nt) =- mGsinhm(y + h)\cos(mx - nt)$, at y=0

$ancos(mx - nt) = mGsinhmh\cos(mx - nt)$

$G = \frac{an\cancel{cos(mx-nt)}}{msinhmh\cancel{\cos(mx-nt)}} = \frac{an}{msinhmh}$

$$G = \frac{an}{msinhmh}$$

$\therefore \varphi = \frac{an\,coshm(y+h)\,\cos(mx-nt)}{msinhmh}$ …………………………………………………………..… (10)

This is the velocity potential of a progressive wave.

3.3.2 THE DERIVATION OF THE VELOCITY PROFILE/PATH OF A PROGRESIVE WAVE

To derive the velocity profile or path, we recall that; $u = \frac{-\partial\varphi}{\partial x}, v = \frac{-\partial\varphi}{\partial y}$

From equation 10 above,

$$\varphi = \frac{an\, coshm(y+h)\cos(mx-nt)}{msinhmh}$$

$$u = \frac{-\partial\varphi}{\partial x} =- [-\frac{an\cancel{m}\, coshm(y+h)\sin(mx-nt)}{\cancel{m}\, sinhmh}] = \frac{an\, coshm(y+h)\sin(mx-nt)}{\sinh mh} \ldots\ldots\ldots\ldots\ldots\ldots.. (11)$$

$$v = \frac{-\partial\varphi}{\partial y} =- \left[\frac{an\cancel{m}sinhm(y+h)\cos(mx-nt)}{\cancel{m}\, sinhmh}\right] =- \frac{an\, sinhm(y+h)\cos(mx-nt)}{\sinh mh} \ldots\ldots\ldots\ldots\ldots\ldots... (12)$$

Next, we integrate both equation (11) and (12) with respect to 't' to get values for x and y, that is; $\dot{x} = u,\ \dot{y} = v$

$$x = \int udt = \int \frac{an\, coshm(y+h)\sin(mx-nt)}{\sinh mh} dt] = [\frac{-a\cancel{n}\, coshm(y+h)\cos(mx-nt)}{sinhmh} * \frac{1}{\cancel{-n}}]$$

$$x = \frac{a\, coshm(y+h)\cos(mx-nt)}{sinhmh} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13.1)$$

$$y = \int vdt = \int -\frac{an\, sinhm(y+h)\cos(mx-nt)}{\sinh mh} dt = [-\frac{ansinhm(y+h)\sin(mx-nt)}{sinhmh} * \frac{1}{-n}]$$

$$y = \frac{asinhm(y+h)\sin(mx-nt)}{sinhmh} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (14.1)$$

Square both equation (13.1) and (14.1)

$$x^2 = \frac{a^2cosh^2m(y+h)cos^2(mx-nt)}{sinh^2mh} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13.2)$$

$$y^2 = \frac{a^2sinh^2m(y+h)sin^2(mx-nt)}{sinh^2mh} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14.2)$$

From equation (13.2) and (14.2), let $A^2 = \frac{a^2cosh^2m(y+h)}{sinh^2mh}$ $and\ B^2 = \frac{a^2sinh^2m(y+h)}{sinh^2mh}$ respectively.

So that we can rewrite equation (13.2) and (14.2) as;

$x^2 = A^2 cos^2\,(mx - nt)$ ………………………………………. (13.3)

Divide through by A^2

$= \frac{x^2}{A^2} = cos^2\,(mx - nt)$ ………………………………………… (13.4)

$y^2 = B^2 sin^2(mx - nt)$ …………………………………………… (14.3)

Divide through by B^2;

$\frac{y^2}{B^2} = sin^2(mx - nt)$ ……………………………………..… (14.4)

Add equation (14.4) and equation (13.4)

$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$……………………………………………………..... (15)

The path of this particle describes ellipse.

3.3.3 SHOWING THAT AN ELLIPSE DEGENERATES INTO A STRAIGHT LINE AT THE BOTTOM:

Recall, $A^2 = \frac{a^2 cosh^2 m(y+h)}{sinh^2 mh}$ and $B^2 = \frac{a^2 sinh^2 m(y+h)}{sinh^2 mh}$.

At the bottom, we let y $=-$ h ……………………………………… (15.1)

for both A^2 and B^2, then we get that;

$A^2 = \frac{a^2 cosh^2 m(-h+h)}{sinh^2 mh} = \frac{a^2}{sinh^2 mh}$ ……………………………………… (15.2)

$B^2 = \frac{a^2 sinh^2 m(-h+h)}{sinh^2 mh} = 0$ ……………………………………………… (15.3)

Then, we can rewrite equation (15) as;

$\frac{x^2}{A^2} + \frac{y^2}{0} = 1$

$= \frac{x^2}{A^2} = 1$

$= x^2 = A^2$ ………………………………………………….………….. (15.4)

$= x^2 = \frac{a^2}{sinh^2 mh}$ ……………………………………..…………… (15.5)

Equation 15.5 shows the equation of a straight line, this implies that equation (15); an ellipse degenerates a straight line at the bottom.

3.4 THE DISPERSION EQUATION

We consider our equation (10) above

$\varphi = \frac{na\, coshm(y+h)\ \cos(mx-nt)}{msinhmh}$

Next, we apply the pressure boundary condition $(\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial\varphi}{\partial y} g = 0)$

$$\therefore \frac{\partial\varphi}{\partial t} = \left[-\frac{na\, coshm(y+h)\sin(mx-nt)}{msinhmh} *- n\right] = [\frac{n^2 a\, coshm(y+h) sin(mx-nt)}{msinhmh}]$$

$\frac{\partial^2\varphi}{\partial t^2} = [-\frac{n^3 acoshm(y+h)\cos(mx-nt)}{msinhmh}]$ ……………………………………………..……. (16)

$\frac{\partial\varphi}{\partial y} g = g\frac{na\, sinh(y+h)\ \cos(mx-nt)}{sinhmh}$ ……………………………………………………. (17.1)

Substitute equation 16 and 17.1 back into the pressure boundary equation we have;

$\left[-\frac{n^3 acoshm(y+h)\cos(mx-nt)}{msinhmh} + \frac{gna\, sinh(y+h)\ \cos(mx-nt)}{sinhmh} = 0\right]$

$= \frac{g\cancel{na}\, sinh(y+h)\ \cancel{\cos(mx-nt)}}{\cancel{sinhmh}} = \frac{n^3\cancel{a}coshm(y+h)\cancel{\cos(mx-nt)}}{m\cancel{sinhmh}}$

$= g\ sinhm(y+h) = \frac{n^2 coshm(y+h)}{m}$ …………………………………………..…. (17.2)

At y=0, we can rewrite (17.2) as;

$= \frac{n^2}{m} = \frac{gsinhmh}{coshnh}$ ……………….. (17.3),

from trigonometry identity we can write equation (17.3) as;

$n^2 = gm\ tanhmh$ …………….. (17.4)

But, recall that phase velocity/speed is given as $c^2 = \frac{n^2}{m^2}$ …………………………. (17.5).

putting equation (17.5) into (17.4) gives;

$c^2 = \frac{g\ tanhmh}{m}$ ……………………………………………………………..…. (18.1)

Equation (18.1) is the dispersion equation and it can also be represented in another form.

Consider $m = \frac{2\pi}{\lambda}$

$c^2 = \frac{g\,\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right)$ ……………………………………………………….………… (18.2)

where 'c' is the wave speed, 'λ' represent the wavelength, 'g' represent the gravitational force and 'h' represent the water depth.

3.5 CONCEPT OF DEEP WATER AND SHALLOW WATER

Since deep sea waves are vital to many offshore constructions and coastal protections strategies, they are fundamental component of ocean engineering. It is important to understand the relationship between their wavelength and water depth in deep water in order to properly build these offshore structures.

The way shallow water waves behave makes them a common sight in coastal areas. The behaviour of these waves is largely dependent on the ratio of wavelength to the water depth. Waves moves more slowly in shallow water. The exact depth of water has little or no effect on the differences between shallow and deep water waves. The ratio of the wavelength to the water depth determines it.

In deep water, the wavelength λ is less than the water depth h. I.e. $\lambda < h,\ \frac{\lambda}{h} \ll 1.$

In shallow water, the wavelength λ is greater than the water depth h, I.e. $\lambda > h,\ \frac{\lambda}{h} \gg 1.$

3.6 SPEED OF WATER WAVE IN DEEP AND SHALLOW WATER

Recall our dispersion equation;

$$c^2 = \frac{g\,\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right)$$

In deep water, $\lambda < h, \frac{\lambda}{h} \gg 1$, and $\frac{2\pi h}{\lambda}$ is big.

Therefore, as $\theta \longrightarrow 0,\ then\ tanh\frac{2\pi h}{\lambda} \longrightarrow 1$, then our wave speed equation becomes;

$$c^2 = \frac{g\,\lambda}{2\pi}$$

$$= c = \sqrt{\frac{g\,\lambda}{2\pi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(18.3)}$$

where the wave speed c is proportional to the square root of wavelength, this implies that c depends on gravitational force, surface tension and wavelength.

In shallow water, we have that $\lambda > h,\ \frac{\lambda}{h} \gg 1$ and $\frac{2\pi h}{\lambda}$ is small. Therefore as $tan\theta \longrightarrow \theta, when$ $\frac{2\pi h}{\lambda}$ is small, we have;

$$c^2 = \frac{g\,\cancel{\lambda}}{\cancel{2\pi}} * \frac{\cancel{2\pi}h}{\cancel{\lambda}}$$

$$= c = \sqrt{gh} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(18.4)}$$

where the wave speed in shallow water tends to a constant i.e. in shallow water, waves move with a speed that is equal to the square root of acceleration due to gravity and water depth. And wave speed is independent of the wavelength.

The speed of the deep water waves depends on the wavelength of the wave. And we conclude that deep water shows dispersion and a wave with a longer wavelength travels at a higher speed. While shallow water waves shows no dispersion, their speed is independent of their wavelength.

3.7 PROGRESSIVE WAVES IN DEEP AND SHALLOW WATER

Recall from equation (10), we have $\varphi = \frac{an\, coshm(y+h)\ \cos(mx-nt)}{msinhmh}$, which is the equation for velocity potential in a progressive wave. We can derive the progressive waves for deep and shallow water.

3.7.1 PROGRESSIVE WAVES IN DEEP WATER

Recall for deep water, $\lambda < h$, $\frac{\lambda}{h} \ll 1$. And equation (10) can be expressed as;

$$\varphi = \frac{an}{m}\ [\frac{\frac{[e^{m(y+h)}+e^{-m(y+h)}]}{2}}{\frac{e^{mh}-e^{-mh}}{2}}] \cos(mx-nt) = \frac{an}{m}\left[\frac{e^{m(y+h)}+e^{-m(y+h)}}{e^{mh}-e^{-mh}}\right] \cos(mx-nt)$$

Since for deep water, $\lambda < h$, $\frac{\lambda}{h} \ll 1$, this implies that $e^{m(y+h)} > e^{-m(y+h)}, and\ e^{mh} > e^{-mh}$, therefore we have that;

$\varphi = \frac{an}{m}\left[\frac{e^{m(y+h)}}{e^{mh}}\right]\cos(mx - nt)$

$\varphi = \frac{an}{m} e^{my} \cos(mx - nt)$ ………………………………………………… (19.1)

This is the progressive wave equation in deep water.

3.7.2 PROGRESSIVE WAVES IN SHALLOW WATER

First recall that $\varphi = \frac{an\, coshm(y+h)\ \cos(mx-nt)}{msinhmh}$ …………………………………… (10),

And the dispersion equation is given as

$c^2 = \frac{g\ tanhmh}{m}$, but $c^2 = \frac{n^2}{m^2}$, therefore the dispersion equation can be rewritten as;

$\frac{n^2}{m^2} = \frac{g\ tanhmh}{m}$

$\frac{n^2}{m} = gtanhmh$

$\frac{1}{m} = \frac{gtanhmh}{n^2}$ ……………………………………………………………..… (19.2)

Put equation (19.2) into equation (10)

$\varphi = \left[\frac{an\ coshm(y+h)\cos(mx-nt)}{sinhmh} * \frac{gtanhmh}{n^2}\right] = \left[\frac{\cancel{an}\ coshm(y+h)\cos(mx-nt)}{\cancel{sinhmh}} * \frac{g\ \cancel{sinhmh}}{\cancel{n}^2\ coshmh}\right]$

$\varphi = \frac{ga\ coshm(y+h)\cos(mx-nt)}{n\ coshmh}$ …………………………………………….... (20)

This is another form of a progressive wave.

In shallow water, we have that $\lambda > h$, $\frac{\lambda}{h} \gg 1$ this implies that $e^{m(y+h)} > e^{-m(y+h)}$, $and\ e^{mh} > e^{-mh}$, therefore;

$$\varphi = \frac{ga}{n}\left[\frac{\frac{\left[e^{m(y+h)}+e^{-m(y+h)}\right]}{2}}{\frac{e^{mh}-e^{-mh}}{2}}\right] = \frac{ga}{n}\left[\frac{e^{m(y+h)}+e^{-m(y+h)}}{e^{mh}-e^{-mh}}\right]$$

Since $e^{m(y+h)} > e^{-m(y+h)}, and\ e^{mh} > e^{-mh}$, it implies that;

$$\varphi = \frac{ga}{n}\left[\frac{e^{m(y+h)}}{e^{mh}}\right]\cos(mx-nt)$$

$$\varphi = \frac{ga}{n}e^{my}\cos(mx-nt) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Equation (21) is the equation of the progressive waves in shallow water.

Note: equation (19.1) and (21) are equations used when we are dealing with deep water while equation (21) is used when we are dealing with shallow water.

3.8 KINETIC AND POTENTIAL ENERGY OF A PROGRESSIVE WAVE ON A DEEP AND SHALLOW WATER

3.8.1 KINETIC ENERGY IN DEEP WATER

Recall that velocity profile is given as $\varphi = \frac{ga}{n}e^{my}\cos(mx-nt)$

Also recall from dispersion equation; $n^2 = mg\ tanhmh$ for deep water, as $\theta \longrightarrow 0,\ then\ tanhm \longrightarrow 1$, therefore $n^2 = mg$……………………………………… (*)

Then $= \frac{an}{m}e^{my}\cos(mx-nt)$,

Then, kinetic energy becomes

$$k = \frac{1}{2}\rho\int_{x=0}^{x=\lambda}\int_{y=-h}^{y=\eta}\overrightarrow{q^2}\,dxdy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22.1)$$

where $\vec{q}$ is the velocity and $\vec{q} = ui + vj,\ \overrightarrow{q^2} = u^2 + v^2, and\ u = \frac{-\partial\varphi}{\partial x}, v = \frac{-\partial\varphi}{\partial y}$

$\therefore \overrightarrow{q^2} = u^2 + v^2 = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$

$k = \frac{1}{2}\rho \int_{x=0}^{x=\lambda} \int_{y=-h}^{y=\eta} \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} dxdy$

By using Green's theorem;

$= \frac{1}{2}\rho \int_0^\lambda \varphi \frac{\partial\varphi}{\partial n}\Big|_{y=0} ds$ ………………………………………………………….. (22.2)

Integrating the undisturbed surface and n generating to y, we have;

$= \frac{1}{2}\rho \int_0^\lambda \varphi \frac{\partial\varphi}{\partial y}\Big|_{y=0} dx$

Recall $\varphi = \frac{an}{m} e^{my} \cos(mx - nt)$

$\frac{\partial\varphi}{\partial y} = \frac{an\cancel{m}}{\cancel{m}} e^{my} \cos(mx - nt)$

$\frac{\partial\varphi}{\partial y} = ane^{my} \cos(mx - nt)$………………………………………………… (22.3)

$\varphi \frac{\partial\varphi}{\partial n} = \left[\frac{an}{m} e^{my} \cos(mx - nt)\right] [ane^{my} \cos(mx - nt)]$

$= \frac{(an)^2}{m} e^{2my} cos^2(mx - nt)$ …………………………………………………… (22.4)

At y=0, $\varphi \frac{\partial\varphi}{\partial y}\Big|_{y=0} = \frac{(an)^2}{m} cos^2(mx - nt)$

But from equation (*), $n^2 = mg$

$\therefore \varphi \frac{\partial\varphi}{\partial y}\Big|_{y=0} = a^2 g cos^2(mx - nt)$ …………………………………………….. (22.5)

$k = \frac{1}{2}\rho \int a^2 g cos^2(mx - nt)$dx

$k = \frac{\rho g a^2}{2} \int cos^2(mx - nt)$dx

$$k = \frac{\rho g a^2}{2} \int_0^{\lambda} \frac{1+cos2(mx-nt)}{2} = \frac{\rho g a^2}{4} \int_0^{\lambda} 1 + \cos{(2mx - 2nt)}\, dx$$

$$k = \frac{\rho g a^2}{4} \; [x|_0^{\lambda} + \int_0^{\lambda} \cos{(2mx - 2nt)}\, dx\,]$$

$$k = \frac{\rho g a^2}{4} [[\lambda] + \left[\frac{\sin{(2mx-2nt)}}{2m}\Big|_0^{\lambda}\right]]$$

$$k = \frac{\rho g a^2}{4} [\lambda + \frac{1}{2m} \sin{(2mx - 2nt)}|_0^{\lambda}]$$

$$k = \frac{\rho g a^2}{4} [\lambda + \frac{1}{2m} (\sin{(2m\lambda - 2nt)} - \sin{(2m(0) - 2nt)})]$$

$$k = \frac{\rho g a^2}{4} \left[\lambda + \frac{1}{2m} (\sin{(2m\lambda - 2nt)} - \sin{(-2nt)})\right]$$

$$k = \frac{\rho g a^2}{4} \left[\lambda + \frac{1}{2m} (\sin{(2m\lambda - 2nt)} + \sin{(2nt)})\right]$$

Also recall, $\lambda = \frac{2\pi}{m}$ this implies that m=$\frac{2\pi}{\lambda}$,

$$\therefore 2m\lambda = 2\lambda * \frac{2\pi}{\lambda} = 4\pi$$

$$k = \frac{\rho g a^2}{4} [\lambda + \frac{1}{2m} (\, \sin{(4\pi - 2nt)}) + \frac{1}{2m} \sin{(2nt)}] \quad \text{……………………………………….. (22.6)}$$

Next, we use trigonometry of double angles on equation 22.6

$$k = \frac{\rho g a^2}{4} [\lambda + \frac{1}{2m} (\, \sin 4\pi cos2nt - cos4\pi sin2nt) + \frac{1}{2m} \sin{(2nt)}]$$

$$k = \frac{\rho g a^2}{4} [\lambda + \frac{1}{2m} (\, - cos4\pi sin2nt) + \frac{1}{2m} \sin{(2nt)}]$$

$$k = \frac{\rho g a^2}{4} [\lambda - \frac{sin2nt}{2m} + \frac{sin2nt}{2m}]$$

where $sin4\pi = 0, cos4\pi = 1$

$$\therefore k = \frac{\rho g a^2 \lambda}{4} \quad \text{……………………………………………………………….. (23)}$$

3.8.2 POTENTIAL ENERGY FOR A PROGRESSIVE IN DEEP WATER

For potential energy we have;

$P.E = \frac{1}{2}\rho g \int_0^\lambda \eta^2 dx$ .... (24.1)

Recall the equation of the surface is given as $\eta = a\sin(mx - nt)$, $\eta^2 = a^2\sin^2(mx - nt)$

$\therefore\ P.E = \frac{1}{2}\rho g \int_0^\lambda a^2\sin^2(mx - nt)dx$

$P.\quad E = \frac{1}{2}\rho g a^2 \int_0^\lambda \sin^2(mx - nt)dx$

$P.E = \frac{\rho g a^2}{2}\int_0^\lambda \frac{1-\cos(2mx-2nt)}{2}$

$= \frac{\rho g a^2}{4}(\int_0^\lambda 1 - \cos(2mx - 2nt)dx$

$= \frac{\rho g a^2}{4}([x|_0^\lambda] - \int_0^\lambda \cos(2mx - 2nt)dx)$

$= \frac{\rho g a^2}{4}[\lambda - \frac{\sin(2mx-2nt)}{2m}\Big|_0^\lambda]$

$= \frac{\rho g a^2}{4}[\lambda - \frac{1}{2m}(\sin(2m\lambda - 2nt) - \sin(- 2nt))]$

$= \frac{\rho g a^2}{4}[\lambda - \frac{1}{2m}(\sin(2m\lambda - 2nt) + \sin(2nt))]$

Also, recall that $\lambda = \frac{2\pi}{m}$ this implies that m=$\frac{2\pi}{\lambda}$,

$\therefore 2m\lambda = 2\cancel{\lambda} * \frac{2\pi}{\cancel{\lambda}} = 4\pi$

$= \frac{\rho g a^2}{4}[\lambda - \frac{1}{2m}(\sin(4\pi - 2nt)) + \sin(2nt)]$

Using the trig of double angles

$= \frac{\rho g a^2}{4}[\lambda - \frac{1}{2m}(\sin 4\pi \cos 2nt - \cos 4\pi \sin 2nt) + \sin(2nt))]$

$= \frac{\rho g a^2}{4}[\lambda - \frac{1}{2m}(- \sin 2nt + \sin 2nt)]$

$P.E = \frac{\rho g a^2 \lambda}{4}$ ………………………………………………………………….. (25)

The total energy is given as; $E = K.E + P.E$

$\frac{\rho g a^2 \lambda}{4} + \frac{\rho g a^2 \lambda}{4}$

$\frac{\rho g a^2 \lambda}{2}$ ………………………………………………………………….. (25.2)

3.8.3 KINETIC ENERGY FOR A PROGRESSIVE WAVE IN SHALLOW WATER

Recall that velocity profile is given as $\varphi = \frac{ga}{n} e^{my} \cos{(mx - nt)}$

Also recall from dispersion equation; $n^2 = mg\ tanhmh$, and for deep water, as $\theta \longrightarrow 0,\ then\ tanhm \longrightarrow mh$, therefore $n^2 = mg * mh = m^2 gh$

$n^2 = m^2 gh$ ………………………………………………………… (**)

We can rewrite the velocity profile as $\varphi = \frac{an}{m^2 h} e^{my} cos(mx - nt)$

Kinetic energy is given as;

$k = \frac{1}{2}\rho \int_{x=0}^{x=\lambda} \int_{y=-h}^{y=\eta} \overrightarrow{q^2}\, dxdy$ …………………………………………… (26.1)

where $\vec{q}$ is the velocity and $\vec{q} = ui + vj,\ \overrightarrow{q^2} = u^2 + v^2, and\ u = \frac{-\partial\varphi}{\partial x}, v = \frac{-\partial\varphi}{\partial y}$

$\therefore \overrightarrow{q^2} = u^2 + v^2 = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$

$k = \frac{1}{2}\rho \int_{x=0}^{x=\lambda} \int_{y=-h}^{y=\eta} \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} dxdy$

By using Green's theorem;

$$= \frac{1}{2}\rho\int_0^\lambda \varphi \frac{\partial\varphi}{\partial n}\Big|_{y=0} ds$$

Integrating the undisturbed surface and n generating to y, we have;

$$= \frac{1}{2}\rho\int_0^\lambda \varphi \frac{\partial\varphi}{\partial y}\Big|_{y=0} dx$$

Recall $\varphi = \frac{an}{m^2h} e^{my} \cos{(mx - nt)}$

$$\frac{\partial\varphi}{\partial y} = \frac{an}{mh} e^{my} cos(mx - nt)$$

$$\varphi\frac{\partial\varphi}{\partial n} = \left[\frac{an}{m^2h} e^{my} \cos{(mx - nt)}\right] \left[\frac{an}{mh} e^{my} cos(mx - nt)\right]$$

$$\varphi\frac{\partial\varphi}{\partial n} = \left[\frac{a^2n^2}{m^3h^2} e^{2my} cos^2(mx - nt)\right]$$

At y=0, $\varphi\frac{\partial\varphi}{\partial n}\Big|_{y=0} = \left[\frac{a^2n^2}{m^3h^2} cos^2(mx - nt)\right]$

But recall $n^2 = m^2gh$

$$\therefore \varphi\frac{\partial\varphi}{\partial n}\Big|_{y=0} = \left[\frac{a^2g}{mh} cos^2(mx - nt)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (26.2)$$

$$k = \frac{1}{2}\rho \int_0^\lambda \frac{a^2n^2}{m^3h^2} cos^2(mx - nt)\, dx$$

$$k = \frac{1\rho a^2 g}{2mh}\int_0^\lambda cos^2(mx - nt)dx$$

$$k = \frac{\rho a^2 g}{2mh}\int_0^\lambda \frac{1 + cos(2mx - 2nt)}{2} dx \quad = \frac{\rho a^2 g}{4mh}\int 1 + cos(2mx - 2nt)dx$$

$$k = \frac{\rho a^2 g}{4mh}\left[[x|_0^\lambda + \int_0^\lambda \cos{(2mx - 2nt)}\, dx\right]$$

$$k = \frac{\rho a^2 g}{4mh}\left[[\lambda] + \left[\frac{\sin{(2mx - 2nt)}}{2m}\Big|_0^\lambda\right]\right]$$

$k = \frac{\rho a^2 g}{4mh}[\lambda + \frac{1}{2m}\sin(2mx - 2nt)|_0^\lambda]$

$k = \frac{\rho a^2 g}{4mh}[\lambda + \frac{1}{2m}(\sin(2m\lambda - 2nt) - \sin(2m(0) - 2nt))]$

$k = \frac{\rho a^2 g}{4mh}\left[\lambda + \frac{1}{2m}(\sin(2m\lambda - 2nt) - \sin(-2nt))\right]$

$k = \frac{\rho a^2 g}{4mh}\left[\lambda + \frac{1}{2m}(\sin(2m\lambda - 2nt) + \sin(2nt))\right]$

$\therefore 2m\lambda = 2\cancel{\lambda} * \frac{2\pi}{\cancel{\lambda}} = 4\pi$

$k = \frac{\rho a^2 g}{4mh}[\lambda + \frac{1}{2m}(\sin(4\pi - 2nt)) + \frac{1}{2m}\sin(2nt)]$ …………………………………… (26.3)

Next, we use trigonometry of double angles on equation 26.3

$k = \frac{\rho a^2 g}{4mh}[\lambda + \frac{1}{2m}(\sin 4\pi cos2nt - cos4\pi sin2nt) + \frac{1}{2m}\sin(2nt)]$

$k = \frac{\rho a^2 g}{4mh}[\lambda + \frac{1}{2m}(-cos4\pi sin2nt) + \frac{1}{2m}\sin(2nt)]$

$k = \frac{\rho a^2 g}{4mh}[\lambda - \frac{sin2nt}{2m} + \frac{sin2nt}{2m}]$

where $sin4\pi = 0, cos4\pi = 1$

$\therefore k = \frac{\rho a^2 g\lambda}{4mh}$ ………………………………………………………… (27)

3.8.4 POTENTIAL ENERGY FOR A PROGRESSIVE WAVE IN A SHALLOW WATER:

For potential energy we have;

$P.E = \frac{1}{2}\rho g\int_0^\lambda \eta^2 dx$ ………………………………………………… (27.1)

Recall the equation of the surface is given as $\eta = a\sin(mx - nt)$, $\eta^2 = a^2\sin^2(mx - nt)$

$\therefore\ P.E = \frac{1}{2}\rho g \int_0^\lambda a^2 \sin^2(mx - nt)dx$

$P.E = \frac{1}{2}\rho g a^2 \int_0^\lambda \sin^2(mx - nt)dx$

$P.E = \frac{\rho g a^2}{2} \int_0^\lambda \frac{1 - \cos(2mx - 2nt)}{2}$

$= \frac{\rho g a^2}{4} (\int_0^\lambda 1 - \cos(2mx - 2nt)dx$

$= \frac{\rho g a^2}{4} ([x|_0^\lambda] - \int_0^\lambda \cos(2mx - 2nt)dx)$

$= \frac{\rho g a^2}{4} [\lambda - \frac{\sin(2mx - 2nt)}{2m} \Big|_0^\lambda]$

$= \frac{\rho g a^2}{4} [\lambda - \frac{1}{2m}(\sin(2m\lambda - 2nt) - \sin(- 2nt))]$

$= \frac{\rho g a^2}{4} [\lambda - \frac{1}{2m}(\sin(2m\lambda - 2nt) + \sin(2nt))]$ ………………………………………… (27.2)

Also, recall that $\lambda = \frac{2\pi}{m}$ this implies that m=$\frac{2\pi}{\lambda}$,

$\therefore 2m\lambda = 2\lambda * \frac{2\pi}{\lambda} = 4\pi$

$= \frac{\rho g a^2}{4} [\lambda - \frac{1}{2m}(\sin(4\pi - 2nt)) + \sin(2nt)]$

Using the trig of double angles

$= \frac{\rho g a^2}{4} [\lambda - \frac{1}{2m}(\sin 4\pi \cos 2nt - \cos 4\pi \sin 2nt) + \sin(2nt))]$

$= \frac{\rho g a^2}{4} [\lambda - \frac{1}{2m}(- \sin 2nt + \sin 2nt)]$

$P.E = \frac{\rho g a^2 \lambda}{4}$ ………………………………………………………………………… (27.3)

The total energy is given as; $E = K.E\ +\ P.E$

$$E = \frac{\rho a^2 g\lambda}{4mh} + \frac{\rho g a^2\lambda}{4}$$

$$E = \frac{\rho g a^2\lambda}{4}\left[\frac{1}{mh} + 1\right] \quad \ldots\ldots (27.4)$$

3.9 STANDING/STATIONARY WAVE

3.9.1 VELOCITY POTENTIAL FOR A STATIONARY WAVE

Let $\eta_1 = \frac{a}{2} sin(mx - nt)$ and $\eta_2 = \frac{a}{2} sin(mx - nt)$

The resultant $\eta_1 + \eta_2 = \frac{a}{2} sin(mx - nt) + \frac{a}{2} sin(mx - nt)$

$$\eta_1 + \eta_2 = \frac{a}{2}[sin(mx - nt) + sin(mx - nt)] \quad \ldots\ldots (28)$$

Recall from the sum of trigonometry identities for two different angles

$\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$, where $A = (mx - nt), B = (mx - nt)$

$$= \frac{a}{2}[2\sin(\frac{(mx-nt)+(mx-nt)}{2})\cos(\frac{(mx-nt)-(mx-nt)}{2})]$$

$$= a[\sin\frac{(2mx)}{2}\cos\frac{(-2nt)}{2}]$$

$$\eta = \sin mx \cos nt \quad \ldots\ldots (28.1)$$

We assume $\varphi = f(y)\sin(mx)\sin(nt)$ ……… (28.2)

Putting equation 25.2 into the Laplace equation $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$

$$\frac{\partial\varphi}{\partial x} = mf(y)cos(mx)sin(nt), \quad \frac{\partial^2\varphi}{\partial x^2} = -m^2 f(y)sin(mx)sin(nt)$$

$$\frac{\partial\varphi}{\partial y} = \frac{dF}{dy} sin(mx)sin(nt)\ , \ \frac{\partial^2\varphi}{\partial y^2} = \frac{d^2 f}{\partial y^2} sin(mx)sin(nt)$$

The Laplace equation becomes; $-m^2 f(y) sin(mx) sin(nt) + \frac{d^2 f}{\partial y^2} sin(mx) sin(nt)$

$= (f''(y) - m^2 f(y)) sin(mx) sin(nt) = 0$

$= (f''(y) - m^2 f(y)) = 0$

We get the indicial root by solving the auxiliary equation by using the operator D method

$= D^2 - m^2 = 0$

$= D^2 = m^2$

$D = \pm m$

This gives the general solution; $f(y) = Ae^{my} + Be^{-my}$…………………………….. (28.3)

Substituting equation (25.3) into equation (25.2), we have;

$\varphi = (Ae^{my} + Be^{-my})\sin(mx)\sin(nt)$……………………………………………… (28.4)

From the physical boundary condition (equation (4), $\frac{\partial \varphi}{\partial y} = 0,\ y = -h$), we have;

$\frac{\partial \varphi}{\partial y} = (mAe^{my} - mBe^{-my})\sin(mx)\sin(nt) = 0$

At $y = -h$, we have

$\left(Ae^{-mh} - Be^{mh}\right) m \sin(mx)\sin(nt) = 0$

$= \left(Ae^{-mh} - Be^{mh}\right) = 0$

$= Ae^{-mh} = Be^{mh}$

$A = \frac{Be^{mh}}{e^{-mh}}$

$A = Be^{2mh}$……………………………………………………………………….. (28.5)

Put equation 28.5 into equation 28.4;

$\varphi = (Be^{2mh}.e^{my} + Be^{-my})\sin(\text{mx})\sin(nt)$=0

Next, we factor out Be^{mh}

$\varphi = Be^{mh}(e^{m(y+h)} + e^{-m(y+h)})\sin(\text{mx})\sin(nt)$ =0

$= 2Be^{mh}[\frac{e^{m(y+h)}+e^{-m(y+h)}}{2}]\sin(\text{mx})\sin(nt)$

$= 2Be^{mh}coshm(y + h)\sin(\text{mx})\sin(nt)$

$\varphi = Dcoshm(y + h)\sin(\text{mx})\sin(nt)$ …………………………………………… (28.6)

where $D = 2Be^{mh}$

Next, using the kinematic boundary condition from equation (2.7), we can find D

$[\frac{\partial\eta}{\partial t} =- \frac{\partial\varphi}{\partial y}]$

$\eta = \text{a}\sin(\text{mx})\cos(nt)$

$\frac{\partial\eta}{\partial t} =- an\sin(\text{mx})\sin(nt)$………………………………………………………… (28.7)

$-\frac{\partial\varphi}{\partial y} =- mDsinhm(y + h)\sin(\text{mx})\sin(nt)$ …………………………………………… (28.8)

Put equation (28.7) and (28.8) into equation 2.7

$-an\sin(\text{mx})\sin(nt) =- mDsinhm(y + h)\sin(\text{mx})\sin(nt)$, at y=0

$\text{an} = \text{mDsinhmh}$

$= \text{D} = \frac{\text{an}}{\text{msinhmh}}$ ……………………………………………………………….. (28.9)

next, we substitute equation (28.9) back into (28.6)

$\varphi = \frac{\text{an}}{\text{msinhmh}}coshm(y + h)\sin(\text{mx})\sin(nt)$ …………………………………… (29)

Next, we use the pressure boundary condition $= \frac{\partial^2\varphi}{\partial t^2} + \frac{\partial\varphi}{\partial y}g = 0$

$$\frac{\partial\varphi}{\partial t}=\frac{n^2acoshm(y+h)\sin(\mathrm{mx})\cos(\mathrm{nt})}{msinhmh}\ ,\ \frac{\partial^2\varphi}{\partial t^2}=-\frac{n^3acoshm(y+h)\sin(\mathrm{mx})\sin(\mathrm{nt})}{msinhmh}\ \ldots\ldots\ldots\ldots\ldots\ldots.(29.1)$$

$$\frac{g\partial\varphi}{\partial y}=\frac{gan\ sinhm(y+h)sin(mx)sin(nt)}{sinhmh}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ (29.2)$$

putting both equations back into the pressure condition we have;

$$-\frac{n^3acoshm(y+h)\sin(\mathrm{mx})\sin(\mathrm{nt})}{msinhmh}+\frac{gan\ sinhm(y+h)sin(mx)sin(nt)}{sinhmh}=0$$

$$-\frac{n^3\cancel{a}coshm(y+h)\cancel{\sin(\mathrm{mx})\sin(\mathrm{nt})}}{m\cancel{sinhmh}}=-\frac{g\cancel{an}\ sinhm(y+h)\cancel{sin(mx)sin(nt)}}{\cancel{sinhmh}}$$

$$\frac{n^2coshm(y+h)}{m}=gsinhm(y+h)$$

$$\frac{n^2}{m}=\frac{gsinhm(y+h)}{coshm(y+h)}$$

At y=o, $\frac{n^2}{m}=\frac{gsinhmh}{coshmh}$

$$\frac{n}{m}=\frac{gsinhmh}{ncoshmh}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ (29.3)$$

Put equation (26.3) into equation (26)

$$\varphi=\frac{g\cancel{sinhmh}*acoshm(y+h)\sin(\mathrm{mx})\sin(nt)}{ncoshmh*\cancel{sinhmh}}$$

$$\varphi=\frac{ag\ coshm(y+h)\sin(\mathrm{mx})\sin(nt)}{n\ coshmh}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ (29.4)$$

Equation (29.4) is the velocity potential of a stationary wave.

3.9.2 DERIVATION OF THE PATH OF A STATIONARY WAVE

$$\mathrm{u}=-\frac{\partial\varphi}{\partial\mathrm{x}}\ ,\ \mathrm{v}=-\frac{\partial\varphi}{\partial\mathrm{y}}$$

Recall from equation (29.4) $\varphi = \frac{ag\ coshm(y+h)\sin(mx)\sin(nt)}{n\ coshmh}$

$$u = -\frac{\partial\varphi}{\partial x} = -\frac{mga\ coshm(y+h)cos(mx)sin(nt)}{n\ coshmh}$$

$$v = -\frac{\partial\varphi}{\partial y} = -\frac{mga\ sinhm(y+h)\sin(mx)\sin(nt)}{n\ coshmh}$$

$$x = \int udt = \int -\frac{mga\ coshm(y+h)cos(mx)sin(nt)}{n\ coshmh}dt$$

$$x = \frac{mga\ coshm(y+h)cos(mx)cos(nt)}{n^2\ coshmh} \text{ ....(29.5)}$$

$$y = \int vdt = \int -\frac{mga\ sinhm(y+h)\sin(mx)\sin(nt)}{n\ coshmh}dt$$

$$y = \frac{mga\ sinhm(y+h)sin(mx)cos(nt)}{n^2\ coshmh} \text{ .... (29.6)}$$

Divide equation (29.6) by equation (29.5)

$$\frac{y}{x} = \frac{\cancel{mga}\ sinhm(y+h)sin(mx)\cancel{cos(nt})}{\cancel{n^2\ coshm}h} * \frac{\cancel{n^2\ coshmh}}{\cancel{mga}\ coshm(y+h)cos(mx)\cancel{cos(nt)}}$$

$$\frac{y}{x} = \frac{sinhm(y+h)sinmx}{coshm(y+h)cosmx} = tanhm(y+h)tanmx$$

$$\frac{y}{x} = tanhm(y+h)tanmx \text{ .... (29.7)}$$

The motion of the stationary wave is rectilinear equation of a straight line.

3.9.3 DEEP AND SHALLOW WATER CONDITIONS IN A STATONARY WAVE

Recall $\varphi = \frac{ag\ coshm(y+h)\sin(mx)\sin(nt)}{n\ coshmh}$

This can be written as;

$$\varphi = \frac{ag}{n}\left[\frac{e^{m(y+h)}+e^{-m(y+h)}}{2} * \frac{2}{e^{mh}+e^{-mh}}\right] sin(mx)sin(nt) = \frac{ag}{n}\left[\frac{e^{m(y+h)}+e^{-m(y+h)}}{e^{mh}+e^{-mh}}\right]sin(mx)sin(nt)$$

For deep water and shallow, $e^{-m(y+h)} \to 0, e^{-mh} \to 0$

$$\varphi = \frac{ag}{n}[\frac{e^{m(y+h)}}{e^{mh}}]sin(mx)sin(nt)$$

$$\varphi = \frac{ag}{n}[\frac{e^{my}.e^{mh}}{e^{mh}}]sin(mx)sin(nt)$$

$$\varphi = \frac{age^{my}sin(mx)sin(nt)}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(30)}$$

3.10 POTENTIAL ENERGY OF A STATIONARY WAVE IN SHALLOW AND DEEP WATER

3.10.1 KINETIC ENERGY OF A STATIOANRY WAVE IN DEEP WATER

Recall from equation (30)

$$\varphi = \frac{ag}{n}e^{my}sin(mx)sin(nt)$$

Also recall from dispersion equation; $n^2 = mg\ tanhmh$ for deep water, as $\theta \longrightarrow 0,\ then\ tanhm \longrightarrow 1$, therefore $n^2 = mg$………………………………………….. (*)

Then $= \frac{an}{m}e^{my}sin(mx)sin(nt)$,

Then, kinetic energy becomes

$$k = \frac{1}{2}\rho\int_{x=0}^{x=\lambda}\int_{y=-h}^{y=\eta}\overrightarrow{q^2}\,dxdy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(30.1)}$$

where $\vec{q}$ is the velocity and $\vec{q} = ui + vj,\ \overrightarrow{q^2} = u^2 + v^2, and\ u = \frac{-\partial\varphi}{\partial x}, v = \frac{-\partial\varphi}{\partial y}$

$$\therefore \overrightarrow{q^2} = u^2 + v^2 = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$$

$k = \frac{1}{2}\rho \int_{x=0}^{x=\lambda} \int_{y=-h}^{y=\eta} [\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}]dxdy$

By using Green's theorem;

$= \frac{1}{2}\rho\int_0^\lambda \varphi \frac{\partial\varphi}{\partial n}\Big|_{y=0} ds$ ………………………………………………………….. (30.2)

Integrating the undisturbed surface dx and n generating to dy, we have;

$= \frac{1}{2}\rho\int_0^\lambda \varphi \frac{\partial\varphi}{\partial y}\Big|_{y=0} dx$

Recall $\varphi = \frac{an}{m} e^{my}\sin(\text{mx})\sin(\text{nt})$

$\frac{\partial\varphi}{\partial y} = \frac{anm}{m} e^{my}\sin(\text{mx})\sin(\text{nt})$

$\frac{\partial\varphi}{\partial y} = ane^{my}\sin(\text{mx})\sin(\text{nt})$……………………………………………………………. (30.3)

$\varphi\frac{\partial\varphi}{\partial n} = \left[\frac{an}{m} e^{my}\sin(\text{mx})\sin(\text{nt})\right] [ane^{my}\sin(\text{mx})\sin(\text{nt})]$

$= \frac{(an)^2}{m} e^{2my}\sin^2(\text{mx})\sin^2(\text{nt})$ ……………………………………………………… (30.4)

At y=0, $\varphi\frac{\partial\varphi}{\partial y}\Big|_{y=0} = \frac{(an)^2}{m}\sin^2(\text{mx})\sin^2(\text{nt})$

But from equation (*), $n^2 = mg$

$\therefore \varphi\frac{\partial\varphi}{\partial y}\Big|_{y=0} = a^2 g\sin^2(\text{mx})\sin^2(\text{nt})$ …………………………………………….. (30.5)

$k = \frac{1}{2}\rho \int a^2 g\sin^2(\text{mx})\sin^2(\text{nt})\ \text{dx}$

$k = \frac{\rho g a^2}{2}\int \sin^2(\text{mx})\sin^2(\text{nt})\ \text{dx}$

$k = \frac{\rho g a^2 \sin^2(\text{nt})}{2}\int_0^\lambda \sin^2(\text{mx})\text{dx} = \frac{\rho g a^2 \sin^2(\text{nt})}{4}[\int_0^\lambda 1 - \cos(2mx)\, dx]$

$k = \frac{\rho g a^2 \sin^2(\mathrm{nt})}{4} \; [x|_0^\lambda - \int_0^\lambda \cos(2mx)\, dx\,]$

$k = \frac{\rho g a^2 \sin^2(\mathrm{nt})}{4} [[\lambda] + \left[\frac{\sin(2mx)}{2m}\Big|_0^\lambda\right]]$

$k = \frac{\rho g a^2 \sin^2(\mathrm{nt})}{4} [\lambda + \frac{1}{2m} \sin(2mx)|_0^\lambda]$

$k = \frac{\rho g a^2 \sin^2(\mathrm{nt})}{4} [\lambda + \frac{1}{2m} (sin(2m\lambda)]$

$k = \frac{\rho g a^2 \sin^2(\mathrm{nt})}{4} \left[\lambda + \frac{1}{2m} (sin(2m\lambda))\right]$

Also recall, $\lambda = \frac{2\pi}{m}$ this implies that m=$\frac{2\pi}{\lambda}$,

$\therefore\ 2m\lambda = 2\lambda * \frac{2\pi}{\lambda} = 4\pi$

$k = \frac{\rho g a^2 \sin^2(\mathrm{nt})}{4} [\lambda + \frac{1}{2m} (\,\sin(4\pi)\,]$ …………………………………………….. (30.6)

$k = \frac{\rho g a^2 \sin^2(\mathrm{nt})}{4} [\lambda]$

$k = \frac{\rho g a^2 \sin^2(\mathrm{nt})\lambda}{4}$

where $sin 4\pi = 0,$

$\therefore\ k = \frac{\rho g a^2 \sin^2(\mathrm{nt})\lambda}{4}$ ……………………………………………………………… (30.7)

3.10.2 POTENTIAL ENERGY FOR A STATIONARY WAVE IN DEEP WATER

For potential energy we have;

$\mathrm{P.E} = \frac{1}{2} \rho g \int_0^\lambda \eta^2 dx$

Recall the equation of the surface is given as

$\eta = a\sin(mx)\cos(nt), \ \ \eta^2 = a^2\sin^2(mx)\cos^2(nt)$

$\therefore \ P.E = \frac{1}{2}\rho g\int_0^\lambda a^2\sin^2(mx)\cos^2(nt)dx$

$P.E = \frac{1}{2}\rho g a^2\cos^2(nt)\int_0^\lambda \sin^2(mx)dx$

$P.E = \frac{\rho g a^2\cos^2(nt)}{2}\int_0^\lambda \frac{1-\cos(2mx)}{2}$

$= \frac{\rho g a^2\cos^2(nt)}{4}(\int_0^\lambda 1-\cos(2mx)dx$

$\frac{\rho g a^2\cos^2(nt)}{4}([x|_0^\lambda] - \int_0^\lambda \cos(2mx-2nt)dx)$

$= \frac{\rho g a^2\cos^2(nt)}{4}[\lambda - \frac{\sin(2mx)}{2m}\Big|_0^\lambda]$

$= \frac{\rho g a^2\cos^2(nt)}{4}[\lambda - \frac{1}{2m}(\sin(2m\lambda)]$

$= \frac{\rho g a^2\cos^2(nt)}{4}[\lambda - \frac{1}{2m}(\sin(2m\lambda)]$

Also, recall that $\lambda = \frac{2\pi}{m}$ this implies that m=$\frac{2\pi}{\lambda}$,

$\therefore 2m\lambda = 2\cancel{\lambda} * \frac{2\pi}{\cancel{\lambda}} = 4\pi$

$= \frac{\rho g a^2\cos^2(nt)}{4}[\lambda - \frac{1}{2m}(\sin(4\pi)]$

$= \frac{\rho g a^2\cos^2(nt)}{4}[\lambda]$

$= \frac{\rho g a^2\cos^2(nt)\lambda}{4}$ …………………………………………………………… (30.8)

If $\cos^2(nt) = 1$, then we have that;

$P.E = \frac{\rho g a^2\lambda}{4}$

To get the energy: $E = P.E + K.E$

$$E = \frac{\rho g a^2 \cos^2(nt)\lambda}{4} + \frac{\rho g a^2 \sin^2(nt)\lambda}{4}$$

$$= \frac{\rho g a^2 \lambda}{4}[\cos^2(nt) + \sin^2(nt)]$$

$$= \frac{\rho g a^2 \lambda}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30.9)$$

This equation is the total energy of a stationary wave in deep water.

3.10.3 KINETIC ENERGY FOR A STATIONARY WAVE IN SHALLOW WATER

Recall that velocity profile is given as $\varphi = \varphi = \frac{ag}{n} e^{my} sin(mx) sin(nt)$

Also recall from dispersion equation; $n^2 = mg\, tanhmh$ and for deep water, as $\theta \longrightarrow 0,\ then\ tanhm \longrightarrow mh$, therefore $n^2 = mg * mh = m^2 gh$

$$n^2 = m^2 gh \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (**)$$

We can rewrite the velocity profile as $\varphi = \frac{an}{m^2 h} e^{my} sin(mx) sin(nt)$

Kinetic energy is given as;

$$k = \frac{1}{2}\rho \int_{x=0}^{x=\lambda} \int_{y=-h}^{y=\eta} \overrightarrow{q^2}\, dxdy$$

where $\vec{q}$ is the velocity and $\vec{q} = ui + vj,\ \overrightarrow{q^2} = u^2 + v^2, and\ u = \frac{-\partial\varphi}{\partial x}, v = \frac{-\partial\varphi}{\partial y}$

$$\therefore \overrightarrow{q^2} = u^2 + v^2 = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$$

$$k = \frac{1}{2}\rho \int_{x=0}^{x=\lambda} \int_{y=-h}^{y=\eta} \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} dxdy$$

By using Green's theorem;

$= \frac{1}{2}\rho\int_0^{\lambda} \varphi \frac{\partial\varphi}{\partial n}\Big|_{y=0} ds$

Integrating the undisturbed surface and n generating to y, we have;

$= \frac{1}{2}\rho\int_0^{\lambda} \varphi \frac{\partial\varphi}{\partial y}\Big|_{y=0} dx$

Recall $\varphi = \frac{an}{m^2h} e^{my} sin(mx) sin(nt)$

$\frac{\partial\varphi}{\partial y} = \frac{an}{mh} e^{my} sin(mx) sin(nt)$

$\varphi\frac{\partial\varphi}{\partial n} = \left[\frac{an}{m^2h} e^{my} sin(mx) sin(nt)\right] \left[\frac{an}{mh} e^{my} sin(mx) sin(nt)\right]$

$\varphi\frac{\partial\varphi}{\partial n} = \left[\frac{a^2n^2}{m^3h^2} e^{2my} sin^2(mx) sin^2(nt)\right]$

At y=0, $\varphi\frac{\partial\varphi}{\partial n}\Big|_{y=0} = \left[\frac{a^2n^2}{m^3h^2} sin^2(mx) sin^2(nt)\right]$

But recall $n^2 = m^2gh$

$\therefore \varphi\frac{\partial\varphi}{\partial n}\Big|_{y=0} = \left[\frac{a^2g}{mh} sin^2(mx) sin^2(nt)\right]$ …………………………………………… (30.10)

$k = \frac{1}{2}\rho \int_0^{\lambda} \frac{a^2n^2}{m^3h^2} sin^2(mx) sin^2(nt)\, dx$

$k = \frac{1\rho a^2 g sin^2(nt)}{2mh}\int_0^{\lambda} sin^2(mx) dx$

$k = \frac{\rho a^2 g}{2mh}\int_0^{\lambda}\frac{1-sin(2mx)}{2} dx \quad = \frac{\rho a^2 g\, sin^2(nt)}{4mh}\int 1 - sin(2mx) dx$

$k = \frac{\rho a^2 g\, sin^2(nt)}{4mh}\left[[x|_0^{\lambda} - \int_0^{\lambda} sin(2mx) dx\right]$

$k = \frac{\rho a^2 g\, sin^2(nt)}{4mh}\left[[\lambda] - \left[\frac{\sin(2mx)}{2m}\Big|_0^{\lambda}\right]\right]$

$k = \frac{\rho a^2 g\, sin^2(nt)}{4mh}\left[\lambda - \frac{1}{2m}\sin(2mx)|_0^{\lambda}\right]$

$$k = \frac{\rho a^2 g\, sin^2(nt)}{4mh}[\lambda - \frac{1}{2m}(sin2m\lambda)]$$

$$k\frac{\rho a^2 g\, sin^2(nt)}{4mh}\left[\lambda - \frac{1}{2m}(\sin(2m\lambda))\right]$$

$$\therefore 2m\lambda = 2\lambda * \frac{2\pi}{\lambda} = 4\pi$$

$$k = \frac{\rho a^2 g\, sin^2(nt)}{4mh}[\lambda]$$

$$\mathrm{k} = \frac{\rho a^2 g\, sin^2(nt)\lambda}{4mh} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30.11)$$

3.10.4 POTENTIAL ENERGY FOR A STATIONARY WAVE IN SHALLOW WATER

For potential energy we have;

$$\mathrm{P.E} = \frac{1}{2}\rho g \int_0^\lambda \eta^2 dx$$

Recall the equation of the surface is given as $\eta = \mathrm{sinmxcosnt}$, $\eta^2 = a^2\sin^2(mx)\cos^2(nt)$

$$\therefore \ \mathrm{P.E} = \frac{1}{2}\rho g \int_0^\lambda a^2\sin^2(mx)\cos^2(nt)dx$$

$$\mathrm{P.E} = \frac{1}{2}\rho g a^2\cos^2(nt)\int_0^\lambda \sin^2(mx)dx$$

$$\mathrm{P.E} = \frac{\rho g a^2\cos^2(nt)}{2}\int_0^\lambda \frac{1-\cos(2mx)}{2}$$

$$= \frac{\rho g a^2\cos^2(nt)}{4}(\int_0^\lambda 1 - \cos(2mx)dx$$

$$= \frac{\rho g a^2\cos^2(nt)}{4}([x|_0^\lambda] - \int_0^\lambda \cos(2mx)dx)$$

$$= \frac{\rho g a^2\cos^2(nt)}{4}[\lambda - \frac{\sin(2mx)}{2m}\Big|_0^\lambda]$$

$$= \frac{\rho g a^2\cos^2(nt)}{4}[\lambda - \frac{1}{2m}(\sin(2m\lambda)]$$

Also, recall that $\lambda = \frac{2\pi}{m}$ this implies that m=$\frac{2\pi}{\lambda}$,

$\therefore 2m\lambda = 2\lambda * \frac{2\pi}{\lambda} = 4\pi$

$= \frac{\rho g a^2 \cos^2(nt)}{4} [\lambda - \frac{1}{2m}(\sin(4\pi)]$

$= \frac{\rho g a^2 \cos^2(nt)}{4} [\lambda]$

$= \frac{\rho g a^2 \cos^2(nt)}{4} [\lambda]$

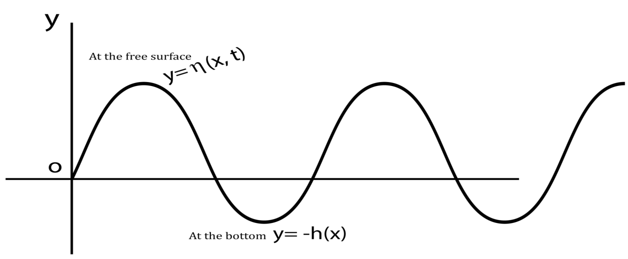
$P.E = \frac{\rho g a^2 \cos^2(nt\lambda)}{4}$ …………………………………………………………… (30.12)

To get the total energy of the system $E = P.E + K.E$

$E = \frac{\rho a^2 g \sin^2(nt)\lambda}{4mh} + \frac{\rho g a^2 \cos^2(nt\lambda)}{4}$

$E = \frac{\rho a^2 g \lambda}{4} [\frac{\sin^2(nt)}{mh} + \cos^2(nt)]$ ……………………………………………… (30.12.2)

3.10.5 THE SHALLOW WATER WAVE THEORY:



At the free surface $y = \eta(x, t)$

$= y - \eta(x, t) = 0$ ……………………………………………………………………… (30.13)

$= y - h(x) = 0$ …………………………………………………………………….. (30.14)

The equations we are to solve for are;

1. **The continuity equation:** $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

From equation (30.13) and (30.14), we have;

$$\frac{d(y-\eta)}{dt} = \frac{\partial(y-\eta)}{\partial t} + u\frac{\partial(y-\eta)}{\partial x} + v\frac{\partial(y-\eta)}{\partial y}$$

$$= -\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + v = 0$$

$v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}$, at y=η(x,t)………………………………………………… (30.13.2)

And

$$\frac{d(y+h)}{dt} = \frac{\partial(y+h)}{\partial t} + u\frac{\partial(y+h)}{\partial x} + v\frac{\partial(y+h)}{\partial y}$$

$v = -u\frac{\partial h}{\partial x}$, at y=-h(x)…………………………………………… (30.14.2)

Next, integrating the continuity equation w.r.t. 'y', we get;

$$\int_{-h(x)}^{\eta} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\, dy$$

$= \int_{-h(x)}^{\eta} \frac{\partial u}{\partial x}\, dy + [v|_{-h(x)}^{\eta}]$ ………………………………………………. (30.15)

From equation (30.13.2) and (30.14.2) we have the values for v at y= η and y=-h, substituting those values into equation (30.15), we get;

$\int_{-h(x)}^{\eta} \frac{\partial u}{\partial x}\, dy + [\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + u\frac{\partial h}{\partial x}]$ ………………………………………………. (30.15.2)

We can rewrite this equation as;

$$\int_{-h(x)}^{\eta} \frac{\partial u}{\partial x}\, dy + u\frac{\partial \eta}{\partial x} + u\frac{\partial h}{\partial x} = -\frac{\partial \eta}{\partial t} \quad \ldots\ldots (30.16)$$

But from Leibniz rule of integration, we have;

$$\frac{\partial}{\partial x}\int_{-h(x)}^{\eta} u dy = \int_{-h(x)}^{\eta} \frac{\partial u}{\partial x}\, dy + u|_{\eta}\frac{\partial \eta}{\partial x} + u|_{-h(x)}\frac{\partial h}{\partial x} \quad \ldots\ldots (30.16.2)$$

Putting equation (30.16.2) into (30.16), we have;

$$\frac{\partial}{\partial x}\int_{-h(x)}^{\eta} u dy = -\frac{\partial \eta}{\partial t} \quad \ldots\ldots (30.17)$$

2. Equation of motion:

The vertical motion is given as; $\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} + g = 0$

Considering steady linear motion, we have;

$$\frac{\partial p}{\partial y} = -\rho g \quad \ldots\ldots (30.18)$$

$$\int \frac{\partial p}{\partial y} dy = \int -\rho g\, dy$$

$$p = -\rho g y + c$$

To find c, at $p = 0, y = \eta$

$$0 = -\rho g \eta + c$$

$c = \rho g \eta$, therefore

$$p = -\rho g y + \rho g \eta$$

$$p = \rho g(\eta + y) \quad \ldots\ldots (30.19)$$

Equation (30.19) satisfies equation (30.15)

$$\frac{\partial p}{\partial x} = \rho g\frac{\partial \eta}{\partial x} \quad \ldots\ldots (30.20)$$

For the horizontal motion, we will have;

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{…………} (30.21)$$

From equation (30.20) put the value for $\frac{\partial p}{\partial x}$ into equation (30.21)

$$\frac{du}{dt} = -\frac{1}{\rho}[\rho g\frac{\partial \eta}{\partial x}]$$

$$\frac{du}{dt} = -g\frac{\partial \eta}{\partial x} \quad \text{…………} (30.22)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x}$$

Considering linear steady motion we have

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \quad \text{…………} (30.23)$$

Next, from equation $p \neq p(y), u \neq u(y), \therefore u = u(x,t), \frac{\partial u}{\partial y} = 0$

From equation (30.17),

$$\frac{\partial}{\partial x}\int_{-h(x)}^{\eta} udy = -\frac{\partial \eta}{\partial t}$$

$$= \frac{\partial}{\partial x}[u]_{-h}^{\eta} = -\frac{\partial \eta}{\partial t}$$

$$= \frac{\partial}{\partial x}u[\eta + h] = -\frac{\partial \eta}{\partial t}$$

$$= \frac{u\partial \eta}{\partial x} + \frac{u\partial h}{\partial x} = -\frac{\partial \eta}{\partial t}$$

By linearity, we have;

$$\frac{\partial [uh]}{\partial x} = -\frac{\partial \eta}{\partial t} \quad \text{…………} (30.24)$$

Equation (30.23) and (30.24) constitute the 2-dimensional shallow wave theory.

3.10.6 DERIVATION OF STURM LIOUVILLE EQUATION

Recall from the shallow wave equation, we have;

$$\frac{\partial u}{\partial t} =- g\frac{\partial \eta}{\partial x} \quad \ldots\ldots (31.1)$$

$$\frac{\partial [uh]}{\partial x} =- \frac{\partial \eta}{\partial t} \quad \ldots\ldots (31.2)$$

Differentiate (31.1) with respect to time

$$\frac{\partial}{\partial t}[\frac{\partial u}{\partial t}] = \frac{\partial}{\partial t}[- g\frac{\partial \eta}{\partial x}]$$

$$= \frac{\partial^2 u}{\partial t^2} =- g\frac{\partial}{\partial x}(\frac{\partial \eta}{\partial t}) \quad \ldots\ldots (31.3)$$

From equation (31.2),

$$\frac{\partial \eta}{\partial t} = \frac{\partial [uh]}{\partial x} \quad \ldots\ldots (31.4)$$

therefore;

$$\frac{\partial^2 u}{\partial t^2} =- g\frac{\partial}{\partial x}(\frac{\partial [uh]}{\partial x})$$

$$\frac{\partial^2 u}{\partial t^2} = g\frac{\partial^2 [uh]}{\partial x^2} \quad \ldots\ldots (31.5)$$

From equation (31.2), differentiating this equation with respect to time t, we have;

$$\frac{\partial}{\partial t}[\frac{\partial [uh]}{\partial x}] = \frac{\partial}{\partial t}[- \frac{\partial \eta}{\partial t}] \quad \ldots\ldots (31.6)$$

$$= \frac{\partial^2 [uh]}{\partial x \partial t} =- \frac{\partial^2 \eta}{\partial t^2} \quad \ldots\ldots (31.7)$$

We can rewrite equation (31.7) as:

$$\frac{\partial}{\partial x}[\frac{\partial[uh]}{\partial t}] =-\frac{\partial^2\eta}{\partial t^2} \quad \ldots\ldots (31.8)$$

Expanding equation (31.8) gives us;

$$=\frac{\partial}{\partial x}[u\frac{\partial h}{\partial t}+h\frac{\partial u}{\partial t}] =-\frac{\partial^2\eta}{\partial t^2} \quad \ldots\ldots (31.9)$$

From $h = h(x), u = u(x, t)$, equation (31.9) becomes;

$$\frac{\partial}{\partial x}[h\frac{\partial u}{\partial t}] =-\frac{\partial^2\eta}{\partial t^2} \quad \ldots\ldots (31.10)$$

Also recall from equation (31.1)

$\frac{\partial u}{\partial t} =- g\frac{\partial\eta}{\partial x}$, so our equation (31.9) is rewritten as;

$$\frac{\partial}{\partial x}[h(- g\frac{\partial\eta}{\partial x})] =-\frac{\partial^2\eta}{\partial t^2} \quad \ldots\ldots (31.11)$$

$$= hg\frac{\partial^2\eta}{\partial x^2} = \frac{\partial^2\eta}{\partial t^2}$$

Which is rewritten as

$$\frac{\partial^2\eta}{\partial t^2} = hg\frac{\partial^2\eta}{\partial x^2} \quad \ldots\ldots (31.12)$$

3.11 DIFFERENCES BETWEEN INTERNAL WAVE AND LONG WAVE

Internal waves and Long waves are two important phenomenon in fluid mechanism. While both types of waves involve the movement of fluids they differ in terms of their characteristics behaviour and impact on the environment.

3.11.1 LOCATION OF PROPAGATION:

Internal waves propagates through a fluid medium usually underneath the surface of lakes or oceans. The interactions between density gradients within these fluids are what causes these waves. While long waves are usually waves that travels across a medium surface such as air waves or water waves.

3.11.2 MECHANISM OF PROPAGATION

The behaviour of these waves also differs in terms of their interactions with boundaries and obstacles. Internal waves can reflect, refract and diffract when encountering changes in the density gradients of the fluid medium hence leading to a complex wave patterns. Long waves on the other hand tends to propagate in a straight line and can pass through obstacles with minimum deviations from their original path. Variations in water densities caused by the disturbances like winds, tides are usually the cause of internal waves. These waves move over interfaces of densities like the ocean's temperature. While wind typically blows over the surface of a body of water to create ling waves especially when it comes to water waves. The surface of the water oscillates as a result of the restoring force of gravity which is the manner in which they spread.

3.11.3 WAVE CHARACTERISTICS:

When compared to long waves, internal waves often have greater frequencies and shorter wavelengths. They might not be easily noticeable at the surface and can show significant vertical displacements. Internal waves travel at a slower speed compared to long weaves. This is because, they are influenced by the density gradient of the fluid which can vary significantly over short distances.

While on the other hand, long waves have exhibit a large wavelengths compared to the depth of fluid mechanism, in other words, they have lower frequencies and longer wavelengths and they are easier to see on the surface and can travel greater distances with minimal energy loss.

3.11.4 EFFECTS AND APPLICATIONS:

In marine life, the distribution of nutrients, underwater acoustics are all impacted by the internal waves, which are very essential to the mixing and circulation of the ocean. Internal waves also affects the dispersal of oil spills and the movement of silts. While long waves are crucial for recreational surfing, coastal engineering and shipping. Also similar to Rosby waves, atmospheric long waves influence climate dynamics and weather patterns.

3.12 DERIVATION OF THE EQUATION OF INTERNAL WAVES IN A STRATIFIED FLUID

In ocean, the density of water varies with depth, exhibiting abrupt transitions over short distances while maintaining relatively consistent regions above and below. This abrupt transition zone is referred to as thermocline. Internal waves can also be present in a continuously stratified fluid, where the vertical density distribution remains continuous in a stable state.

We will assume that the wave motion is non-viscous. Additionally, we will consider small amplitudes, allowing us to neglect the nonlinear terms. Furthermore, we will assume that the motion frequency is slightly higher than the Coriolis frequency, hence it does not influence the motion.

To derive the basic equation governing the motion of small disturbances in a stratified fluid, first, we consider the principle of conservation of momentum, also known as newton's second law applied in the vertical direction;

$$\frac{\rho\partial v}{\partial t} + \nabla \cdot (\rho\vec{v}\vec{v}) = -\nabla p + \rho g + \mu\nabla^2\vec{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (31.13)$$

where: ρ Is the density of the fluid, $\vec{v}$ is the velocity vector, p is the pressure, g is the gravitational acceleration vector, μ is the dynamic viscosity.

Next, we make several simplifications for a stratified fluid experiencing small disturbances; for the Boussinesq approximation, we assume small density variations and take densities as constant except in the buoyancy term ρg, also we neglect the effect of viscosity $\mu\nabla^2\vec{v}$ while aiming at the disturbances primarily in the vertical direction.

Our momentum equation now reduces to

$$\frac{\partial}{\partial t}(\rho v_z) = -\frac{\partial p}{\partial z} + \rho g \quad \text{.. (31.14)}$$

where v_z is the vertical component of velocity.

Next we apply the continuity equation;

$$\nabla.\vec{v} = 0 \quad \text{... (31.15)}$$

Since we are dealing with the vertical direction we have that

$$\frac{\partial v_z}{\partial z} = 0 \quad \text{... (31.16)}$$

Integrating equation (31.16) with respect to z, we have that:

$$v_z = \frac{\partial \xi}{\partial t} \quad \text{... (31.17)}$$

where ξ is the vertical displacement of fluids from their equilibrium position. Next we put equation (31.17) back into equation (31.14)

$$\frac{\partial}{\partial t}(\rho \frac{\partial \xi}{\partial t}) = -\frac{\partial p}{\partial z} + \rho g \quad \text{... (31.18)}$$

Given that $p = p_0 + p'$, where p_0 is the hydro-static pressure and p' is the perturbation pressure, and also assuming there is no heat transfer (adiabatic condition), we can write that;

$\frac{\partial p'}{\partial z} = c^2 \frac{\partial \rho}{\partial z}$ ………………………………………………………….. (31.19)

where c is the speed of sound in the fluid, substituting this relation into the momentum equation, we get;

$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial z^2}$ ………………………………………………………… (31.20)

Equation (31.20) governs the motion of small disturbances such as internal waves in a stratified fluid.

3.13 DERIVATION OF THE EQUATION OF LONG WAVE:

Waves having wavelengths significantly longer than the water depths are commonly referred to as long waves. Although they can occur in shallow and deep water, their interactions with seafloor and coastlines make shallow water habitats especially important for them. Long waves are less influenced by the bottom at deeper water, and linear wave theory describes their behaviour more precisely. However, because of the seafloor's effect, the wave speed drops and the wave profile shifts in shallow water, resulting in phenomena including shoaling, refraction and breaking. Consequently, as these effects are more noticeable in shallow water, the derivation of equation for long waves frequently focuses on shallow water.

To derive the equation governing the propagation of long waves, we start the shallow water equations which are the simplification of the fundamental equations of fluid dynamics I.e. the Navier-Stoke equation which describes the motion of fluids I.e. in the Navier-Stoke equation, the viscous effects are neglected and the pressure variations due to depths are well balanced by gravity. The shallow water equations are given as;

$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0$ ……………………………………………………….. (31.21.1)

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h}{\partial x} = 0$ ………………………………………………………….. (31.21.2)

$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial h}{\partial y} = 0$ ……………………………………………..……….. (31.21.3)

where ‘h’ is the water depth, ‘u’ and ‘v’ are the velocity components and ‘g’ is the acceleration due to gravity.

Next, we examine the case where the horizontal length scale is far greater than the vertical length scale this implies that the horizontal derivatives are smaller compared to the vertical derivatives, neglecting the horizontal advection terms that is the terms involving horizontal derivatives, the momentum equation becomes

$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0$ ………………………………………………………….. (31.21.4)

$\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial y} = 0$ ………………………………………………………….. (31.21.5)

We further assume that the horizontal velocity component are uniform over water depth, this enables us to express the velocity components as; $u = \frac{H}{h}\vec{u}$, $v = \frac{H}{h}\vec{v}$, where $\vec{u}$ and $\vec{v}$ are the depth-averaged horizontal components. Next, we neglect the time derivatives I.e.

$\frac{\partial \vec{u}}{\partial t} = 0 \; and \; \frac{\partial \vec{v}}{\partial t} = 0$

Substituting the above into the shallow water equation we get;

$\frac{\partial h}{\partial t} + H(\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{v}}{\partial t}) = 0$ …………………………………………………………… (31.22.1)

$\frac{\partial \vec{u}}{\partial t} + g\frac{\partial h}{\partial x} = 0$ …………………………………………………………… (31.22.2)

$\frac{\partial \vec{v}}{\partial t} + g\frac{\partial h}{\partial y} = 0$ …………………………………………………………… (31.22.3)

These equation describes the propagation of long waves in shallow water where the depth variations are small compared to the horizontal length scales. They foundations to modelling phenomena like tsunamis and tidal waves.

3.14 KELVIN WAVES:

3.14.1 INTRODUCTION:

In the ocean and the atmosphere, a large scale wave motion of practical significance known as Kelvin wave exist. It is a special kind of gravity wave that was initially observed by Sir William Thompson in 1879 and later named after Sir Kelvin. This wave is caused by the Earth's rotation and become stuck near the equator or at lateral vertical limits like coasts or mountain regions. For kelvin waves to occur, it depends on the equator or vertical boundaries, oscillations obtained by gravity or stable stratification, land large Coriolis acceleration. The distinctive characteristic of this wave to other waves is the unidirectional propagation (movement). This waves travels cyclically around a closed surface, towards the equator along a western boundary and towards the north along the eastern boundary. When the wave is farthest from the boundary, its amplitude decreases exponentially. Kelvin waves always go towards the east direction near the equator where they reach their maximum amplitude and also rapidly diminish with increasing latitudes.

3.14.2 DEFINITION OF KELVIN WAVE:

A Kelvin wave is a kind of wave that travels through the atmosphere or along the interface between fluids with differing densities, such as the ocean's surface and deeper depths. These waves have rotational motion and arises from the interaction of pressure gradients and the Coriolis Effect.

3.14.3 FORMATION OF KELVIN WAVE:

Kelvin waves are formed by the complex interplay between various factors, these factors include;

3.14.3.1 WIND FORCING:

On the ocean surface, continuous wind stress can cause kelvin waves, surface water is forced to travel in the direction of the wind when wind blows over a certain area on a regular basis. There is a gradient in the surface elevation caused by this wind-induced motion, with higher water levels downwind and lower water levels upwind. The creation of Kelvin waves is facilitated by this gradient.

3.14.3.2 CORIOLIS EFFECT:

Moving objects are deflected to the left in the Southern Hemisphere and to the right in the Northern Hemisphere due to the Coriolis Effect, which is brought on by the Earth's rotation. The Coriolis Effect affects the direction of fluid motion caused by wind forcing in the setting of Kelvin waves. It causes the surface water to deviate to the right/left of the wind direction in the Northern/Southern Hemisphere resulting the formation of a cyclonic/anticyclonic circulation pattern. The Coriolis force interacting with the wind-driven motion creates the circumstances for the production of Kelvin waves in this cyclonic/anticyclonic circulation pattern.

3.14.3.3 PRESSURE GRADIENT:

The differences in pressure between various places give rise to the pressure gradient force. This force moves fluid motion from high-pressure to low-pressure zones in the case of Kelvin waves. Kelvin waves are initiated and propagated in part by the pressure gradient connected to the wind-induced surface elevation gradient. It supplies the energy required for fluid motion, which keeps the wave going.

3.14.3.4 FEATURES OF THE TOPOGRAPHY:

Changes in ocean currents or physical features like undersea mountains, coasts or continental shelves can also cause Kelvin waves. Ocean currents may create Kelvin waves as a way of adapting to new circumstances when they come against obstructions. Kelvin waves can also be produced by the interaction of ocean currents with coasts.

3.14.3.5 BALANCE OF FORCES:

Kelvin waves also form when the Coriolis Effect and the pressure gradient force reach a balance. Waves propagate along the interface between several fluid layers as a result of the Coriolis Effect deflecting the fluid motion caused by the pressure gradient force. The long term sustained propagation of Kelvin waves across vast distances depends on this balance between the two forces.

In conclusion, pressure gradients, wind forcing, the Coriolis Effect and topographic characteristics all work together to create Kelvin waves, which propagates along the interface of fluids with varying densities in the ocean or atmosphere. Predicting and researching patterns, climatic variability and ocean circulation patterns require an understanding of these processes.

3.14.4 TYPES OF KELVIN WAVES:

There are two major types of Kelvin waves, namely;

3.14.4.1 COASTAL KELVIN WAVES:

Wind stress along the continent's coastlines cause coastal Kelvin waves. The sea levels and ocean currents are altered as a result of these waves as they move windward (in the direction of the wind) along the coast. Coastal erosion, marine ecosystems, storm surges can all be affected by coastal Kelvin waves, which are important to coastal oceanography. Coastal Kelvin waves occur as a result of the changes in the wind patterns, atmospheric pressures and coastal topography and they can move both parallel to the coastline (alongshore) and away from the coastlines (offshore). They play important role in the coastal ocean dynamics like transportation of heat, sediment along coastlines and nutrients. This type of Kelvin wave affect coastal ecosystems and influence phenomena like coastal down welling and upwelling.

3.14.4.2 EQUATORIAL KELVIN WAVE:

These are kind of waves that travel through the atmosphere and water along the equator. It is affected by the rotation of the Earth and they are also distinguished by their eastward movement. These waves propagate thousands of kilometres along the equator affecting the sea temperature, distribution of heat. This Kelvin wave aid in influencing the ocean circulation pattern and climate variability through the redistributing of heat and momentum within the ocean.

3.14.5 IMPORTANCE FOR THE STUDYING OF KELVIN WAVE.

1. Understanding kelvin waves is essential for forecasting and assessing marine phenomena including tides, wave dynamics and climate trends

2. Engineers use Kelvin waves extensively in the constructions and upkeep of offshore constructions like wind farms and oil rigs.

3. Through an understanding of the ways in which Kelvin waves impact wave dynamics and sea surface temperature, engineers may better predict loads and pressures that offshore structures are likely to experience.

4. These waves are needed for the studying of oceanic circulation patterns in addition to their uses in offshore engineering

5. Through the examining of Kelvin wave behaviour, scientists can learn more about the processes behind heat transport and ocean currents.

6. Understanding climate changes and how it affect s the marine ecosystems requires the knowledge of this wave.

7. Also, underwater sounds makes use of this wave; through the examination of the movement of these waves over the oceans, engineers may create more precise models to forecast the behaviour of sounds waves beneath the surface

8. Applications like underwater communications, underwater object detection and navigation depends on the knowledge of this waves.

CHAPTER FOUR

NUMERICAL EXAMPLES ON LINEARIZED WATER WAVE:

PROBLEM ONE: Discretize the Euler-Stokes equation for water wave problem as a linear system Ax=b where A is the matrix, x is the wave vector to be found, b is the constant vector (which forms the wall or boundary of the ocean).

Solution:

Euler equation for a 2D is given by;

Continuity equation (mass equation);

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum equation

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \rho g$$

$$\rho[\frac{\partial}{\partial t} + \vec{v}.\nabla\vec{v}] = -\nabla p + \rho g$$

Rewriting in x, and y direction (taking g as a constant) we have;

X-direction;

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + \vec{u}\frac{\partial \vec{u}}{\partial x} + \vec{v}\frac{\partial \vec{u}}{\partial y}\right] = -\frac{\partial p}{\partial x}$$

$$= \frac{\partial \vec{u}}{\partial t} + \vec{u}\frac{\partial \vec{u}}{\partial x} + \vec{v}\frac{\partial \vec{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Y-direction;

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + \vec{u}\frac{\partial \vec{v}}{\partial x} + \vec{v}\frac{\partial \vec{v}}{\partial y}\right] = -\frac{\partial p}{\partial y}$$

$$= \frac{\partial \vec{v}}{\partial t} + \vec{u}\frac{\partial \vec{v}}{\partial x} + \vec{v}\frac{\partial \vec{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Combining all the equations;

$\frac{\partial\vec{u}}{\partial t}+\vec{u}\frac{\partial\vec{u}}{\partial x}+\vec{v}\frac{\partial\vec{u}}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$............................ (1)

$\frac{\partial\vec{v}}{\partial t}+\vec{u}\frac{\partial\vec{v}}{\partial x}+\vec{v}\frac{\partial\vec{v}}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}$ .... (2)

Next, for the spatial discretization; we divide the domain into a uniform grids with grid spacing $\Delta x, and\ \Delta y$. Discretize the velocity component at each grid points (ij) as $u_{ij}, and\ v_{ij}$. The pressure is discretized at each point $(ij)\ as\ p_{ij}$.

For Temporal discretization; we discretize the time with time step Δt . Using backward difference $\frac{\partial\vec{u}}{\partial t}=\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t},\ \frac{\partial\vec{v}}{\partial t}=\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta t}$

We discretize this equation using Central difference for spatial derivatives and backward differences for time derivatives. The equations can be written as;

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}+v_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i,j-1}^{n}}{\Delta y}=-\frac{1}{\rho}\frac{p_{i+1,j}^{n}-p_{i,j}^{n}}{\Delta x}$$

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\frac{v_{i,j}^{n}-v_{i-1,j}^{n}}{\Delta x}+v_{i,j}^{n}\frac{v_{i,j}^{n}-v_{i,j-1}^{n}}{\Delta y}=-\frac{1}{\rho}\frac{p_{i,j+1}^{n}-p_{i,j}^{n}}{\Delta y}$$

For the pressure equation, this is gotten from discretizing the continuity equation;

$$\frac{p_{i+1,j}^{n}-p_{i,j}^{n}}{\Delta x}+\frac{p_{i,j+1}^{n}-p_{i,j}^{n}}{\Delta y}=\frac{\rho}{\Delta t}\left(\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}+\frac{v_{i,j}^{n}-v_{i,j-1}^{n}}{\Delta y}\right)$$

Next, we apply boundary conditions:

For the velocity, apply the no slip condition that is $u=v=0$, at the boundaries

For the Pressure, we assume a constant pressure $P_0=10$

For convergence, we will use a tolerance of 10^{-6}

Next, we solve using the Guass-siedel iterative method, updating the velocities u and v

X-direction;

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i+1,j}^{n}-p_{i,j}^{n}}{\Delta x} - u_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x} - v_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i,j-1}^{n}}{\Delta y} \quad \text{....................(3)}$$

Y-direction;

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i,j+1}^{n}-p_{i,j}^{n}}{\Delta y} - u_{i,j}^{n}\frac{v_{i,j}^{n}-v_{i-1,j}^{n}}{\Delta x} - v_{i,j}^{n}\frac{v_{i,j}^{n}-v_{i,j-1}^{n}}{\Delta y} \quad \text{....................(4)}$$

Iterative processes

1. Start by applying the boundary conditions. Set $u = v = 0$ and pressure $P_0 = 10$
2. Iterate until convergence by;
   - Updating u and v using the updated pressure from th4e previous iteration
   - Update pressure using the newly updated u and v
   - Repeat until the change in u,v and p between consecutives iterations is a predefined tolerance

Next, we arrange these equations to solve for updated velocities $u^{n+1}, v^{n+1,}$ at the grid point using the Guass-Siedel iterative method. We iterate until convergence;

Equation (3) and (4) can be written as

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{\Delta t}{\rho}\left(\frac{p_{i+1,j}^{n} - p_{i,j}^{n}}{\Delta x}\right) - u_{i,j}^{n}\frac{\Delta t}{\Delta x}\left(u_{i,j}^{n} - u_{i-1,j}^{n}\right) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}\left(u_{i,j}^{n} - u_{i,j-1}^{n}\right)$$

$$v_{i,j}^{n+1} = v_{i,j}^{n} - \frac{\Delta t}{\rho}\left(\frac{p_{i,j+1}^{n} - p_{i,j}^{n}}{\Delta y}\right) - u_{i,j}^{n}\frac{\Delta t}{\Delta x}\left(v_{i,j}^{n} - v_{i-1,j}^{n}\right) - \quad v_{i,j}^{n}\frac{\Delta t}{\Delta y}\left(v_{i,j}^{n} - v_{i,j-1}^{n}\right)$$

Let $a = \frac{\Delta t}{\Delta x},\ b = \frac{\Delta t}{\Delta y}$

Let $\Delta u_{i,j,}^{n} = \left(u_{i,j}^{n} - u_{i-1,j}^{n}\right) = \left(u_{i,j}^{n} - u_{i,j-1}^{n}\right)$

$\Delta v_{i,j}^{n} = \left(v_{i,j}^{n} - v_{i-1,j}^{n}\right) = \left(v_{i,j}^{n} - v_{i,j-1}^{n}\right)$

$\Delta p_{i,j}^{n} = \left(p_{i+1,j}^{n} - p_{i,j}^{n}\right) = \left(p_{i,j+1}^{n} - p_{i,j}^{n}\right)$

The matrix form becomes;

$$\begin{pmatrix} u_{i,j}^{n+1} \\ v_{i,j}^{n+1} \end{pmatrix} \approx \begin{pmatrix} u_{i,j}^{n} \\ v_{i,j}^{n} \end{pmatrix} - \left(\begin{pmatrix} a\Delta u_{i,j}^{n} & b\Delta u_{i,j}^{n} \\ a\Delta v_{i,j}^{n} & b\Delta v_{i,j}^{n} \end{pmatrix}\begin{pmatrix} u_{i,j}^{n} \\ v_{i,j}^{n} \end{pmatrix}\right) - \begin{pmatrix} \frac{a}{\rho}\Delta p_{i,j,}^{n} \\ \frac{b}{\rho}\Delta p_{i,j}^{n} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Which of the form $Ax = b$.

Next we compute the numerical simulation using Matlab. To do this; we need to update the pressure field, and set boundary condition for the velocities and pressure.

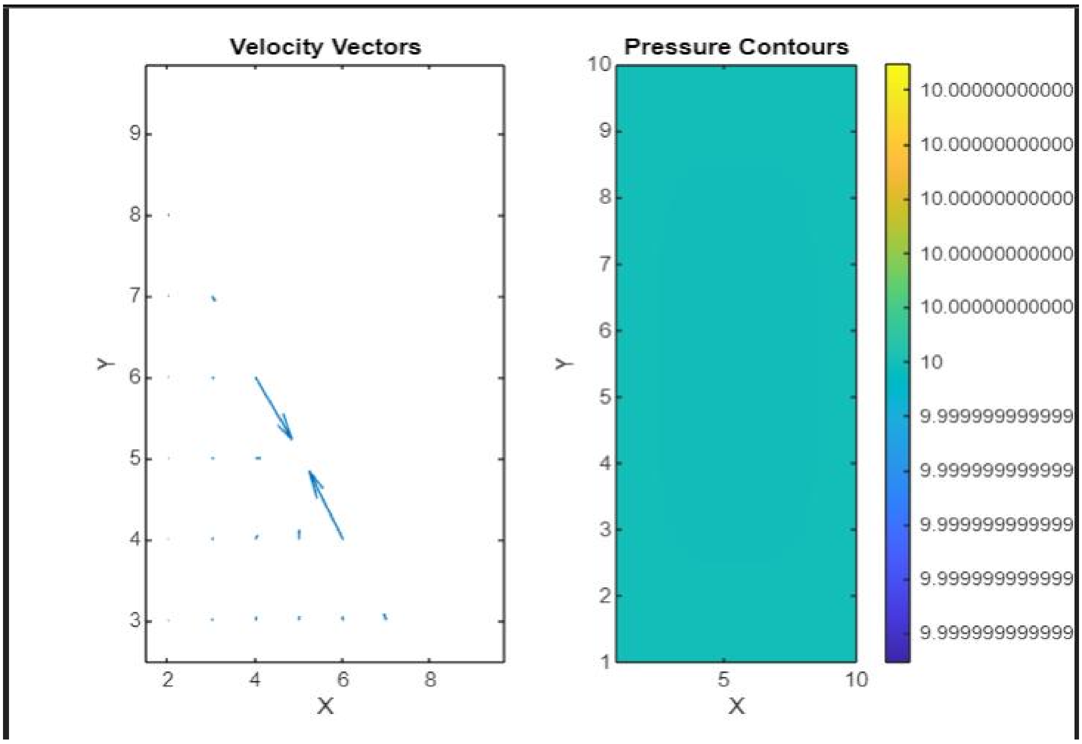
Boundary conditions for velocity;

1. No slip boundary condition $u = v = w = 0$ at solid boundaries

For the pressure we assume a constant pressure $P_0 = 10$, Matlab code solving the discretization of Euler-Stoke equation using the Guass-Siedel iterative method;

```
% Constants and parameters
rho = 1.0;  % Density
dx = 1.0;   % Grid spacing in x-direction
dy = 1.0;   % Grid spacing in y-direction
dt = 0.1;   % Time step
tolerance = 1e-6;   % Convergence tolerance
max_iterations = 1000;  % Maximum number of iterations

% Grid dimensions
nx = 10;    % Number of grid points in x-direction
ny = 10;    % Number of grid points in y-direction

% Initialize arrays for velocity components u and v, and pressure p
u = zeros(nx, ny);
v = zeros(nx, ny);
p = zeros(nx, ny);

% Boundary conditions
% Velocity: No-slip boundary condition (u = v = 0)
u(:, 1) = 0;    % Bottom boundary
u(:, end) = 0;  % Top boundary
u(1, :) = 0;    % Left boundary
u(end, :) = 0;  % Right boundary
v(:, 1) = 0;    % Bottom boundary
v(:, end) = 0;  % Top boundary
v(1, :) = 0;    % Left boundary
v(end, :) = 0;  % Right boundary

% Pressure: Constant pressure at boundaries (p = 10)
p(:, 1) = 10;   % Bottom boundary
p(:, end) = 10; % Top boundary
p(1, :) = 10;   % Left boundary
p(end, :) = 10; % Right boundary

% Main iterative loop
for itr = 1:max_iterations
   % Copy arrays for previous iteration
   u_old = u;
   v_old = v;
   p_old = p;

   % Update velocity components u and v
   for i = 2:nx-1
      for j = 2:ny-1
```

```
        u(i, j) = (u_old(i, j) * (1/dt - (u_old(i, j) - u_old(i-1, j)) / dx - (v_old(i, j) - v_old(i, j-1))
/ dy) ...
              - (1/rho) * (p_old(i+1, j) - p_old(i, j)) / dx) * dt;
        v(i, j) = (v_old(i, j) * (1/dt - (u_old(i, j) - u_old(i-1, j)) / dx - (v_old(i, j) - v_old(i, j-1))
/ dy) ...
              - (1/rho) * (p_old(i, j+1) - p_old(i, j)) / dy) * dt;
      end
   end

   % Update pressure p
   for i = 2:nx-1
      for j = 2:ny-1
        p(i, j) = ((p_old(i+1, j) + p_old(i-1, j)) * dy^2 + (p_old(i, j+1) + p_old(i, j-1)) * dx^2) /
(2 * (dx^2 + dy^2));
      end
   end

   % Check for convergence
   if max(max(abs(u - u_old))) < tolerance && max(max(abs(v - v_old))) < tolerance &&
max(max(abs(p - p_old))) < tolerance
      disp(['Converged after ', num2str(itr), ' iterations.']);
      break;
   end
end

% Visualize velocities and pressure
[X, Y] = meshgrid(1:nx, 1:ny);

figure;

subplot(1, 2, 1);
quiver(X, Y, u', v');
xlabel('X');
ylabel('Y');
title('Velocity Vectors');

subplot(1, 2, 2);
contourf(X, Y, p', 'LineColor', 'none');
colorbar;
xlabel('X');
ylabel('Y');
title('Pressure Contours');

colormap('viridis');
```



The velocity vectors indicate the direction and magnitude of the fluid flow at each grid point. Each arrow represents the velocity vector at a specific grid point, and the length of the arrow corresponds to the magnitude of the velocity. The direction of the arrow indicates the direction of the flow.

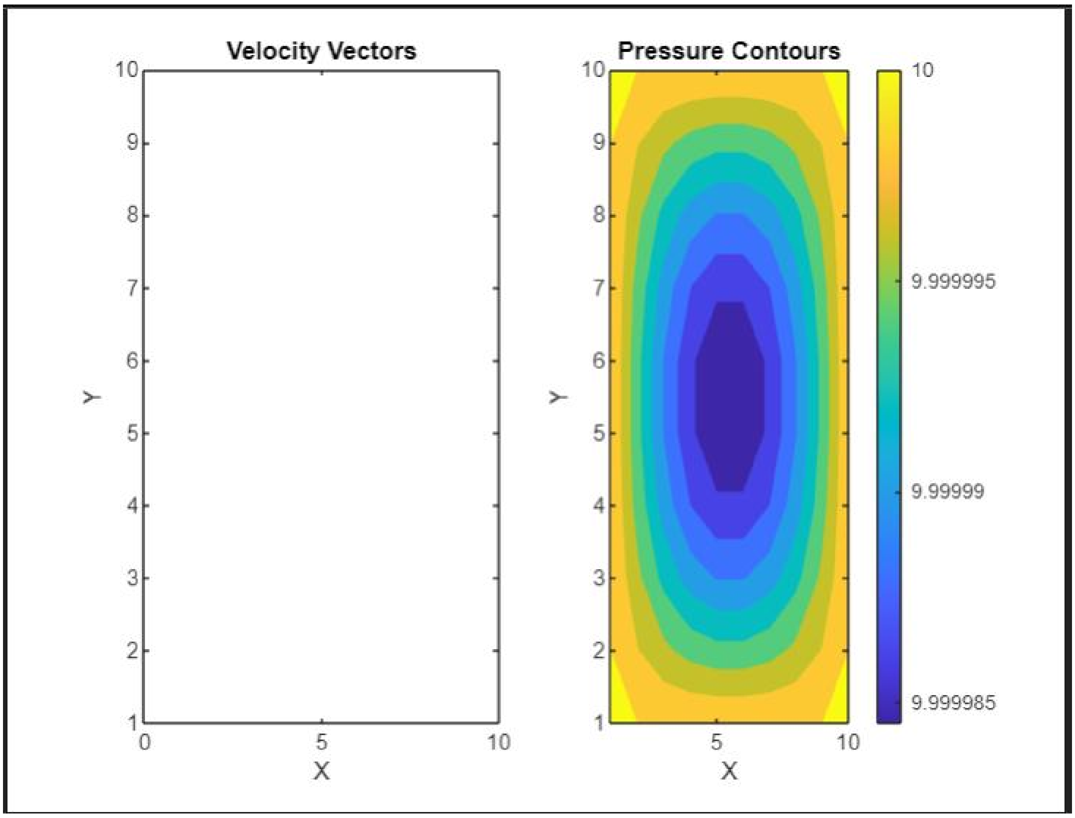
As for the pressure contours, the colour represents the pressure value at each grid point. In the pressure contour plot, regions with the same colour have the same pressure value. Generally, darker regions represent higher pressure, while lighter regions represent lower pressure. The contour lines represent areas with constant pressure values. The spacing between contour lines indicates the rate of change of pressure; closer contour lines indicate a steeper pressure gradient.

MATLAB CODE USING JACOBI ITERATIVE METHOD FOR EULER DICRETIXATION

```
% Constants and parameters
rho = 1.0;  % Density
dx = 1.0;   % Grid spacing in x-direction
dy = 1.0;   % Grid spacing in y-direction
dt = 0.1;   % Time step
```

```
tolerance = 1e-6;   % Convergence tolerance
max_iterations = 1000;  % Maximum number of iterations

% Grid dimensions
nx = 10;    % Number of grid points in x-direction
ny = 10;    % Number of grid points in y-direction

% Initialize arrays for velocity components u and v, and pressure p
u = zeros(nx, ny);
v = zeros(nx, ny);
p = zeros(nx, ny);

% Boundary conditions
% Velocity: No-slip boundary condition (u = v = 0)
u(:, 1) = 0;    % Bottom boundary
u(:, end) = 0;  % Top boundary
u(1, :) = 0;    % Left boundary
u(end, :) = 0;  % Right boundary
v(:, 1) = 0;    % Bottom boundary
v(:, end) = 0;  % Top boundary
v(1, :) = 0;    % Left boundary
v(end, :) = 0;  % Right boundary

% Pressure: Constant pressure at boundaries (p = 10)
p(:, 1) = 10;   % Bottom boundary
p(:, end) = 10; % Top boundary
p(1, :) = 10;   % Left boundary
p(end, :) = 10; % Right boundary

% Main iterative loop
for itr = 1:max_iterations
    % Copy arrays for previous iteration
    u_old = u;
    v_old = v;
    p_old = p;

    % Update velocity components u and v
    for i = 2:nx-1
        for j = 2:ny-1
            u(i, j) = (1/dt - (u_old(i, j) - u_old(i-1, j)) / dx - (v_old(i, j) - v_old(i, j-1)) / dy) ...
                      * u_old(i, j) - (1/rho) * (p_old(i+1, j) - p_old(i, j)) / dx * dt;
            v(i, j) = (1/dt - (u_old(i, j) - u_old(i-1, j)) / dx - (v_old(i, j) - v_old(i, j-1)) / dy) ...
                      * v_old(i, j) - (1/rho) * (p_old(i, j+1) - p_old(i, j)) / dy * dt;
        end
    end

    % Update pressure p
    for i = 2:nx-1
        for j = 2:ny-1
```

```
            p(i, j) = ((p_old(i+1, j) + p_old(i-1, j)) * dy^2 + (p_old(i, j+1) + p_old(i, j-1)) * dx^2) /
(2 * (dx^2 + dy^2));
        end
    end

    % Check for convergence
    if max(max(abs(u - u_old))) < tolerance && max(max(abs(v - v_old))) < tolerance &&
max(max(abs(p - p_old))) < tolerance
        disp(['Converged after ', num2str(itr), ' iterations.']);
        break;
    end
end

% Visualize velocities and pressure
[X, Y] = meshgrid(1:nx, 1:ny);

figure;

subplot(1, 2, 1);
quiver(X, Y, u', v');
xlabel('X');
ylabel('Y');
title('Velocity Vectors');

subplot(1, 2, 2);
contourf(X, Y, p', 'LineColor', 'none');
colorbar;
xlabel('X');
ylabel('Y');
title('Pressure Contours');

colormap('viridis');
```



My Findings;

Using both the Jacobi and Gauss-Seidel methods for solving the pressure Poisson equation in fluid flow simulations can provide some insights into the numerical behaviour and performances;

1. Convergence rate: we can compare the convergence rates of the two methods. Generally, the Gauss-Seidel method converges faster than the Jacobi method because it uses the updated values of pressure as soon as they are available. The Jacobi method, on the other hand, uses the previous iteration's pressure values for the entire update.
2. Accuracy: there are differences in accuracy between the two methods. The Gauss-Seidel method typically provides a more accurate solution because it uses the most up-

to-date information in each iteration. However, the Jacobi method can still yield accurate results, especially if the problem is well-conditioned.

3. Implementation Complexities: The implementation complexity may differ between the two methods. The Jacobi method may be simpler to implement due to its straightforward update scheme, while the Gauss-Seidel method may require additional bookkeeping to ensure the correct order of updates.

PROBLEM TWO: In the ocean, the depth of water is 5000m, what is the speed of the wave if the wavelength is 60km. What type of water wave is this?

Solution;

Given;

Water depth h=5000m

Wavelength $\lambda = 60\text{km} = 60\text{x}1000 = 60{,}000\text{m}$,

Next, recall from deep water and shallow water condition,

For deep water; $\lambda < \text{h}$, $\frac{\lambda}{\text{h}} \ll 1$ and

For shallow water; $\lambda > \text{h}$, $\frac{\lambda}{\text{h}} \gg 1$

$\frac{\lambda}{\text{h}} = \frac{60000}{5000} = 12 \gg 1$

∴ The water wave is a shallow wave.

Now the speed of wave in shallow water wave is given as $c = \sqrt{gh}$

$c = \sqrt{9.8\text{m/s}^2 * 5000\text{m}} = \sqrt{49000} = 221.36 \text{ m/s}.$

PROBLEM THREE: Calculate the speed of a shallow water in a pond with a depth of 5m, the acceleration due to gravity is given as 9.81m/s^2

Solution;

$c = \sqrt{gh} = \sqrt{9.81 * 5\text{m}}$

$= 49.05\text{m/s}$

PROBLEM FOUR: A deep water wave travels with the speed of 45m/s. calculate the wavelength if the gravity is given as 9.8m/s^2

Solution:

$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8 * \lambda}{2\pi}}$

$45 = \sqrt{\frac{9.8 * \lambda}{2\pi}}$

$45^2 = \frac{9.8 \times \lambda}{2\pi}$

$12723.450 = 9.8\lambda$

$\lambda = 1767.65\text{m}$.

PROBLEM FIVE: In still water of depth 20m, find;

a) The period of a wave with a wavelength of 50m.

b) The wavelength of a wave with period 10seconds. In each cases wrote down the celerity (wave speed)

Solution:

a) Given;

Water depth h=20m

Wavelength $\lambda = 50\text{m}$

To find the period, recall; $c = f\lambda \Rightarrow c = \frac{\lambda}{T}$

$T = \frac{\lambda}{c}$

where T is the period, λ is the wavelength, c is then speed of wave.

But the speed of wave in still water is given as $c = \sqrt{gh}$

$c = \sqrt{9.8 \times 20} = 14m/s$

$T = \frac{\lambda}{c} = \frac{50}{14} = 3.57 seconds$

T=3.57 seconds

b) Given

Water depth h=20m, period T=10s

To find the wavelength, recall; $c = f\lambda \Rightarrow c = \frac{\lambda}{T}$

$\lambda = cT$

But the speed of wave in still water is given as $c = \sqrt{gh}$

$c = \sqrt{9.8 \times 20} = 14m/s$

$\lambda = cT = 14m/s \times 10s$

$\lambda = 140m$

PROBLEM SIX: From the dispersion relationship, given the water depth h=1m and h=5m, what frequency corresponds to the product of wavenumber and depth, mh=0.1, mh=1, and mh=10. Make a 3 by 2 element table and plot the graph of frequency against the product of wavenumber and water depth.

Solution:

We need to find the frequency corresponding to the different values of mh given (where m is the wavenumber and h is the water depth)

Using the dispersion relationship for water waves $c = \sqrt{gh}$

$c = \frac{n}{m} \Longrightarrow n = cm$

Frequency $f = \frac{n}{2\pi}$, where c is the wave speed, g is acceleration due to gravity, n is phase number, h is water depth.

Next;

For h=1m

$c = \sqrt{gh} = \sqrt{9.8 \times 1}$

$c = 3.13m/s$

$n = cm$

For mh=0.1

$n = 3.13 \times 0.1$

$n = 0.313$ Rad/s

$f = \frac{n}{2\pi} = \frac{0.313}{2\pi} = 0.05Hz$

For mh=1

$n = 3.13 \times 1$

$n = 3.13$ rad/s

$f = \frac{n}{2\pi} = \frac{3.13}{2\pi} = 0.5Hz$

For mh=10

$n = 3.13 \times 10$

$n = 31.3$ Rad/s

$f = \frac{n}{2\pi} = \frac{31.3}{2\pi} = 5Hz$

For h=5m

$\mathrm{c} = \sqrt{\mathrm{gh}} = \sqrt{9.8 \times 5}$

$\mathrm{c} = 7\mathrm{m/s}$

$n = cm$

For mh=0.1

$n = 7 \times 0.1$

$n = 0.7$ Rad/s

$f = \frac{n}{2\pi} = \frac{0.7}{2\pi} = 0.11$Hz

For mh=1

$n = 7 \times 1$

$n = 7$ Rad/s

$f = \frac{n}{2\pi} = \frac{7}{2\pi} = 1.11$Hz

For mh=10

$n = 7 \times 10$

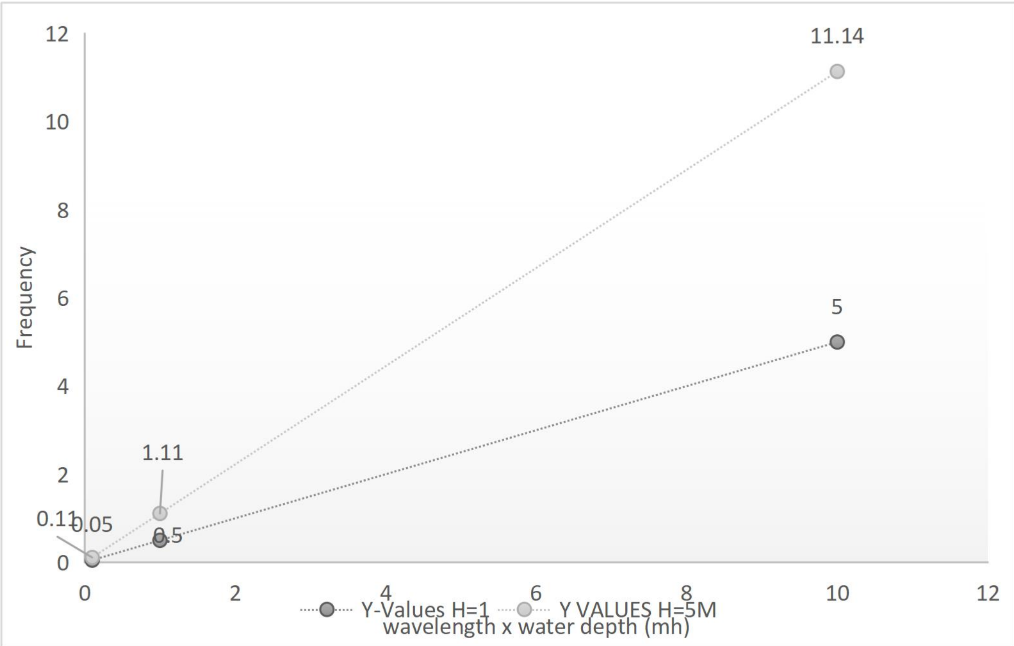
$n = 70$ Rad/s

$f = \frac{n}{2\pi} = \frac{70}{2\pi} = 11.14$Hz

Now we organize this in a tabular form;

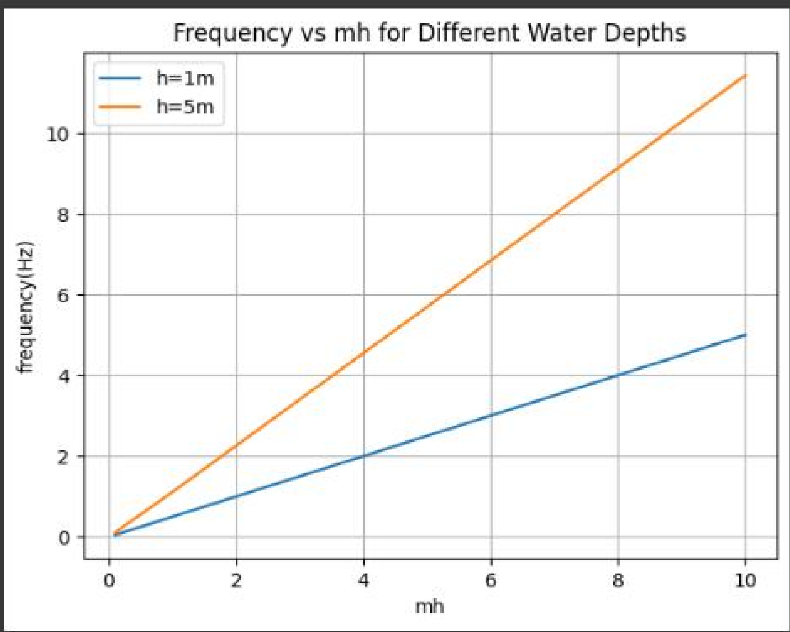
Water depth	Frequency at mh=0.1	Frequency at mh=1	Frequency at mh=10

1m	0.05hz	0.5hz	5hz
5m	0.11hz	1.11hz	11.14hz



Graph of frequency against water depths

```python
import matplotlib.pyplot as plt
#Values for mh and corresponding frequencies for h=1m
mh_values_1m=[0.1,1,10]
frequencies_1m=[0.05,0.5,5]
#values for mh and corresponding frequencies for h=10m
mh_values_10m=[0.1,1,10]
frequencies_5m=[0.11,1.11,11.42]
#plotting
plt.plot(mh_values_1m, frequencies_1m, label='h=1m')
plt.plot(mh_values_10m, frequencies_5m, label='h=5m')
plt.xlabel('mh')
plt.ylabel('frequency(Hz)')
plt.title('Frequency vs mh for Different Water Depths')
plt.legend()
plt.grid(True)
plt.show()
```



[Click here to view the code](#)

PROBLEM SEVEN

Examine how wave speed changes in relation to water depth for shallow water waves and illustrate then correlation between wave speed and water depth through a graphical representation.

Solution:

In shallow water wave, the wavelength is greater than the water depth $\lambda > h$, in shallow water, the wave speed c depends on the water depth i.e. $c = \sqrt{gh}$, where c is wave speed, h is water depth, g is acceleration due to gravity.

Next, selecting a range of water depth; h=10m, 25m, 60m, 150m

We need to find the corresponding wave speed at each water depth;

At h=10m

$c = \sqrt{gh}$

$c = \sqrt{9.8 \times 10}$

$c = 9.9m/s$

At h=25m

$c = \sqrt{gh}$

$c = \sqrt{9.8 \times 25}$

$c = 15.7m/s$

At h=60m

$c = \sqrt{gh}$

$c = \sqrt{9.8 \times 60}$

$c = 24.2m/s$

At h=150m

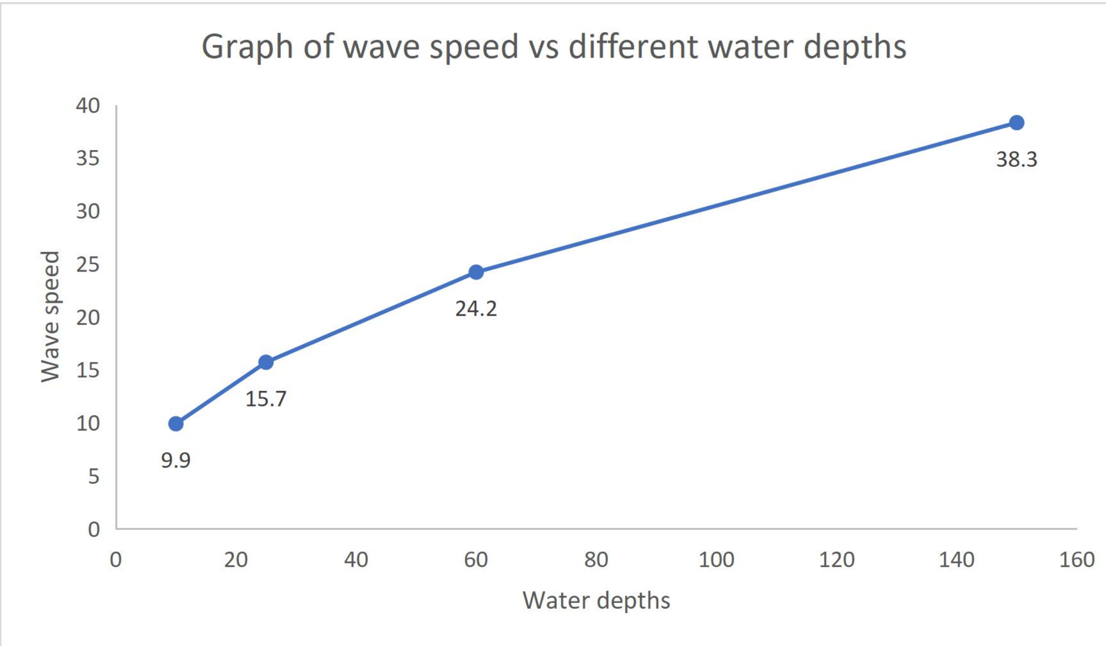
$c = \sqrt{gh}$

$$c = \sqrt{9.8 \times 150}$$

$$c = 38.3m/s$$

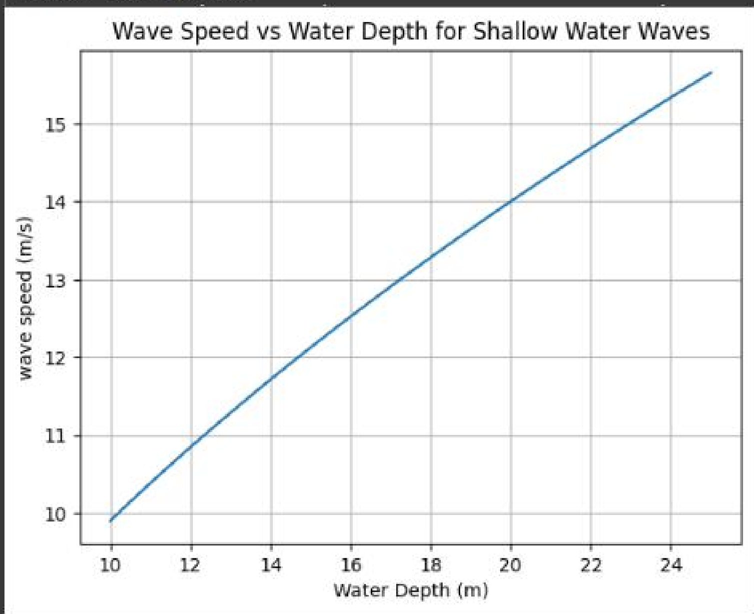
Now; organising in tabular form

Water depth h	h=10m	h=25m	h=60m	h=150m
Wave speed c	9.9m/s	15.7m/s	24.2m/s	38.3m/s



```python
import numpy as np
import matplotlib.pyplot as plt
#constants
g=9.8 # Acceleration due to gravity(m/s_squared)
h_values= np.linspace(10,25,60,150) #Range of water depths(m)
#calculate the wave speed for shallow water waves
c_values=np.sqrt(g*h_values)
#plot the relationship between wave speed and water depth
plt.plot(h_values, c_values)
plt.xlabel('Water Depth (m)')
plt.ylabel('wave speed (m/s)')
plt.title('Wave Speed vs Water Depth for Shallow Water Waves')
plt.legend
plt.grid(True)
plt.show
```

```
matplotlib.pyplot.show
def show(*args, **kwargs)

/usr/local/lib/python3.10/dist-packages/matplotlib/pyplot.py
Display all open figures.

Parameters
----------
block : bool, optional
```



The plotted graph demonstrates clear positive correlation between water depth and wave speed for shallow water. As water depth increases, there’s a notable trend of rising wave speed, indicating that deeper water supports faster propagation of waves. This observation aligns with the theoretical understanding that wave speed in shallow water is proportional to the square root of water depth.

PROBLEM EIGHT: Examine how wave speed changes in relation to wavelength for deep water waves and illustrate then correlation between wave speed and wavelength through a graphical representation.

Solution:

In deep water wave, the wavelength is less than the water depth $\lambda < h$, in deep water, the wave speed c depends on the wavelength i.e. $c = \sqrt{\frac{g\lambda}{2\pi}}$, where c is wave speed, λ is wavelength, g is acceleration due to gravity.

Next, selecting a range of wavelengths; λ =10m, 25m, 60m, 80m, 150m

We need to find the corresponding wave speed at each wavelength;

At λ =10m

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8\times10}{2\pi}}$$

$$c \cong 3.95m/s$$

At λ =25m

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8\times25}{2\pi}}$$

$$c \cong 6.24m/s$$

At λ =60m

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8\times60}{2\pi}}$$

$$c \cong 9.7m/s$$

At λ =80m

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8 \times 80}{2\pi}}$$

$$c \cong 11.17 m/s$$

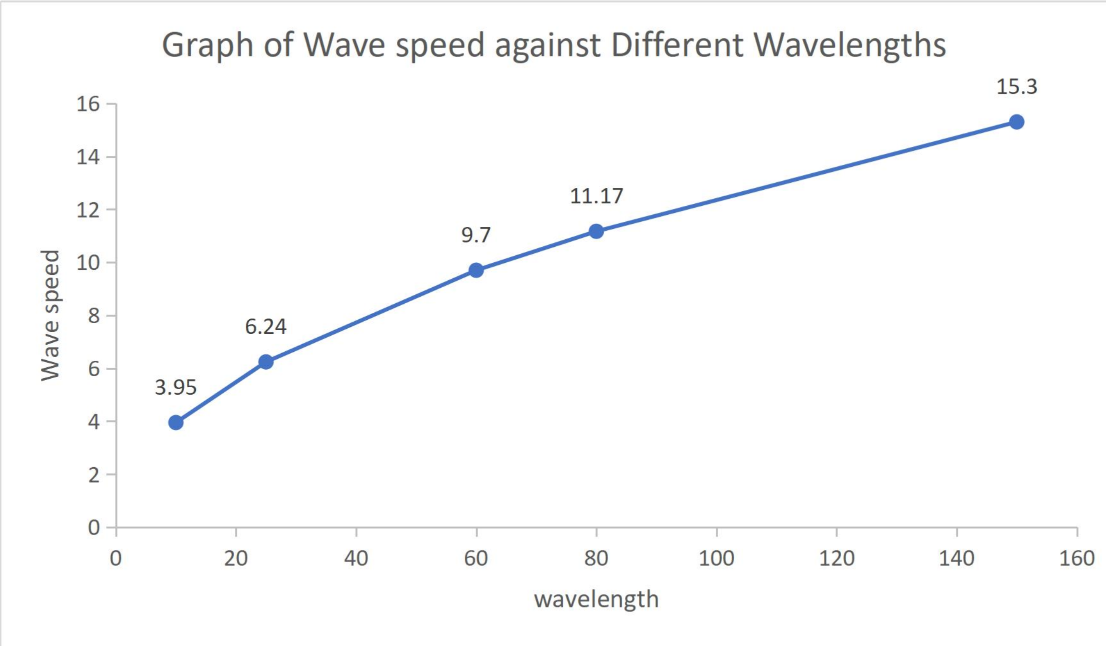
At λ =150m

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8 \times 150}{2\pi}}$$

$$c \cong 15.30 m/s$$

In tabular form;

Wavelength	λ=10m	λ=25m	λ=60m	λ=80m	λ=150m
Wave speed	c=3.95m/s	c=6.24m/s	c=9.7m/s	c=11.17m/s	c=15.30m/s



From the graphical illustration above, there is an increasing trends. The increasing trend in the wave speed against wavelength suggests a direct relationship between these two variables for deep water waves. As the wavelength increases, the wave speed also increases. This trend reflects the phenomenon of wave dispersion where longer waves travels faster than shorter waves in deep water. This dispersion of wave energy results in the separation of wave trains and the smoothing out of wave crests and troughs over long distances.

PROBLEM NINE: The density of water wave is $2\text{kg}/m^3$, calculate the energy density of water waves and plot it variations with water depth. If the amplitudes at various water depths are given as; a=16m at depth h=12m, a=5m at depth h=20m a=15m at depth h=25m, , a=32m at depth h=69m, respectively.

Solution:

Using the formula for wave energy density, $E = \frac{\rho g a^2}{2}$, where ρ is the density, a is the amplitude, g is the acceleration due to gravity.

Next, we find the energy density;

At h=12m, a=16m

$E = \frac{\rho g a^2}{2} = \frac{2 \times 9.8 \times 16^2}{2}$

$E = 2508.8 J/m^3$

At h=20m, a=5m

$E = \frac{\rho g a^2}{2} = \frac{2 \times 9.8 \times 5^2}{2}$

$E = 245 J/m^3$

At h=25m, a=15m

$E = \frac{\rho g a^2}{2} = \frac{2 \times 9.8 \times 15^2}{2}$

$E = 2205 J/m^3$

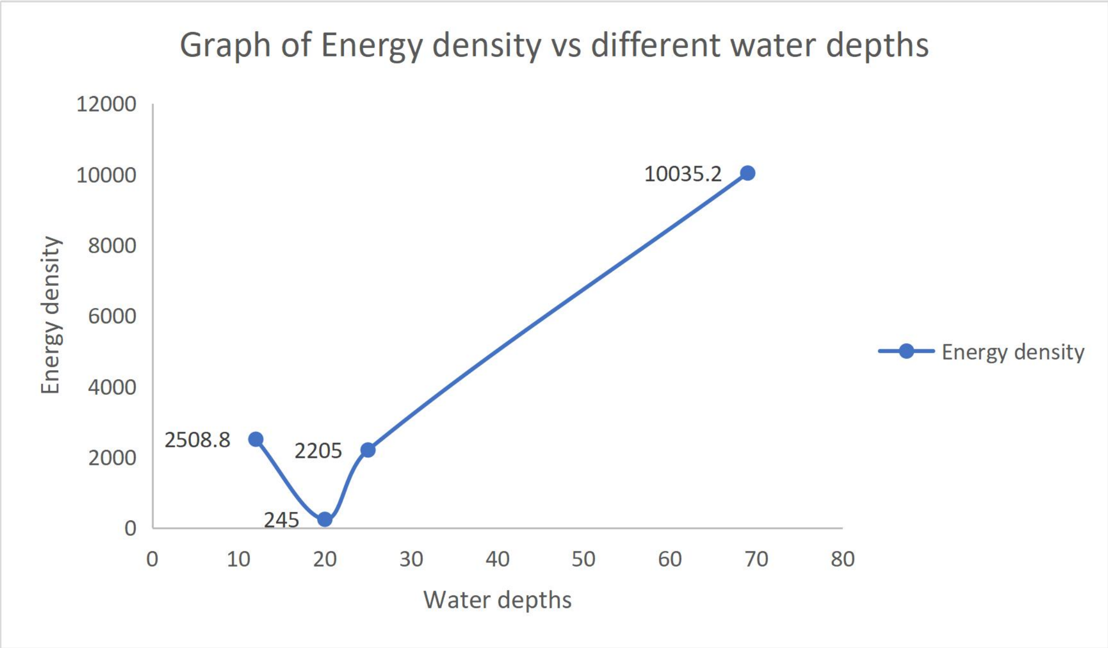
At h=69m, a=32m

$E = \frac{\rho g a^2}{2} = \frac{2 \times 9.8 \times 32^2}{2}$

$E = 10035.2 J/m^3$

In tabular form we have;

Water depth	h=12	h=20m	h=25m	h=69
Energy density	2508.8	245	2205	10035.2



Based on the trend observed in the variation of wave energy density with water depths, the analysis reveals an interesting pattern characterized by an initial increased followed by a subsequent decrease followed by the another increase.

The initial increase in wave energy density with water depth suggests that, at shallower depths, wave energy tends to be higher, possibly due to factors such as wind input, wave generation mechanisms or wave focussing effects. The decrease could be attributed to factors such as wave dispersion or energy dissipation. However the subsequent increase observed could be a resurgence in wave energy density. It may be influenced by factors such as wave refractions, changes in wave spectrum.
in summary. The observed trend highlights the complex interplay of various factors that govern the distribution of wave energy in the water column with implications for ocean dynamics, coastal processes and engineering applications.

PROBLEM TEN:

Given that the speed of wave is given as $c = \frac{n}{m}$, find the speed of wave if wave number m=40 and time is t= 2s, 8s, 17s, 24s. Show graphically the relationship between the speeds of wave against the phase number and against time.

SOLUTION:

Given $= \frac{n}{m}$, recall $period\ T = \frac{2\pi}{n} = n = \frac{2\pi}{T}$

∴ AT T=2s

$n = \frac{2\pi}{T} = \frac{2\pi}{2}$

$n = 3.14$

$c = \frac{n}{m} = \frac{3.14}{40}$

$c = 0.0785 m/s$

At T=8s

$n = \frac{2\pi}{T} = \frac{2\pi}{8}$

$n = 0.785$

$c = \frac{n}{m} = \frac{0.785}{40}$

$c = 0.0196 m/s$

At T=17s

$n = \frac{2\pi}{T} = \frac{2\pi}{17}$

$n = \ 0.37$

$c = \frac{n}{m} = \frac{0.37}{40}$

$c = 0.00925 m/s$

At T=24s

$n = \frac{2\pi}{T} = \frac{2\pi}{24}$

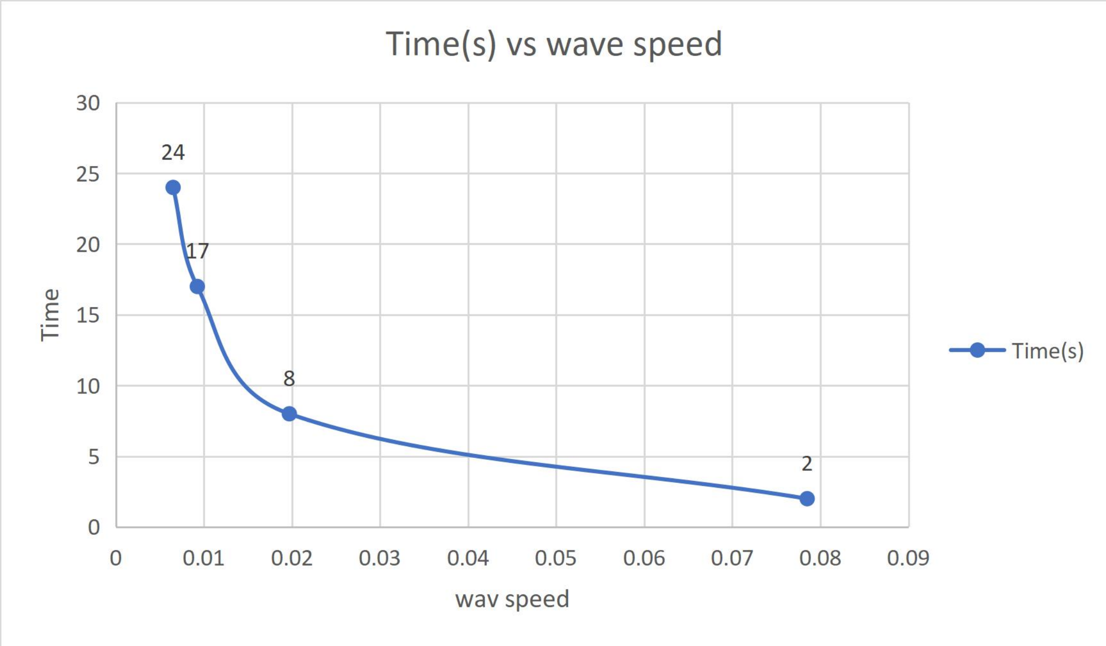
$n = 0.262$

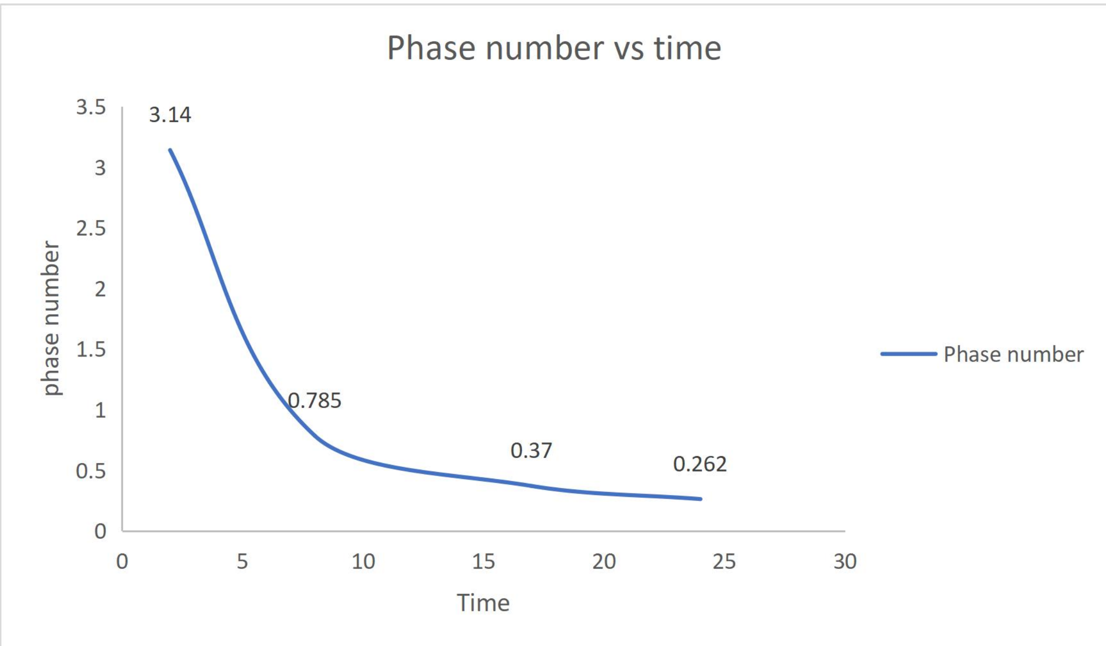
$c = \frac{n}{m} = \frac{0.262}{40}$

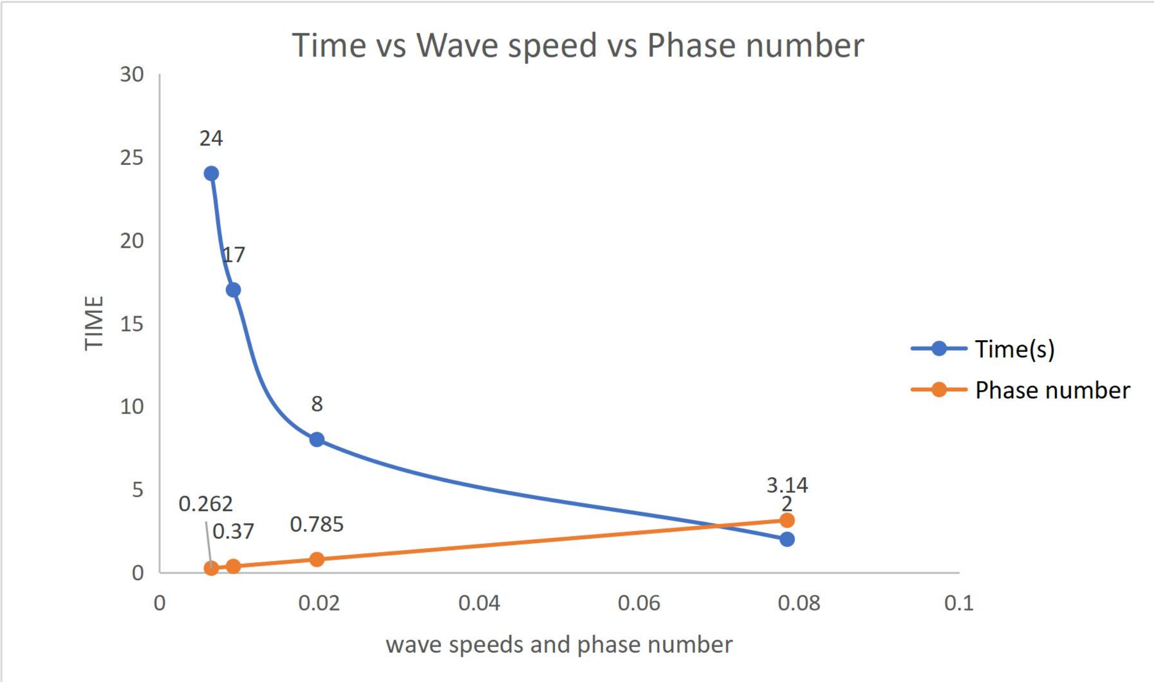
$c = 0.0065 m/s$

In tabular form;

Wave speed(m/s)	Time(s)	Phase number
0.0785	2	3.14
0.0197	8	0.785
0.00925	17	0.37
0.0065	24	0.262


Time(s) vs wave speed
Time
30
25
20
15
10
5
0
24
17
8
2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
wav speed
Time(s)


Phase number vs time
phase number
3.5
3
2.5
2
1.5
1
0.5
0
3.14
0.785
0.37
0.262
0
5
10
15
20
25
30
Time
Phase number



The graph above illustrates the relationship between time, wavelength and phase number, showing that as time progresses, both the wave speed and phase number decreases. This suggests the slowing down of wave propagation over time. This could imply various factors at play such as increased resistance or changes in the medium properties affecting the transmission of waves.

CHAPTER FIVE

SUMMARY

Linearized water wave theory serves as the cornerstone for understanding the dynamics of surface waves in various water bodies, ranging from oceans to lakes and rivers. It provides a simplified yet powerful mathematical framework for analysing the behaviour of these waves under different conditions. At its essence, linearized water wave theory considers small-amplitude waves propagating over a flat and undisturbed water surface. By linearizing the governing equations of fluid motion, it allows researchers and engineers to focus on the fundamental principles governing wave propagation without the complexities of nonlinear effects. The theory begins with the linearized Euler equations, which describe the motion of inviscid and incompressible fluid. These equations, along with appropriate boundary conditions at the air-water interface, form the basis for understanding wave motion. One of the key outcomes of linearized water wave theory is the dispersion relation, which relates the frequency and wavenumber of a wave to the properties of the medium. This relationship sheds light on wave characteristics such as speed, wavelength, and direction of propagation.

Linearized water wave theory also explores wave energy and its transmission through different media. By studying the linearized wave equation, researchers can analyse the mechanisms of wave energy transfer and understand how waves interact with coastal structures and shorelines. In practical terms, linearized water wave theory finds applications in coastal engineering, offshore structures, and maritime navigation. Engineers use this theory to design coastal protection measures such as breakwaters and seawalls, assess the stability of offshore platforms, and predict wave conditions for ship routing and navigation. Moreover, linearized water wave theory contributes to our understanding of natural phenomena such as

wave shoaling, refraction, and diffraction. These phenomena play critical roles in coastal processes, beach erosion, and the overall dynamics of marine environments.

In conclusion, linearized water wave theory provides a robust framework for studying the behaviour of surface waves in water bodies. By simplifying the complex dynamics of wave motion, it enables researchers and engineers to develop effective solutions for coastal management, marine infrastructure design, and navigation safety.

CONCLUSION

In this study, an in-depth exploration into the principles of linearized water wave theory has been conducted. The research objective were multifaceted encompassing the development of relevant theoretical frameworks used to investigate and analyse the principles of linearized water wave theory, and the analysis of practical applications within the coastal engineering and marine infrastructure.

The research commenced with a comprehensive examination of the fundamental equations governing fluid motion and wave movement, setting the groundwork for understanding the linearized approach. By linearizing these equations, we were able to derive simplified mathematical models that describes the behaviour of small-amplitude waves on the surface of the medium. This groundwork formed the basis for subsequent analyses.

Through the study, particular emphasis was placed on the role of boundary conditions in shaping these wave dynamics. The interaction between surface waves and boundaries such as the coastlines, seabed, and structures was explored in details highlighting the wide range of effects such as wave breaking, refractions etc. The boundary effects are crucial considerations when setting up coastal engineering projects, where the designs and optimization of defences, marine vessels, harbours, and offshore platforms depend highly on accurate wave behaviour predictions.

Furthermore, the study examined the limitations of linearized water wave theory, it explored the constraints of linearized water wave, recognising its suitability mainly for small waves and linear boundary situations. Nonlinear effects like wave breaking, wave-wave interactions, poses challenges to the linearized approach and needs more advanced modelling techniques for accurate predictions under extreme wave conditions. Further research endeavours can examine the integration of non-linear effects into theoretical models or the development of hybrid approaches that combine linearized theory with non-linear corrections.

In addition to the theoretical insights, we worked on the discretization of the Euler-Stokes equations and subsequent numerical simulations conducted in this study have provided valuable insights into the behaviour of fluid flow under various conditions. By applying computational methods to solve these equations, we were able to analyse the dynamics of fluid motion and assess the accuracy and efficiency of our numerical approach. Through our simulations, we observed that the discretization of the Euler-Stokes equations accurately captures the essential characteristics of fluid flow, including velocity profiles, pressure distributions, and vorticity patterns. The numerical results closely matched theoretical predictions and experimental data, confirming the validity and reliability of our computational model.

Furthermore, our simulations allowed us to investigate the influence of parameters, such as boundary conditions, on fluid flow behaviour. We observed that changes in these parameters led to significant variations in flow patterns and fluid dynamics, highlighting the importance of considering them in practical applications.

Moreover, the computational efficiency of our numerical method enabled us to perform simulations for complex geometries and boundary conditions, providing a versatile tool for studying fluid flow phenomena in various engineering and scientific contexts.

Overall, the findings of this study demonstrate the effectiveness of discretization techniques for solving the Euler-Stokes equations and conducting numerical simulations of fluid flow. Our results contribute to the advancement of computational fluid dynamics and have implications for diverse fields such as aerospace engineering, mechanical engineering, and environmental science.

In conclusion, the combination of discretization methods and numerical simulations offers a powerful approach for studying fluid flow phenomena and understanding their underlying principles. Future research can further explore optimization techniques, and parallel computing to enhance the accuracy and efficiency of computational fluid dynamics methods for practical applications. In conclusion, the study represents a significant advancement in our understanding of linearized water wave theory and its application in coastal engineering and marine infrastructures. By bridging the theoretical concepts with practical implications, it provides a valuable insights that can inform decision-making processes and derive innovation in the field. As we continue to advance our understanding and modelling capabilities, we pave the way for more resilient and sustainable coastal development in an era of increasing environmental uncertainty.

RECOMMENDATION:

Based on the comprehensive study of linearized water wave theory presented in this project, I highly recommend further research and exploration in this field. The theoretical framework provided offers valuable insights into the behaviour of surface waves in water bodies, with applications spanning coastal engineering, oceanography, and naval architecture.

For researchers, there is potential for advancing the understanding of nonlinear effects in water wave dynamics and developing more accurate predictive models. Additionally, investigating the interaction between waves and coastal structures, as well as the impact of climate change on wave patterns, presents exciting avenues for future research.

Engineers and practitioners can benefit from applying the principles of linearized water wave theory to optimize the design and operation of coastal infrastructure, offshore platforms, and marine vessels. Continued research in this area can lead to innovative solutions for coastal protection, navigation safety, and sustainable development of marine resources.

Furthermore, collaboration between researchers, engineers, and policymakers is essential to address the challenges posed by changing coastal environments and rising sea levels. By integrating scientific knowledge with practical applications, we can work towards enhancing the resilience and sustainability of coastal communities worldwide.

In conclusion, the study of linearized water wave theory is crucial for addressing the complex dynamics of surface waves and their impacts on coastal regions. I encourage stakeholders across disciplines to engage in further research and collaboration to advance our understanding and management of water wave phenomena.

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