OPTION PRICING

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A PROJECT PRESENTED TO THE DEPARTMENT OF MATHEMATICS, FACULTI OF PHYSICAL SCIENCES, UNIVERSITY OF BENIN, BENIN CITY, IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE AWARD OF BACHELOR OF SCIENCE DEGREE IN APPLIED MATHEMATICS

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UNDERTAKING

This research work was carried out by **Victor Ejiro UMUKORO**. I have not copied the work of any other author. All work used have been duly referenced and acknowledged.

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Victor Ejiro UMUKORO **Date**

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Christopher Adizua DIABUAH. **Date**

CERTIFICATION

This is to certify that this project work was carried out by **CHRISTOPHER ADIZUA DIABUAH** matriculation number **PSC1607475** Department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, Under my supervision.

________________ ___________________

Dr. C. I. Nkeki Date
Project Supervisor

______________________ _______________

Prof. A.A. Osagiede Date
Head of Department

DEDICATION

I dedicate this project to God Almighty, for his love and favour upon me and who has been the source of every knowledge and inspiration. You are the source of my strength and the reason of my continual existence, the helm in my heart. You alone is worthy of my praises and I submit my entirety to you.

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ABSTRACT

The derivation and solution of the celebrated Black-Scholes Option Pricing Formula is set out in rather more detail than has appeared in the literature so far. One problem with the Black-Scholes analysis is that the mathematical skills required in the derivation and particularly in the solution of the model are fairly advanced and probably unfamiliar to most economists. In this project, we will derive the Black-Scholes pricing model of a European option by calculating the expected value of the option. We will assume that the stock price is log-normally distributed and that the universe is risk neutral. Then, using Ito's Lemma, we will justif5' the use of the risk-neutral rate in these initial calculations. Finally, we will prove put-call parity in order to price European put options, and apply the concepts of the Black-Scholes formula to value an option with pricing equity.

CHAPTER ONE

INTRODUCTION TO OPTION PRICING

1.0 INTRODUCTION

An option is a contract that gives the holder the right but not the obligation to buy or sell a specific amount of an underlying asset at a fixed price (called the strike or exercise price) at or before the maturity date of the option. The act of making this transaction is referred to as exercising the option. The given date is called the expiration date. A call option gives the right to buy the shares while a put option gives the right to sell the shares.

Options have been traded for centuries, but they remained relatively obscure financial instruments until the introduction of a listed options exchange in 1973. Since then, Options trading has enjoyed an expansion unprecedented in American securities markets.

1.1 BACKGROUND/HISTORY OF OPTION PRICING

Option pricing theory has a long and illustrious history, but it also underwent a revolutionary change in 1973. At that time. Fischer Black and Myron Scholes presented the first completely satisfactory equilibrium option pricing model, called the Black-Scholes Model. In the same year, Robert Merton extended their model in several important ways. These path breaking articles have formed the basis for many subsequent academic studies.

As these studies have shown, option pricing is relevant to almost every area of finance.

For example, virtually all corporate securities can be interpreted as portfolios of puts and calls on the assets of the firm. Indeed, the option pricing applies to a very general class of economic problems such as, the valuation of contracts where the outcome to each party depends on a quantifiable uncertain future event.

Unfortunately, the mathematical tools employed in the Black-Scholes and Merton articles are quite advanced and have tended to obscure the underlying economics. However, thanks to a suggestion by William Sharpe, it is possible to derive the same results using only elementary mathematics.

1.2 SIGNIFICANCE OF STUDY

Option pricing is a very importance tool in the financial market. One of the basic importance's of the study of option pricing to its investors is that they can carry out a riskless transaction which brings about a profit to the investors.

Secondly, with the application of Black Scholes formula in the option pricing, stakeholders can make certain predictions which would bring about a positive return to the stakeholders. In other word, stakeholders use the Black Scholes formula to determine the value of options.

Lastly, investors can also benefits from option pricing using the concept of implied volatility. Where implies volatility is a forecast of future market prices and acts as an indicator to current market sentiment.

1.3 AIMS AND OBJECTIVES

The aims and objectives of this study are as follows. To;

i. Explicitly explain the concept of option pricing.

ii. Consider in (i) how is related to financial market.

iii. Discuss the Black- Scholes formula to solve our problems.

iv. Discussing the Binomial option pricing formula relating it to problem solving in the financial market.

v. Analyze and giving economic implications of our results as it relates to option pricing.

1.4 SCOPE OF STUDY

The extent to which this study cover includes: the basics of option pricing, the introduction of the black Scholes model to solve option pricing problems, and its applications.

1.5 BASIC CONCEPTS OF OPTION PRICING

1.6 DEFINITION OF TERMS

OPTION: An option is a contract that gives the holder the right but not the obligation to buy or sell a specific amount of an underlying asset at a fixed price (called the strike or exercise price) at or before the maturity date of the option.

HOLDER: An investor who purchases an option and makes a premium payment to the writer.

WRITER: An investor who sells an option and collects the premium payment from the buyer. Writers are obligated to buy or sell if the holder chooses to exercise the option.

PREMIUM: The total cost of the option. An option holder pays the premium to the option writer in exchange for the right, but not the obligation to exercise the option. In general, the options premium is its intrinsic value combined with its time value.

DERIVATIVE: An investment product that derives its value from an underlying asset. Options are derivatives.

CONTRACTS: An option that represents 100 shares of an underlying stock.

UNDERLYING SECURITY: The security that is subject to being bought or sold upon the exercise of an option.

CALL: An option that gives the holder the right to buy the underlying security at a particular price for a specified, fixed period of time.

PUT: An option that gives the holder the right to sell the underlying security at a particular price for a specified, fixed period of time.

STRIKE PRICE: The agreed upon price at which an option can be exercised. The strike price for a call option is the price at which the security can be bought (prior to the expiration date). The strike price for a put option is the price at a security can be sold (before the expiration date). The strike price is also called the exercise price.

AMERICAN OPTIONS: An option that can be exercised at any point during the life of the contract. Most exchange traded options are American.

EUROPEAN OPTIONS: An option that can only be exercised during a particular time period just before its expiration.

LISTED OPTION: A put or calls option that is traded on an options exchange. The terms of the option including strike price and expiration dates are standardized by the exchange.

IN – THE – MONEY: An option that has an intrinsic value. A call option is considered in – the – money if the underlying security is higher than the strike price.

AT – THE – MONEY: An option whose strike price is equal to the market price of the underlying security.

OUT – THE – MONEY: An option with no intrinsic value that would be worthless if it expires on that day. A call option is out – of– the – money when the strike is higher than the market price of the underlying security. A put option is out – of – the – money when the strike price is lower than the market price of the underlying security.

EARLY EXERCISE: The exercise of an option before its expiration date.

Early exercise can occur with American-style options.

1.7 FACTORS DETERMINING THE VALUE FR1CE) OPTION

The value of an option is determined by a number of variables relating to the underlying asset and financial markets.

1. **Current Value (price) of the Underlying Asset:** Options are assets that derive value from an underlying asset. Consequently, changes in the value of the underlying asset affect the value of the options on that asset. Since calls provide the right to buy the underlying asset at a fixed price, an increase in the value of the asset will increase the value of the calls. Puts, on the other hand, become less valuable as the value of the asset increase.
2. **Variance in Value of the Underlying Asset:** The buyer of an option acquires the right to buy or sell the underlying asset at a fixed price. The higher the variance in the value of the underlying asset, the greater the value of the option will become. This is true for both calls and puts. While it may seem counter-intuitive that an increase in a risk measure (variance) should increase value, options are different from other securities since buyers of options can never lose more than the price they pay for them; in fact, they have the potential to earn significant returns from large price movements.

3. **Dividends Paid on the Underlying Asset:** The value of the underlying asset can be expected to decrease if dividend payments are made on the asset during the life of the option. Consequently, the value of a call on the asset is a decreasing function of the size of expected dividend payments. and the value of a put is an increasing function of expected dividend payments. There is a more intuitive way of thinking about dividend payments, for call options. It is a cost of delaying exercise on in-the-money options. To see why, consider an option on a traded stock. Once a call option is in the money, i.e. the holder of the option will make a gross payoff by exercising the option, exercising the call option will provide the holder with the stock and entitle him or her to the dividends on the stock in subsequent periods. Failing to exercise the option will mean that these dividends are foregone.

4. **Strike Price of Option:** A key characteristic used to describe an option is the strike price. In the case of calls, where the holder acquires the right to buy at a fixed price, the value of the call will decline as the strike price increases. In the case of puts, where the holder has the right to sell at a fixed price, the value will increase as the strike price increases.

5. **Time to Expiration on Option:** Both calls and puts become more valuable as the time to expiration increases. This is because the longer time to expiration provides more time for the value of the underlying asset to move, increasing the value of both types of options.

6. **Riskless Interest Rate Corresponding to Life of Option:** Since the buyer of an option pays the price of the option up front, an opportunity cost is involved. This cost will depend upon the level of interest rates and the time to expiration on the option. The riskless interest rate also enters into the valuation of options when the present value of the exercise price is calculated, since the exercise price does not have to be paid (received) until expiration on calls (puts). Increases in the interest rate will increase the value of calls and reduce the value of puts.

1.8 PROPERTIES OF OPTION PRICING

The level of the strike price and the value of option: Call options with lower strike prices are more valuable and the put options with higher strike prices are more valuable to the holders.

Intrinsic value versus time value: Option premiums (values) have two components:

(i) Intrinsic value

(ii) Time value

Intrinsic Value:

- If the current stock price is above the strike price of a call (or below the strike price of a put), the option has intrinsic value. An option with intrinsic value is said to be in — the — money.
- If the current stock price is equal to or below the strike price of a call (or equal to or above the strike price of a put), the option has no intrinsic value.

So, an option with no intrinsic value is said to be at-the-money if current market price of the stock is equal to the strike price.

An option with no intrinsic value is said to be out — of — the — money if current market price of the stock is below the strike price of a call and above the strike price of a put.

Market scenario	Call	Put
Market price>strike price	In-the-money	Out-of-the-money

Market price>strike price	At-the-money	At-the-money
Market price = strike price	Out-of-the-money	In-the-money

Time – value: Positive premium?

Answer: Because, it shall has a time value. The difference between an option's price (premium) and its intrinsic value is called the option's time value.

Hence, Time Value Premium — Intrinsic Value

Hence, Intrinsic value and Time Value:

Call: Intrinsic Value = Market Price — Strike Price

Time Value = Premium — Intrinsic Value

Premium = Intrinsic Value + Time Value

Put: Intrinsic Value Strike Price — Market Price

Time Value = Premium — Intrinsic Value

Premium = Intrinsic Value + Time Value

1.9 WHEN DO WE EXERCISE OPTIONS?

Call options -

Closing stock price> strike price

Put options -

Closing stock price <strike price

1.10 THE IN AND OUT OF OPTIONS

In-The-Money Calls:

Stock price is above strike price

In-the-money calls have intrinsic value

EXAMPLE:

With a stock price of $63, the 60 Call is in-the-money. money by $3, and it has $3 (per share) of intrinsic value.

Out-of-The-Money Calls:

Stock price below strike price

Out-of-the-money calls do not have intrinsic value

EXAMPLE:

With a stock price of $63, the 65 Call is out-of-the-money. of-the-money by $2, and it has no intrinsic value.

At-the-Money Calls:

Stock price equal to strike price

At-the-money calls do not have intrinsic value

EXAMPLE:

With a stock price of $60, the 60 Call is at-the-money.

1.11 AMERICAN VERSUS EUROPEAN OPTIONS: VARIABLES RELATING TO EARLY EXERCISE

A main distinction between American and European options is that American options can be exercised at any time prior to its expiration, while European options can be exercised only at expiration. The possibility of early exercise makes American options more valuable than otherwise similar European options. It also makes them more difficult to value. There is one compensating factor that enables the former to be valued using models designed for the latter. In most cases, the time premium associated with the remaining life of an option and transactions costs makes early exercise sub-optimal. In other words, the holders of in-the- money options will generally get much more by selling the option to someone else than by exercising the options.

While early exercise is not optimal general]y, there are at least two exceptions to this rule. One is a case where the underlying asset pays large dividends, thus reducing the value of the asset, and any call options on that asset. In this case, call options may be exercised just before an ex-dividend date if the time premium on the options is less than the expected decline in asset value as a consequence of the dividend payment. The other exception arises when an investor holds both the underlying asset and deep in-the-money puts on that asset at a time when interest rates are high. In this case, the time premium on the put may be less than the potential gain from exercising the put early and earning interest on the exercise price.

CHAPTER TWO

2.0 INTRODUCTION

The studying of option pricing whose evolution through literature to be consider in this chapter is very large and still growing vast in the financial market. In this chapter, we shall consider the vast literature on option pricing using the Garch Black — Scholes formula.

2.1 LITERATURE REVIEW

Some of the relevant literatures are reviewed in the following.

Adesi (2007) proposed a method for pricing options based on GARCH models with filtered historical innovations. They found that their model outperformed other GARCH pricing models and Black-Scholes models empirically for S&P 500 index options. Their model was validated by empirically obtaining decreasing state price densities per unit probability.

Also, their model explained implied volatility smiles by the negative asymmetry of the filtered historical innovations. The study also provides empirical evidence and quantifies the deterioration of the delta hedging in the presence of large volatility shocks.

Cristofferson (2004) extended their results in the presence of conditional skewness

Shiu (2004) proposed a method for pricing derivatives under the GARCH assumption for underlying assets in the context of a dynamic version of Gerber-Shius option-pricing model. Instead of adopting the notion of local risk-neutral valuation relationship (LRNVR) they employ the concept of conditional Esscher Transforms to identify a martingale measure under the incomplete market setting. Under the conditional normality assumption for the stock innovation, the pricing result is consistent with that of Duan. In line with the Gerber-Shius option pricing model, they also justify the pricing result within the dynamic framework of utility maximization problems which makes the economic intuition of the pricing result more appealing. Numerical results for the comparison of the model with the Black Scholes-Merton option pricing model are also presented.

Dash (2012) applied the GARCH options pricing model for options traded on the National Stock Exchange, India. They used the GARCH (1. 1) model to obtain volatility projections, and calculated option prices using these volatility projections in the Black-Scholes-Merton model. They found that the implied Volatility (for both calls and puts) were overestimated, and that call and put option prices were predominantly overvalued, and, further, that put options were more overpriced than call options. They also found that the overestimation of volatility and overvaluation of options prices increased with higher market capitalization and moderate/higher trading volume of the underlying stocks.

Duan (1995) introduced the GARCH option pricing model, linking econometric models with the options pricing literature.

Heston and Nandi (2000) developed a closed-form option valuation formula for a spot asset whose variance follows a GARCH (p, q)-process that can be correlated with the returns of the spot asset. They found empirically for S&P500 index options that their model had lower valuation errors than the Black. Scholes Merton model with implied volatilities. They argued that the GARCH mode] was able to simultaneously capture the correlation of volatility with spot returns and the path dependence in volatility.

Hao and Yang (2011) presented a scenario-based risk measure for a portfolio of European-style derivative securities over a fixed time horizon under the regime- switching Black-Scholes economy. The study derived a closed-form expression for the risk measure for vanilla European options and barrier options, and this approach can be applied to some other exotic options. The results of the study provide some guidelines and insights for portfolios containing different kinds of derivatives.

Jacobs and Christofferson (2004) compared a range of GARCH models with different lags, using option prices and returns. They found that, in contrast to the returns-based objective function, using an option price-based objective function favored a more parsimonious model.

Jacobs (2004) suggested that index option prices differ systematically from those predicted by the Black-Scholes-Merton model. In particular, out-of-the- money put prices and in-the-money call prices were higher than predicted by the Black-Scholes-Merton model. They inverse Gaussian GARCH model, which performed better than the usual BSM model for out-of-the-money puts on the S&P 500 Index.

Singh (2011) empirically investigated the forecasting performance of closed- form discrete time GARCH option pricing model with benchmark Black-Scholes and its version practitioner Black-Scholes model for pricing S&P CNX Nifty 50 index option of India, relative to market price using error metrics, moneyless maturity-wise. They found that the practitioner Black-Scholes model outperforms the other two models, and reduced the price bias between model and market.

Varma (2002) evaluated the volatility pricing of the index options with the help of the Black-Scholes-Merton option pricing formula and the GARCH (1, 1) model and has found severe mispricing in Indian Index options. He has also established the significant difference in volatility smiles for call and put options.

Lehar (2002) examined the performance of two extensions of the Black Scholes-Merton framework, the GARCH and the stochastic volatility option pricing model. They found empirically for FTSE 100 option prices that GARCH dominated over the stochastic volatility and the Black-Sdholes-Merton model. However, they found significant errors in the prediction of the market risk from hypothetical derivative positions in all the models.

In Grossmann (1976), Grossman provided some of the most influential insights into the 'role of information in markets. He constructed a simple model of a market with a single risky asset and traders who can be either uninformed or become informed by incurring some cost. He reasoned that, in a perfect market with costly information, there must be noise so that agents can earn a return on their investment in information gathering. Otherwise the market will break down because it lacks both an equilibrium where agents earn a return on their information and one where agents do not gather information.

Figlewski and Webb (1993) echoed the arguments of Manaster and Rendleman (1982) in reasoning that option markets give traders who cannot or will not engage in short sales (e.g., due to transaction costs) an opportunity to sell short indirectly.

They argued that the option market maker who is the counterparty of such a transaction will usually hedge by performing a short sale herself, subject to lower transaction costs and fewer constraints. Starting from this assumed mechanism, the authors conjectured that the existence of options shouZ m ‘ositively related to the average level of short interest. They tested this hypothesis empirically using a sample of 342 stocks with uninterrupted data from 1969 to 1985 from the Standard & Poor’s 500 index (S&P 500), taken from the CRSP tapes. The results show that relative short interest was significantly higher for stocks that had traded options than for those without, in each year of the sample.

Jennings and Starks (1986) examined quarterly earnings announcements from NYSE-listed stocks of the S&P 500 from June 15 to August 21, 1981, and from October 4 to December 31, 1982, to find what effect the trading of options on a stock had on the price impact of earnings announcements. They found that the prices of non-option companies took longer to adjust following earnings announcements than that of companies which were the underlying of option trading, supporting the notion that the latter were more efficient.

Skinner (1990) arrived at similar results when he found that optioned stocks at the Chicago Board Options Exchange (CBOE) and the American Stock Exchange (AMEX) were being followed by a larger number of analysts than stocks without options written on them. He took that as an explanation for his second finding, namely that the stock price reaction upon the release of accounting earnings information for newly optioned stocks, as compared to levels prior to options being written on their shares, declined both in absolute terms and conditional on unexpected earnings, with significance at the 1%-level.

Easley (1998) showed that option volumes led stock price changes and carried information about future stock price changes, an interdependence that was later complemented by the results of

Jayaraman (2001). The latter reported that, for their sample period of 1986—1996, the CBOE led equity markets in terms of volume. Pan and Poteshman (2003) came to the same conclusion and reported that the effect was particularly evident for small stocks (which can generally be assumed to be less informational efficient) and remained consistent at the annual level over a period of 12 years.

Lee and Yi (2001) found that informed traders preferred trading on the CBOE to trading on the NYSE, but not for all volumes. They calculated that large-volume informed trades were more frequent at the NYSE and argued that the reason for this observation may have been that large trades at the CBOE tended not to be anonymous, while they were more so at the NYSE. They argued that, since market makers at the CBOE could distinguish between informed and uninformed traders for larger orders, they increased the spread for informed traders, thus making the CBOE less attractive for such large informed orders. Furthermore, their results suggested that informed investors were attracted to options with lower option deltas, i.e., larger leverage.

Chakravarty (2004) focused on a slightly different aspect of the topic and argued that informed insiders sometimes trade in option markets, a conjecture that they arrived at after reviewing insider trading convictions in option markets. They employed an approach first applied by Hasbrouck (1995), which allowed them to measure directly the share of price discovery across 60 stocks listed at the NYSE that possessed options exclusively at the CBOE over a period from 1988 to 1992. With this method, they calculated implied stock prices from call option prices and compared them to actual prices in the stock market. The results showed that an average of between 17% and 18% of the price discovery occurred in the option market, with estimates for individual stocks ranging from close to 12—23% — numbers that they found to be significantly different from zero at the 1%-level. They also observed that the information share of out-of-the-money options seemed to be higher than for in- or at-the-money options, and that option market price discovery appeared to be an increasing function of volume — evidence that is consistent with informed traders who value both leverage and liquidity.

Cao (1999) proposed a model which implied that the introduction of options caused an increase in the prices of the underlying asset and the market index, decreased the price response of the asset upon new public information, and increased the number of analysts following the underlying asset (consistent with Skinner (1990ff His empirical evidence backed up the predictions of the model, supporting his hypothesis that the installation of an options market induced investors to acquire more precise information, because it gave them additional opportunities to profit from trading on it.

Taken together, the evidence suggests relatively strongly that the presence of derivatives markets in general and option markets in particular tend to increase the efficiency and market quality in the market for the underlying stock. It was these results that formed part of the motivation for the experiments described in the following chapters.

CHAPTER THREE

DERIVATION OF THE MODEL EQUATIONS

3.0 INTRODUCTION

In this chapter, we will derive the Black-Scholes partial differential equation and ultimately solve the equation for a European call option and the European put option. First, we will derive a model for the movement of a stock, which will include a random component and Brownian motion. From this model, we will derive the Black-Scholes partial differential equation, and I will use boundary conditions for a European call option to solve the equation. Lastly, I will apply the concept of call put parity to solve the European put option.

3.1 STOCK PRICE MODEL

The stock prices move randomly because of the efficient market hypothesis. There are different forms of this hypothesis, but all say the same two things. First, the history of the stock is frilly reflected in the present price. Second, markets respond immediately to new information about the stock. With the previous two assumptions, changes in a stock price follow a Markov process. A Markov process is a stochastic process where only the present value of the variable is relevant for predicting the future. So, our stock model states that our predictions for the future price of the stock should be unaffected by the price one week, one month, or one year ago.

As stated, a Markov process is a stochastic process. In the financial world, stock prices are restricted to discrete values, and changes in the stock price can only be realized during specified trading hours. Nevertheless, the continuous-variable, continuous-time model proves to more useful than a discrete model.

Another important observation is to note that the absolute change in the price of a stock is by itself, not a useful quality. For example, an increase of one dollar in a stock is much more significant on a stock worth $10 than a stock worth $100. The relative change of the price of a stock is information that is more valuable. The relative change will be defined as the change in the price divided by the original price.

Now consider the price of a stock S at time t. Consider a small time interval dt during which the price of the underlying asset S changes by an amount dS. The most common model separates the return on the asset, $\frac{dS}{s}$ into two parts. The first part is completely deterministic, and it is usually the risk free interest rate on a Treasury bill issued by the government.

This part yields a contribution of

$$\mu dt$$

to $\frac{dS}{s}$. Here u is a measure of the average rate of growth of the stock, also known as the drift. In this model ji is assumed to be the risk free interest rate on a bond, but it can also be represented as a function of S and t. The second part of the model accounts for the random changes in the stock price due to external effects, such as unanticipated news. It is best modeled by a random sample drawn from a normal distribution with mean zero and contributes

$$\sigma \, d \, B$$

to $\frac{dS}{s}$. In this formula is defined as the volatility of the stock, which measures the standard deviation of the returns. Like the term μ, σ can be represented as a function of S and t. The B in dB denotes Brownian motion, which will be described in the next section. It is important to note that μ and σ can be estimated for individual stocks using statistical analysis of historical prices. This is not of interest for our model. It is only important that [t and a are functions of S and 1. putting this information together; we obtain the stochastic differential equation

$$\frac{dS}{s} = \mu\ d\ t + \sigma\ d\ B \tag{1}$$

Note that if the volatility is zero the model implies

$$\frac{d\,S}{S} = \mu\, d\ t$$

where S is the price of the stock at t and S_0 is the price of the stock at t = 0. This equation shows that when the variance of an asset is zero, the asset grows at a continuously compounded rate of μ per unit of time.

Theorem 1. (Ito's lemma): Let *f(S, t)* be a function of two variables, and let S satisfies the stochastic process $dS\ \mu Sdt + \sigma SdB_t$, then

$$df\ (S,t) = \frac{\partial f}{\partial S} dS + \frac{\partial\ f}{dt} dt + \frac{1}{2}\sigma^2 S^2 \left(\frac{\partial^2 f}{\partial s^2}\right) dt \tag{2}$$

We will use this case of Ito's lemma to compute the Black-Scholes partial differential equation.

3.2 BLACK SCHOLES MODEL

Here we are going to derive the price of a derivative security, V(S, t). The model for a stock we derived in (3.1) satisfies an Ito process.

Therefore, we let the function V(S, t) be twice differentiable in S and differentiable in t. Applying equation (2) we have

$$dV(S,t) = \frac{\partial v}{\partial s} dS + \frac{\partial v}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial s^2} dt \tag{3}$$

Substituting into equation (3) for dS with Equation 1 we have

$$dV(S,t) = \frac{\partial V}{\partial S} (\mu S dt + \sigma S dB) + \frac{\partial V}{\partial t} dt + \frac{1}{2} \mu^2 S^2 \frac{\partial^2 V}{\partial S^2} dt$$

This simplifies to

$$dV(S,t) = \sigma S dB \frac{\partial V}{\partial S} + \left(\mu S \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{4}$$

Now set up a portfolio long one option, V, and short an amount $\frac{\partial V}{\partial S}$ stock. Note from above that this portfolio is hedged. The value of this portfolio, π, is

$$\pi = V - \frac{\partial V}{\partial S} S \tag{5}$$

The change, d, in the value of this portfolio over a small time interval dt is given by

$$d\pi = dV = \frac{\partial V}{\partial s} dS \tag{6}$$

Now substituting equation (4) and (1) into equation (6) for dV and dS we get

$$d\pi = \sigma S dB \frac{\partial V}{\partial S} + \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt - \frac{\partial v}{\partial t} \left(\mu S dt + \sigma S dB \right) \tag{7}$$

This simplifies to

$$d\pi = \left(\frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{8}$$

It is important to note that this portfolio is completely riskless because it does not contain the random Brownian motion term. Since this portfolio contains no risk it must earn the same as other short-term risk-free securities. If it earned more than this, arbitrageurs could make a profit by shorting the Thee securities and using the proceeds to buy this portfolio. If the portfolio earned less arbitrageurs could make a risk-free profit by shorting the portfolio and buying the risk-free securities. It follows for a risk-frees portfolio that

$$d\pi = r\pi dt \tag{9}$$

Where r is the risk free interest rate. Substituting for dπ and π from equations (10) and (10) yields

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left(V - S \frac{\partial V}{\partial S} \right) dt \tag{11}$$

Further simplification yields the Black-Scholes differential equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{12}$$

3.3 BOUNDARY CONDITIONS FOR EUROPEAN OPTIONS

Having the payoff value for call and put options, it is also called the value maximum and minimum values for the call and put options. The minimum value of any option is zero because no option can be exercised less than zero.

The maximum value for a call is the current value of the underlying asset. While the maximum value for the put option is the present value of the exercised price. Therefore, the boundary condition for European call option with payoff at T is given as

$$c(S,T)=\max(S-K,0) \tag{13}$$

While the value of the European put with payoff at T is given as

$$p(S,T)=\max(K-S,0) \tag{14}$$

3.4 SOLUTION TO THE BLACK SCHOLES FORMULA

Here, we give the formulas for European calls and puts. We verify that the formulas we give are the solution of the Black Scholes equation.

Theorem 2: The value of the vanilla European call is given by

$$c(S,t) = SN(d_1) - K_e^{-(T-t)}N(d_2) \tag{15}$$

Where

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}S^2}\, ds \tag{16}$$

The cumulative distribution function for the standard normal distribution,

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{17}$$

Proof: We first check that c(S,t) in (15) really satisfies that Black Scholes equation (12). We first note that for $\omega = t$ or S, we have

$$\frac{\partial \mathrm{N}(\mathrm{d_i})}{\partial \omega} = \frac{\partial N(d_i)}{\partial d_i}\frac{\partial d_i}{\partial \omega} = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial d_i}\int_{-\infty}^{d_i} e - \frac{1S^2}{2}ds.\frac{\partial d_1}{\partial \omega} = \frac{e - \frac{d_i 2}{2}}{\sqrt{2\pi}}\frac{\partial d_i}{\partial \omega}$$

We can check that

$$\frac{\partial d_1}{\partial t} = \frac{d_1}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)$$

And

$$\frac{\partial d_2}{\partial t} = \frac{d_2}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)$$

Hence we have

$$\frac{\partial c}{\partial t} = S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - K_e^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial t}$$

$$= \frac{Se - d_i^2}{\sqrt{2\pi}}\left[\frac{d_1}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)\right] - rKe^{-r(T-t)}N(d_2)$$

$$- \frac{Ke^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left[\frac{d_2}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\right] \quad (18)$$

Also, since

$$\frac{\partial d_i}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}, i = 1,2.$$

We have

$$\frac{\partial c}{\partial S} = N(d_1) + S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - K_{e-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial s}$$

$$= N(d_1) + e\frac{-\frac{d_1^2}{2}}{\sqrt{2\pi}} - Ke^{-r(T-t)}e - \frac{d_2^2}{\sqrt{2\pi}}\frac{1}{S\sigma\sqrt{T-t}} \quad (19)$$

Differentiating it once more, we get

$$\frac{\partial^2 c}{\partial S^2} = \frac{e^{-\frac{d_1^2}{2}}}{S\sigma\sqrt{2\pi}\sqrt{T-t}} - \frac{d_1 e^{-d_1^2}}{S\sigma^2\sqrt{2\pi}(T-t)} + \frac{Ke^{-r}(T-t)e^{-\frac{d_2^2}{2}}}{S^2\sigma^2\sqrt{2\pi}(T-t)}$$

$$= \frac{2}{S^2\sigma^2}\left\{\frac{Se^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2(T-t)} - \frac{d_1}{2(T-t)}\right) + \frac{Ee^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}} + \frac{d_2}{2(T-t)}\right)\right\} \quad (20)$$

By substituting (15), (17) to (20) into the left hand side of the Black-Scholes equation (12), we see that it is indeed identically equal to zero.

For the boundary condition (15), we first note that by (17), d_1, d_2 as $S \to 0$. Obviously $N(-\infty) = 0$. Hence

$$c(0,t) = 0N(-\infty) - Ke^{-r(T-t)}N(-\infty) = 0$$

For the boundary condition (15), we note again that d_1, $d_2 \to \infty$ as $S \to \infty$ where $N(\infty)$ 1. Hence

$$c(S,t) \to Sn(\infty) - Ke^{-r(T-t)}N(\infty) \approx -Ke^{-r(T-t)}$$

As $S \to \infty$

Finally, we consider the final condition (15). At t = T, if S > K, then d_1, d_2 $\to \infty$.

Hence c(S, T) = S − K. If S < K, then d_1, $d_2 \to \infty$. Hence c(S, T) = 0. If S = K, by continuity, c(S, T) = 0.

THEOREM 3: The value of vanilla European put is given by

$$p(S,t) = Ke^{-r(T-t)}N(-d_1) - SN(-d_2) \qquad (21)$$

Where d_1 and d_2 are given in (17).

Proof: we can verify that the formula (21) does satisfy the Black-Scholes equation and the boundary and final conditions for European puts as we did in the proof of Theorem 1. However, there is a better way to verify that. We can derive (21) immediately by using the put-call parity formula.

3.5 CALL-PUT PARITY

Here will be dealing with the assumption of arbitrage-free pricing. Arbitrage clearly exists in the real world. One example is simply different prices for different assets in different markets. One might be able to buy something in one market and sell it for a higher price in a different market and make a profit. However, instances like these are usually eliminated quickly. People notice when there are arbitrage opportunities, and they pounce on them, thus adjusting prices so that these opportunities vanish.

Another instance of arbitrage can occur when dealing with two stocks. Suppose stock A and B are worth the same at time $t = 0$, but at time T. A is worth twice as much as before and B is still at its initial value. Then, one can create a portfolio that is long A and short B. Clearly, the portfolio has positive value at time I but zero value at time $t = 0$.

No-arbitrage means no free lunch; that a person can't make a riskless profit when he starts out with some portfolio with no value. Such an assumption makes things simpler and provides for a certain amount of necessary order in the world of financial mathematics. Without it, we would not be able to come up with a unique price for options.

Theorem 4 (Put-Call Parity): Let C(t) be the value of a European call option on an asset S with strike price K and expiration T. Let P(t) be the value of a European put option on the same asset S with the same strike price and expiration. Finally, let S have a final value at expiration of S_T and let B (t, T) represent the value of a risk-free bond at time t with final value 1 at expiration time T. If these assumptions hold and there is no arbitrage, then

$$C(t)+KB(t,T)=p(t)+S_c \tag{22}$$

Proof: Consider first a portfolio X that consists of one put option and one share of S. At time T, portfolio X has value

$$X_v=\begin{cases}K, & \text{if } S_t \leq K \text{ as the option will be worth } K-S_T \text{ and the share } S_T\\ S_T, & \text{if } S_T \leq K \text{ as the option will be worth } 0 \text{ and the share } S_T\end{cases}$$

Now consider a portfolio Y that consists of one call option and K bonds that pay 1 at time T with certainty. Then, at time T, portfolio Y has value

$$Y_v = \begin{cases} K, & \text{if } S_t \le K \text{ as the option will be worth } 0 \text{ and the bonds } K \\ S_T, & \text{if } S_T \le K \text{ as the option will be worth } S_T - K \text{ and the bonds } K \end{cases}$$

We can see that whatever value S takes at time T, portfolios X and Y have the same value. Thus, from Theorem 3, at any time t < T, the portfolios must also have the same value. It follows then that

$$C(t) + KB(t,T) = P(t) + S(t)$$

Now, it is straightforward to obtain the price of a European put option with strike price K and expiration time I. Recall that

$$C(S, t) = SN(\infty) - {}^{Ke\text{-}r(T\text{-}t)}N(\infty)$$

Also note that, if we assume the interest rate r is constant, which we have implicitly done in this chapter, $B(S, t) = e^{-rT}$. And so.

$$P(S,t) = C\,(S,t) - S_t + Ke^{-rT}$$

Hence,

$$P(S, t) = SN(\infty) - S + Ke^{-r(T-t)} - Ke^{-r(T-t)}N(\infty)$$

$$= -S(1 - N(\infty) + Ke^{-r(T-t)}(1 - N(\infty))$$

$$p(S,t) = Ke^{-r(T\_t)}N(\_d_1) - sN(-d_2) \qquad (32)$$

a.

CHAPTER FOUR

4.0 INTRODUCTION

In this chapter, I applied the Black Scholes model to explain the European call and put options. I also show how time and volatility could affect the value of an option with appropriate examples. Lastly, I also applied the Black Scholes Merton model to the concept default with appropriate examples. In this chapter, I considered default as bankruptcy.

4.1 APPLICATION OF BLACK SCHOLES MODEL TO EUROPEAN OPTIONS

In this section, I applied the black Scholes model to valuation of European call option on a stock.

In other to achieve this concept, let use the given inputs:

Stock price, S = \$10

Strike price, K = \$10

Volatility, d = 30% = 0.3

Riskless rate, r = 0.4% = 0.4

Time to maturity, T = 1 year

Recall from equation (24) the value for European call option is given by

$$c(S,t) = SN(d_1) - Ke^{(-rT)}N(d2)$$

Where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Using the inputs given above,

$$d_1 = \frac{\ln(1) + \left(0.04 + \frac{1}{2}(0.3)^2\right)(1)}{0.3\sqrt{1}}$$

$$= \frac{0.085}{0.3} = 0.28$$

$$d_2 = 0.28 - 0.3\sqrt{1}$$

$$= -0.02$$

From standard normal distribution table, we have;

$$N(d_1) = 0.6103$$

$$N(d_2) = 0.492$$

Therefore,

$c(S, t) = 10(0.6103) - 10 \times e^{(004)*1)}(0.492)$

$= 6.103 - 9.60789 \times 0.492$

$= 1.38$

Therefore the value of the European call in 1 year is 1.38

To calculate the value of the European put, we have that

$p(S, t) = Ke^{-rT}N(d_2) - SN(d_1)$

Where

- $d_2 = 002$

- $d_1 = 0.142$

From the standard normal distribution table we have

$N(-d_2) = 0.50798$

$N(-d_1) = 0.38974$

Therefore,

$p(S,t) = 10 \times e^{-004*(1)}(0.50798) - 10(0.38974)$

$= 9.6\ (0.50798) - 3.8974$

$= 0.98$

The value of the European put is given as $0.98

EXAMPLE 2

Let us the same inputs in the example above. But in this example, we choose our maturity, T i.e. 2 months. Let calculate the value of the option with maturity 3 months.

Solution:

Our inputs is given below as

Stock price, S = $10

Strike price, K = $10

Volatility, σ = 30% = 0.3

Riskless rate, r = 0.4% = 0.4

Time to maturity, T = 3 months = 3/12 = 0.25

Therefore,

$$c(S,t) = SN(d_1) - Ke^{(-rT)}N(d_2)$$

Where

$$d_1 = \frac{\ln(1) + \left(0.04 + \frac{0.3^2}{2}\right)(0.25)}{0.3\sqrt{0.25}}$$

$$= \frac{0.085 \times 0.25}{0.15}$$

$$= 0.142$$

And

$$d_2 = 0.142 - 0.3\sqrt{0.25}$$

$$= 0.142 - 0.15 = -0.008$$

From standard normal distribution function, we have;

$$N(d_1) = 0.5596$$

$$N(d_2) = 0.4681$$

Substituting all this values we have

$$c(S,t) = 10(0.5596) - 10 \times e^{-(0.04*0.25)}(0.4681)$$

$$= 5.596 - 9.9 \times 0.4681$$

$$= 0.962$$

Therefore, with the two examples given, we can now deduce that the longer the maturity dates of the European option the more valuable the options. Also, the higher the volatility the more valuable the option.

4.2 APPLICATION OF THE BLACK SCHOLES F(Ib4ULA TO EQUITY VALUATION

In this section, I considered equity as a call option. If the value of the firm equity is less than the value of the debt payment D the firm defaults. Let c be the value of the equity and S is the value of the firm assets.

The black Scholes Merton model frame work says that the current value of equity is

$$c = SN(d_1) - De^{-rT} N(d_2) \tag{33}$$

Whereby d_1 and d_2 have been defined

Since, we say that equity is the same as a call option on the firm's asset, the strike price equals the value of the firm liabilities.

Hence, equation (33) can now be written as

$$c(S, t) = SN(d_1) - Ke^{-rT}N(d_2)$$

where the debt payment of the firm is equal to the strike price of the option.

To understand this concept let us use an appropriate example for illustration.

EXAMPLE 3

Let assume that you have a firm whose assets are currently valued at \$100million and that the standard deviation in this asset value is 40%. Let us also assume that the face value of debt is \$80million. (It is zero coupon debt with 10 years left to maturity), if the 10years Treasury bond rate is 10% how much is the equity worth?

What should be the interest rate on debt be?

Solution

The parameters of equity as a call option are as follows;

Value of the underlying asset = value of the firm = S = \$100million

Exercise price = face value of outstanding debt = K = \$80million

Life of the option = life of zero—coupon debt = l0yrs

Variance in the value of the underlying asset = variance in the firm value $= \sigma^2 = 0.16 \rightarrow \sigma = 0.4$

Riskless rate = Treasury bond rate corresponding to option life = r = 10%

Therefore, based on these input, the Black-Scholes model gives;

Value of the call $= SN(d_1) - Ke^{rt}N(d_2)$

Where

$$d_1 = \frac{\ln\left(\frac{100}{80}\right) + \left(0.1 + \frac{0.16}{2}\right)(10)}{0.4\sqrt{10}}$$

$$= \frac{0.223 + 0.18(10)}{1.2649}$$

= 1.5993

Therefore,

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$= 1.5993 = 0.4\sqrt{10}$$

$$d_2 = 0.334$$

From standard normal distribution table we have;

$$N(d_1) = 0.9451$$

$$N(d_2) = 0.6310$$

Hence,

Value of call = 100(0.9451) – 80 × $e^{-(0.1)(10)}$ × (0.6310)

= 94.51 – 18.571

= 75.94

Value of outstanding debt = \$100 - \$75.94

= \$24.60milion

$$\text{Interest rate on debt} = \left(\frac{80}{24.06}\right)^{\frac{1}{2}} - 1$$

$$= 1.1277 - 1$$

$$= 0.1277$$

Interest rate on debt = 12.77%

4.3 VALUING OF EQUITY IN A TROUBLED FIRM

In this section, a trouble firm is a firm whose value falls, S, fell below the face value of the outstanding debt, K. the troubled firm can be considered as a default or a bankruptcy firm. To understand this concept we use an appropriate example for illustration.

EXAMPLE 4

Using the parameters of equity as a call option is as follows;

Value of the firm = S = \$50 million

Face value of outstanding debt = K \$80 million

Life of zero — coupon debt = 10 years

Variance in firm value = 0.16

Treasury bond rate corresponding to option life = 10%

<u>Solution</u>

The Black and Scholes model gives

$$c(s, t) = SN(d_1) - Ke^{-rt}N(d_2)$$

Where,

$$d_1 = \left(\ln\left(\frac{50}{50}\right) + \left(0.1 + \frac{0.16}{2}\right)(10) \right)$$

$$= \frac{-0.46 + 1.8}{1.265}$$

$$= 1.0515$$

Therefore,

$$d_2 = 1.0515 - 1.265$$

$$= -0.2135$$

From standard normal distribution table we have;

$$N(d_1) = 0.8534$$

$$N(d_2) = 0.4155$$

Hence,

Value of call = $50(0.8534) - 80\ e^{-(0.1x10)} \times (0.4155)$

$= \$30.44$ million

Value of the bond = \$50 - \$30.44

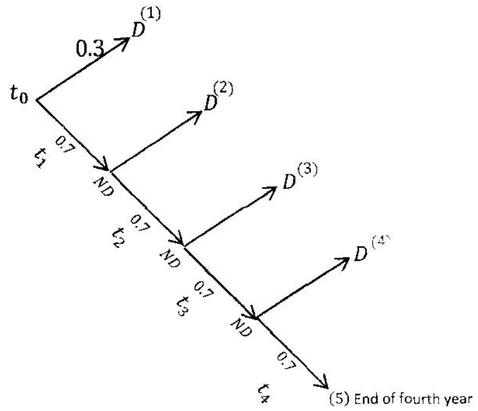
= \$19.56 million

Let us also consider another example. This example explains the probability of default we will use the Binomial tree formula instead of the Black Scholes formula so as to see a different way of solving default problem.

EXAMPLE 5

The probability that a firm will default in a year is 0.3. Assuming the probability of default remains the same for the subsequent years, compute the probability that the firm will be in default at the end of 4 years.

Solution



Hence

Probability of default = prob of [1 + 2 + 3 + 4]

$$= 0.3 + (0.7 \times 0.3) + (0.7)^2 \times 0.3 + (0.7)^3 \times 0.3$$

Therefore, from the theory of probability

Prob of D + prob of ND= 1

At node 5 the firm is in a healthy state, so to calculate the probability at node 5 gives us the probability of non-default.

Therefore,

$$P(ND) = 0.7 \times 0.7 \times 0.7 \times 0.7$$

$$= (0.7)^4$$

Hence,

P(D) = 1-P(ND) [by probability theory]

$= 1 - (0.7)^4$

$= 0.7599$

For n years we have;

$P(D) = 1 - [1 - P(D) \text{ in a year}]^n$

$= 1 - (1 - P_d)^n$

Therefore we have;

$P(D) = 1 - [1 - P(D \text{ in a year})]^4$

$= 1 - [1 - (0.3)]^4$

$= 0.7599$

CHAPTER FIVE

CONCLUSION

The important point to note about the derivation of the Black-Scholes differential equation is that we never specified a specific type of derivative security we were trying to and the price for until we set up boundary conditions for a European call. This means that a person can use the Black-Scholes differential equation to solve for the price of any type of option only by changing the boundary conditions.

The Black-Scholes model truly revolutionized the world of finance. For the first time the model has given traders, and investors have a standard way to value options. The model also has also caused a huge growth in the importance of financial engineering in the world of finance.

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