MARSHALL-OLKIN LOMAX-WEIBULL DISTRIBUTION WITH PROPERTIES AND APPLICATION

BY

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A PROJECT WORK SUBMITED TO THE DEPARTMENT OF STATISTICS, FACULTY OF PHYSICAL SCIENCES, UNIVERSITY OF BENIN, BENIN CITY, IN PARTIAL FULFILLMENT OF THE AWARD OF BACHELOR OF SCIENCE (B.Sc. HONOURS) DEGREE I N STATISTICS.

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CERTIFICATION

This is to certify that the project work was carried out by ISAIAH NOAH ONIMISI with Matriculation No. PSC2106398 in the Department of Statistics, Faculty of Physical Sciences, University of Benin, in partial fulfillment for the requirement for the award of the bachelor of Science (B.Sc.) Degree in Department of Statistics.

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EXTERNAL EXAMINER: ______________________________

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UNDERTAKING

This project was carried out by **ISAIAH NOAH ONIMISI** with the Matriculation number **PSC2106398**. I have not copied the work of any author(s). All texts used have been duly cited and acknowledged.

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ISAIAH NOAH ONIMISI DATE

DEDICATION

This project is dedicated to almighty God for his Grace, Mercy, strength and guidance and to my late grandmother Mrs. Jimoh Adisa and my parents Mr. and Mrs. ISAIAH for being extremely supportive during my studies.

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ABSTRACT

This study introduces a new distribution known as Marshall-Olkin Lomax-Weibull Distribution, a novel lifetime model that extends the flexibility of an existing Lomax-Weibull distribution in reliability analysis and survival studies.

The proposed distribution combines the Marshall-Olkin transformation with the Lomax-Weibull distributions resulting to an additional scale parameter added to the four parameter Lomax-Weibull Distribution, and enhancing its ability to model diverse hazard rate behaviours, including increasing, decreasing, bathtub, and upside-down bathtub shapes and monotonic and nonmonotonic failure data which could be obtained from complex systems used in diverse scientific fields.

Some statistical properties of the proposed lifetime distribution are considered. Parameter estimation of the Marshall-Olkin Lomax-Weibull distribution is obtained using maximum likelihood Estimation. Comparison with other traditional models, applicability and flexibility of the new distribution in lifetime analysis is illustrated with the aid of a real life example. The real world applications demonstrate the superiority of the Marshall-Olkin Lomax-Weibull distribution in fitting complex datasets compared to traditional models.

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Statistical distributions are essential tools for analyzing and modeling real-world data. Among these, the Weibull distribution is widely used because it can handle different types of data, especially in reliability and survival analysis. However, the Weibull distribution sometimes struggles to accurately describe data with unusual patterns, such as those with long tails or sudden changes. This limitation has encouraged researchers to create new, more flexible distributions.

Similarly, the Lomax distribution is known for its ability to handle data with heavy tails, making it useful in fields like economics and risk analysis.

The Marshall-Olkin family of distributions, introduced by Marshall and Olkin, is one way to improve existing distributions. By adding extra parameters, this family makes the distributions more adaptable to complex data patterns. To address these challenges, this study proposes the Marshall-Olkin Lomax-Weibull (MOLW) distribution, a new model that combines the strengths of the Marshall-Olkin, Lomax-Weibull distributions. This new distribution is designed to be more flexible and capable of fitting a wider range of data. It can model data with different shapes, patterns, and tail behaviors better than its individual components.

The MOLW distribution is analyzed in terms of its mathematical properties and is tested using real-world data to show its effectiveness. By developing this distribution, the study aims to provide researchers and practitioners with a more powerful tool for analyzing complex datasets.

1.2 Statement of the Problem

Many existing statistical distributions, such as the Weibull and Lomax distributions, are widely used for modeling data in fields like reliability, survival analysis, and economics. However, these models often struggle to handle datasets with complex patterns, such as extreme values, heavy tails, or sudden changes. This limitation makes it challenging to accurately describe and analyze real-world data.

Although the Marshall-Olkin family of distributions has introduced improvements by adding flexibility to existing models, there is still a need for a more comprehensive distribution that combines the strengths of multiple models. Specifically, a distribution that integrates the flexibility of the Marshall-Olkin family with the adaptability of the Weibull and Lomax distributions could better handle a wider variety of data patterns.

This study seeks to address this problem by proposing the Marshall-Olkin Lomax-Weibull (MOLW) distribution. The goal is to create a new statistical model that can overcome the limitations of current distributions and provide a better tool for analyzing complex datasets in various applications.

1.3 Aims and Objectives

Aim

To propose and analyze the Marshall-Olkin Lomax-Weibull (MOLW) distribution as a new statistical model that combines the strengths of the Marshall-Olkin, Lomax-Weibull distributions, providing a more flexible and accurate tool for modeling complex datasets.

Objectives:

1. Develop the MOLW Distribution: To define the probability density function, cumulative distribution function, and other mathematical properties of the Marshall-Olkin Lomax-Weibull distribution.
2. Analyze Properties: To study the theoretical characteristics of the MOLW distribution, such as moments, hazard rate, to demonstrate its flexibility and adaptability.
3. Model Real-World Data: To apply the MOLW distribution to real-world datasets and compare its performance with existing models.

1.4 Significance of the Study

This study is significant because it addresses the need for more flexible and accurate statistical models for analyzing real-world data. Existing distributions, like the Weibull and Lomax distributions, are widely used but have limitations when dealing with complex datasets that involve heavy tails, skewed patterns, or sudden changes. By proposing the

Marshall-Olkin Lomax-Weibull (MOLW) distribution, this study offers a new tool that combines the strengths of these distributions to overcome such challenges.

The proposed MOLW distribution is important for several reasons:

1. Improved Data Analysis: It provides a more adaptable model that can handle a wider variety of data patterns, improving the accuracy of data analysis in fields like reliability engineering, survival studies, and risk assessment.
2. Practical Applications: The MOLW distribution can be applied to real-world problems, such as predicting failure times, analyzing survival rates, and modeling extreme events, making it valuable for researchers and practitioners.
3. Advancement of Statistical Knowledge: This study contributes to the development of advanced statistical methods by introducing a novel distribution that expands the toolkit available for analyzing complex datasets.
4. Enhanced Decision-Making: With a better-fitting model, decision-makers in industries like healthcare, engineering, and finance can make more informed and reliable decisions based on accurate data analysis.

Overall, the study is a valuable contribution to statistics and its applications, offering both theoretical insights and practical benefits.

1.5 Organization of the Study

This study is organized into five main chapters, each focusing on different aspects of the development, analysis, and application of the Marshall-Olkin Lomax-Weibull (MOLW) distribution.

Chapter 1 (Introduction) chapter provides an overview of the study, including the background, statement of the problem, aims and objectives, significance of the study, organisation of the study and definition of terms. It sets the foundation for understanding the need for the proposed MOLW distribution.

Chapter 2 (Literature Review) reviews existing studies on statistical distributions, particularly the Weibull, Lomax, Lomax-Weibull and Marshall-Olkin families. It highlights their applications, strengths, and limitations, demonstrating the need for a more flexible hybrid model.

Chapter 3 (Methodology) chapter defines the proposed MOLW distribution, including its probability density function, cumulative distribution function, and theoretical properties. It explains how the distribution is constructed and its mathematical framework.

Chapter 4 (Applications and Data Analysis) applies the MOLW distribution to real-world datasets and compares its performance with existing models. The results are analyzed to show how the MOLW distribution provides a better fit for complex data patterns.

Chapter 5 (Conclusion and Recommendations) summarizes the findings of the study, discusses its contributions to statistical modeling, and provides recommendations for future research and practical applications of the MOLW distribution.

This structure ensures a clear and systematic exploration of the proposed distribution, from theoretical development to practical applications.

1.6 Definition of Terms

1. Marshall-Olkin Distribution: A family of statistical distributions that introduces additional flexibility to baseline distributions by incorporating extra parameters. It is often used to model systems with sudden changes or external shocks.

2. Lomax Distribution: Also known as the Pareto Type II distribution, this is a probability distribution commonly used in economics, reliability analysis, and risk management to model heavy-tailed data.

3. Weibull Distribution: A widely used statistical model in reliability and survival analysis. It is known for its flexibility in modeling failure times and life data with varying hazard rates.

5. Probability Density Function (PDF): A mathematical function that describes the likelihood of a random variable taking on specific values. For the MOLW distribution, the PDF is used to model how probabilities are distributed across different outcomes.

6. Cumulative Distribution Function (CDF): A mathematical function that shows the probability that a random variable will take a value less than or equal to a given value. The CDF provides insights into the overall distribution of data.

7. Hazard Rate Function: A measure of the likelihood of an event (e.g., failure) occurring at a particular time, given that it has not yet occurred. It is a key property in reliability and survival analysis.

8. Heavy-Tailed Data: Data with extreme values or outliers that occur more frequently than would be expected in a normal distribution. Heavy tails are common in fields like finance, economics, and risk analysis.

9. Flexibility: The ability of a statistical model to adapt to a variety of data patterns, including skewness, heavy tails, and varying rates of change.

10. Real-World Applications: Practical uses of statistical models in fields like engineering, healthcare, finance, and environmental studies to analyze and interpret complex datasets.

These terms provide a clear understanding of the key concepts and ideas related to the study on the Marshall-Olkin Lomax-Weibull distribution.

CHAPTER TWO
LITERATURE REVIEW

2.1 INTRODUCTION

This literature review examines the evolution of the Marshall-Olkin Lomax-Weibull distribution, highlights the theoretical and practical contributions made by recent studies. By investigating its applications and performance in comparison to other competing models, the review aims to provide a comprehensive understanding of the MOLW distribution's potential and the directions for future research.

2.2 STATISTICAL DISTRIBUTIONS OF LIFETIME MODELLING

✧ Lomax Distribution

Lomax (1954); The Lomax distribution, also known as the Pareto Type II distribution, is a continuous probability distribution commonly used in modeling heavy-tailed data. It is a generalization of the exponential distribution, with an additional shape parameter that allows for greater flexibility in modeling data with long tails.This distribution has found applications in various fields, including reliability analysis, survival analysis, business modeling, and actuarial science. The Lomax distribution is particularly suitable for modeling lifetime data and economic phenomena where extreme events or large values are of interest.

✧ Weibull Distribution

Gulati & Balakrishnan (2000); The Weibull distribution is one of the most versatile statistical models in reliability analysis and survival studies due to its ability to characterize a wide range of failure behaviors. The Weibull distribution is widely applied in fields such as engineering, hydrology, meteorology, and biomedical sciences due to its simplicity and adaptability.

In summary, the Weibull distribution remains a cornerstone in statistical modeling due to its versatility, ease of application, and ability to describe a variety of real-world phenomena. Its rich theoretical foundation and practical utility have made it an indispensable tool across diverse fields of study.

✧ **Lomax-Weibull Distribution**

Osagie and Osemwenkhae (2020); Lomax-Weibulldistribution is a four parameter life-time distribution obtained from mixing two classical distributions such as Lomax distribution and Weibull distribution using the method of constructing competing risk models. This distribution is designed to combine the flexible modeling capabilities of the Weibull distribution with the heavy-tail behavior of the Lomax (Pareto type II) distribution, making it especially useful in applications like survival analysis, reliability engineering, and finance, where both long-tailed behaviors and various failure rates.

✧ **Type I half Logistic Lomax Distribution**

Sule, (2024); The Type I Half Logistic Lomax Distribution is a recent development in statistical modeling that combines the flexibility of the Lomax distribution (also known as Pareto Type II) with the Half Logistic distribution. This novel distribution is particularly useful for modeling lifetime data that exhibit heavy tails and skewness, which are common in fields like reliability engineering, survival analysis, and finance.

It provides a flexible tool for capturing a wide range of complex hazard rate patterns. Its applications in survival analysis, reliability engineering, and financial modeling make it an important distribution for researchers and practitioners dealing with real-world data that exhibits extreme values or long-tail behaviors.

✧ **Extended Power Lomax Distribution**

Qura *et al* (2023); The Extended Power Lomax (EPL) distribution is a recent extension of the well-known Lomax distribution (also called Pareto Type II distribution). The Lomax distribution is commonly used to model lifetime data, particularly when the data exhibit heavy tails and skewness. However, the basic Lomax distribution may not be flexible enough to model all types of real-world phenomena, especially those that exhibit more complex failure patterns. The Extended Power Lomax distribution was introduced to address this limitation by providing a more flexible model with additional parameters to better capture varying tail behaviors and hazard rates

Its ability to accommodate varying hazard rates, skewness, and kurtosis makes it an essential tool for researchers and practitioners in fields such as reliability engineering, actuarial science, medical statistics, and finance. By introducing additional parameters, the EPL distribution ensures better fit and more accurate modeling of real-world data.

✧ Marshall-Olkin Distribution

Marshall and Olkin (1997) proposed a new method to establish more flexible family of distributions by adding a parameter to the baseline distributions. The Marshall-Olkin distributions are a family of probability distributions that generalize the classical exponential and weibull distributions. It is used in various fields such as reliability theory, survival analysis and queuing theory due to its ability to model dependence structures in systems where multiple components fail simultaneously.

✧ Marshall-Olkin Extended Power Function Distribution

Azad and Mohsin, (2020); The Marshall-Olkin Extended Power Function (MOEPF) distribution is an extension of the classic Marshall-Olkin distribution, which is widely used in reliability analysis and survival modeling. The traditional Marshall-Olkin model is employed to describe the dependence structure between random variables, particularly in systems where components may fail simultaneously due to a common external shock. The Marshall-Olkin Extended Power Function distribution introduces more flexibility by incorporating a power function into the Marshall-Olkin framework, allowing it to better model real-world data with complex dependence structures and heavy tails.

With its ability to capture varying hazard rates, skewness, kurtosis, and heavy tails, the MOEPF distribution offers a powerful tool for modeling failure times and survival data in a wide range of applications, including reliability engineering, survival analysis, and financial risk modeling.

2.3 REVIEW OF MODIFICATION AND GENERALIZATION OF MOL-W DISTRIBUTION

The Marshall-Olkin Lomax-Weibull (MOLW) distribution as a composite model that integrates the Marshall-Olkin mechanism with the Lomax-Weibull distributions, aiming to provide a flexible tool for modeling various types of data, particularly in reliability and survival analysis. Over time, researchers have proposed modifications and generalizations to enhance its applicability and flexibility.

The Marshall-Olkin Extended Weibull (MOEW) distribution

Hanan et al (2016); The Marshall-Olkin Extended Weibull (MOEW) distribution is an extension of the Weibull distribution, introduced to provide more flexibility for modeling real-world data in reliability and survival analysis. This extension introduces an additional parameter using the Marshall-Olkin mechanism, which allows for incorporating dependence structures or "shocks" into the model.

Marshall-Olkin Power Lomax Distribution:

Almalki and Nadarajah (2014); Introduced the Marshall-Olkin Power Lomax distribution, which combines the Marshall-Olkin scheme with the Lomax distribution. This model provides a heavy-tailed alternative to the exponential, Weibull, and gamma distributions, offering flexibility in modeling data with decreasing failure rates. Widely used in fields that require modeling of extreme values or heavy-tailed data, such as reliability engineering, actuarial science, and risk analysis.

The Marshall-Olkin Power Generalized Weibull (MOPGW) Distribution,

Elbatal et al. (2013) This is an advanced statistical model that combines the Marshall-Olkin mechanism with the Power Generalized Weibull (PGW) distribution. This extension enhances the flexibility of the PGW distribution by adding parameters that improve its ability to model diverse data patterns in reliability, survival analysis, and other fields. Provides greater flexibility and adaptability compared to traditional Weibull-based models. Captures a wide range of data behaviors due to its ability to model different hazard rate shapes.

Marshall-Olkin Weibull–Burr XII (MOWB-XII) distribution

Najwan et al (2023); This is a five-parameter lifetime distribution that combines the Weibull–Burr XII model with the Marshall–Olkin-G family. This distribution is particularly useful in reliability and lifespan statistics due to its flexibility in modeling various types of data. This new lifetime distribution is designed to model complex data structures, particularly in reliability studies.

Marshall-Olkin Lehmann Lomax Distribution:

Alzaghal and Ghosh; (2020). proposed a four-parameter lifetime distribution called the Marshall-Olkin Lehmann Lomax distribution. This model extends the Lomax distribution by incorporating the Marshall-Olkin mechanism and the Lehmann alternative, providing greater flexibility in modeling lifetime data.

2.4 METHODS OF PARAMETER ESTIMATION

Parameter estimation is a critical task in statistics, aiming to infer the values of unknown parameters of a population distribution or model based on sample data. Various methods exist for parameter estimation, each with unique characteristics, advantages, and drawbacks. Below, we explore some of the most widely used parameter estimation techniques and the reasons why Maximum Likelihood Estimation (MLE) is often preferred over others.

1. Method of Moments (MoM)

Arnold, (1986); The method of moments involves equating sample moments (such as the sample mean, variance, etc.) to the corresponding theoretical moments of a distribution and solving for the parameters. MoM estimators are generally less efficient than MLE estimators, particularly when the sample size is small.

2. Least Squares Estimation (LSE)

(Seber and Lee, 2012); LSE is typically used in regression analysis. It minimizes the sum of the squared differences between the observed data and the model's predicted values. LSE is highly sensitive to outliers, which can skew the estimates significantly. LSE assumes that the errors are normally distributed and homoscedastic (constant variance), which may not always hold in real data.

3. Bayesian Estimation

Carlin & Louis, (2000); Bayesian estimation treats parameters as random variables with prior distributions. The parameters are estimated by updating the prior distribution using observed data to form a posterior distribution. The parameter estimate is typically the mean or mode of the posterior distribution. Bayesian estimation can be computationally intensive, especially for high-dimensional or complex models, and often requires numerical methods like Markov Chain Monte Carlo (MCMC). The choice of prior can significantly influence the results, potentially introducing subjectivity into the analysis.

4. Maximum Likelihood Estimation (MLE)

Cox and Hinkley, (1974); MLE is a popular statistical method for estimating the parameters of a probabilistic model. MLE is grounded in the principle of likelihood, which is the probability of observing the data as a function of the model parameters. The method involves maximizing the likelihood function with respect to the parameters. MLE is widely used due to its desirable properties, such as asymptotic unbiasedness and efficiency under regular conditions.

(McCullagh & Nelder, 1989). MLE's flexibility makes it applicable in various statistical models, including linear regression, survival analysis, and more complex models like generalized linear models.

Preference of MLE over other methods of Estimation

- ✓ **Properties of MLE**

Several properties make MLE a widely used estimation method:

I. Asymptotic Unbiasedness:

(Fisher, 1921); Under regular conditions, MLE estimators are asymptotically unbiased, meaning they approach the true parameter value as the sample size grows

II. Efficiency:

(Lehmann & Casella, 1998); MLE estimators are efficient in large samples, meaning they achieve the Cramer-Rao lower bound, which ensures minimum variance among unbiased estimators.

III. Consistency:

(van der Vaart, 1998); MLE estimators are consistent, meaning they converge in probability to the true parameter values as the sample size increases .

- ✓ **Applications of MLE**

MLE has widespread applications in numerous fields such as:

I. **Econometrics:** Estimating parameters of economic models, such as the parameters in a Cobb-Douglas production function (Hendry, 1995).

II. **Machine Learning**: MLE is used in training probabilistic models like Gaussian mixture models (Bishop, 2006).

III. **Survival Analysis:** In medical statistics, MLE estimates the parameters of survival models like the Weibull or Cox proportional hazards model (Kalbfleisch & Prentice, 2002).

MLE remains a cornerstone of statistical inference due to its efficiency, consistency, and asymptotic properties. However, challenges such as computational complexity and issues with small samples necessitate consideration of alternative methods in some cases.

CHAPTER THREE

METHODOLOGY

3.1 INTRODUCTION

This chapter emphasize on the method of transformation by adding an extra parameter to a four parameter Lomax-Weibull distribution so as to propose a new distribution with five parameters. The method entails the application of Marshal-Olkin family of distribution to add extra parameter to the Lomax-Weibull distribution.

3.2 DEVELOPMENT OF PROPOSED DISTRIBUTION

This section entails the construction of the proposed distribution called Marshall-Olkin Lomax-Weibull distribution using the Marshall-Olkin family of distribution.

Osagie and Osenwenkhae (2020) constructed the density function of the Marshall-Olkin Lomax-Weibull distribution which is defined by

$$f_{LW}(x) = [\theta\beta + \varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{-(\theta+1)}e^{-\varphi x^{\lambda}};$$

$x>0,\ \theta>0,\ \beta>0,\ \varphi>0,\ \lambda>0$ -- *(3.1)*

and the corresponding cumulative distribution function obtained a

$$F_{LW}(x) = 1-(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}};$$

$x>0,\ \theta>0,\ \beta>0,\ \varphi>0,\ \lambda>0$ --- *(3.2)*

Marshall and Olkin (1997) defined the density function of the Marshall-Olkin family of distribution as

$$g(x,\alpha) = \frac{\alpha f(x)}{\left[1-\bar{\alpha}\bar{F}(x)\right]^2} \qquad -\infty < x < \infty, 0 < \alpha < \infty$$ -------------- *(3.3)*

Where, the baseline survival function is given as

$$\bar{F}(x) = 1 - F(x)$$ --- *(3.4)*

And the corresponding survival function of the Marshall Olkin family of distribution is defined as

$$\bar{G}(x) = \frac{\propto \bar{F}(x)}{1-\bar{\propto}\bar{F}(x)} \quad \text{---} \; (3.5)$$

Where,

f(x) and F(x) are respectively the density function and cumulative distribution of the baseline distribution and

$$\bar{\alpha} = 1-\alpha \quad \text{---} \; (3.6)$$

Inserting (3.1) and (3.2) into (3.3) and (3.5), the density function of Marshall-Olkin Lomax-Weibull (MOL-W) distribution is defined as

$$f_{MOL-W}(x) = \frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{-(\theta+1)}e^{-\varphi x^{\lambda}}}{\left[1-\bar{\propto}(1-\left[1-(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}\right])\right]^{2}};$$

$$f_{MOL-W}(x) = \frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{-(\theta+1)}e^{-\varphi x^{\lambda}}}{\left[1-\bar{\propto}(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}\right]^{2}};$$

$x>0, \propto > 0, \theta>0, \beta>0, \varphi>0, \lambda>0$ -- (3.7)

and the corresponding survival function is defined as

$$\bar{F}_{MOL-W}(x) = \frac{\propto[1-(1-(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}})]}{\propto+\bar{\propto}(1-(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}})};$$

$$\bar{F}_{MOL-W}(x) = \frac{\propto(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}}{1-\bar{\propto}(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}};$$

$x>0, \propto > 0, \theta>0, \beta>0, \varphi>0, \lambda>0$ --- (3.8)

Applying (3.4) the MOL-W distribution has its cumulative distribution given as

$$F_{MOL-W}(x) = 1 - \bar{F}_{MOL-W}(x) \quad \text{---} \; (3.9)$$

Inserting (3.8) into (3.9) the cumulative distribution of the MOL-W distribution is

$$F_{MOL-W}(x) = 1 - \left[\frac{\propto (1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}}{1-\bar{\propto}(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}}\right]$$

$$F_{MOL-W}(x) = \frac{1-(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}{1-\bar{\propto}(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}} \; ;$$

x>0, $\propto > 0$*, θ>0, β>0, φ>0, λ>0* --- *(3.10)*

3.3 SERIES REPRESENTATION OF THE DENSITY FUNCTION

The density function defined in (3.7) can further be express in a series representation using the generalized binomial expansion for any positive real number |s|<1 reported by George and Thobias (2017) as

$$(1-s)^{-n} = \sum_{l=0}^{\infty} \binom{n+l-1}{l} s^{l}$$

And

$$(1+s)^{-n} = \sum_{l=0}^{\infty} \binom{n+l-1}{l} (-1)^{l} s^{l}$$

Thus, we have

$$(1+\beta x)^{-(\theta+1)} e^{-\varphi x^{\lambda}} \left[1 - \bar{\propto}\, (1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}\right]^{-2}$$

$$= (1+\beta x)^{-(\theta+1)} e^{-\varphi x^{\lambda}} \sum_{i=0}^{\infty} \binom{i+1}{i} \bar{\propto}^{i} \, (1+\beta x)^{-\theta i} e^{-\varphi x^{\lambda} i}$$

$$= \sum_{j=0}^{\infty} \binom{i+1}{i} \bar{\propto}^{i} \, (1+\beta x)^{-(\theta i+\theta+1)} e^{-\varphi x^{\lambda}(i+1)}$$ ------------------------- *(3.11)*

Now,

$$(1+\beta x)^{-(\theta i+\theta+1)} = \sum_{j=0}^{\infty} \binom{\theta i+\theta+j}{j} (-1)^{j} \beta^{j} x^{j}$$ ---------------------- *(3.12)*

Insering *(3.12)* into *(3.11)*, thus *(3.11)* becomes

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+1}{i}\binom{\theta i+\theta+j}{j}\overline{\propto}^{i}\,(-1)^{j}\beta^{j}x^{j}e^{-\varphi x^{\lambda}(i+1)}$$ - ----------------------- *(3.13)*

Making $\Omega = \binom{i+1}{i}\binom{\theta i+\theta+j}{j}\overline{\propto}^{i}\,(-1)^{j}\beta^{j}$, ------------------------------------ *(3.14)*

Inserting *(3.14)* into *(3.13)*, thus *(3.13)* becomes

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega x^{j}e^{-\varphi x^{\lambda}(i+1)}$$ ------------------------------------ *(3.15)*

Inserting *(3.15) into (3.7), the density function of MOL-W distribution is given as*

$$f_{MOL-W}(x) = \propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)]\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega x^{j}e^{-\varphi x^{\lambda}(i+1)}$$ ----------------- *(3.16)*

Making $\mu = \varphi(i+1)$

Inserting *(3.17)* into *(3.16)* thus we have,

$$f_{MOL-W}(x) = \propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)]\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega x^{j}e^{-\mu x^{\lambda}}$$ ------------------- *(3.17)*

Now, expanding *(3.17)*

$$f_{MOL-W}(x) = \propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)]\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega x^{j}e^{-\mu x^{\lambda}}$$

Is now

$$f_{MOL-W}(x) = \propto\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega x^{j}e^{-\mu x^{\lambda}} + \propto\varphi\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega x^{j+\lambda-1}e^{-\mu x^{\lambda}} + \propto$$

$$\beta\varphi\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega x^{j+\lambda}e^{-\mu x^{\lambda}}$$ -- *(3.18)*

3.4 STATISTICAL PROPERTIES OF THE MOL-W DISTRIBUTION

This section explains the various properties of MOL-W distribution with their possible plots and interpretations.

3.4.1 Properties of the MOL-W distribution density function and Cumulative function.

It is known that the properties of a valid continuous density function f(x) as reported by Zafar and Anjum (2018) are given as

(i) $\mathbf{f(x) > 0, for \ x > 0}$

(ii) $\int_{-\infty}^{\infty} \boldsymbol{f(x)dx = 1 \ for \ -\infty < x < \infty}$

(i) Proving that $\boldsymbol{f_{MOL-W}(x) > 0}, \mathbf{for \ x > 0}$

From *(3.7)* we have

$$f_{MOL-W}(x) = \frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{-(\theta+1)}e^{-\varphi x^{\lambda}}}{\left[1-\overline{\propto}(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}\right]^2}$$

for $x > 0, \propto [\theta\beta + \varphi\lambda x^{\lambda-1}(1 + \beta x)](1 + \beta x)^{(\theta+1)} e^{\varphi x^{\lambda}} > 0$

and

$$\left[(1 + \beta x)^{\theta} e^{\varphi x^{\lambda}} - \overline{\propto}\right]^2 > 0$$

Hence,

$$\frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{(\theta+1)}e^{\varphi x^{\lambda}}}{\left[(1+\beta x)^{\theta}e^{\varphi x^{\lambda}}-\overline{\propto}\right]^2} > 0 \quad \text{---------------------------------- } (3.19)$$

Divide *(3.19)* through by $(1 + \beta x)^{2\theta} e^{2\varphi x^{\lambda}}$

thus we have *(3.19)* as

$$\frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{-(\theta+1)}e^{-\varphi x^{\lambda}}}{\left[1-\overline{\propto}(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}\right]^2} > 0$$

It follows that for $f_{MOL-W}(x) > 0$, for $x > 0$ ---------------------------------- *(3.20)*

(ii) Prooving that $\int_0^{\infty} \boldsymbol{f_{MOL-W}(x)dx = 1}$,

from *(3.7)*, we have

$$\int_0^{\infty} \frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{-(\theta+1)}e^{-\varphi x^{\lambda}}}{\left[1-\overline{\propto}(1+\beta x)^{-\theta}e^{-\varphi x^{\lambda}}\right]^2} dx$$

Multiply through by $(1 + \beta x)^{2\theta} e^{2\varphi x^{\lambda}}$, We have

$$\int_0^\infty \frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{\theta-1}e^{\varphi x^\lambda}}{\left[(1+\beta x)^\theta e^{\varphi x^\lambda}-\overline{\propto}\right]^2}dx \quad \text{-------------------------------------- (3.21)}$$

Let $y = (1+\beta x)^{2\theta}e^{2\varphi x^\lambda}$ -------------------------------------- *(3.22)*

Inserting *(3.22)* into *(3.21),*

$$\int_1^\infty \frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{\theta-1}e^{\varphi x^\lambda}}{[y-\overline{\propto}]^2}dy \quad \text{------------------------------------- (3.23)}$$

Differentiating *(3.22)* the result is given as

$$dx = \frac{dy}{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{\theta-1}e^{\varphi x^\lambda}} \quad \text{------------------------------------- (3.24)}$$

For *(3.22),* when $x \to \infty, y \to \infty$ and when $x \to 0, y \to 1$ ----------------------- *(3.25)*

Inserting *(3.24)* and *(3.25)* into *(3.23),* the result is

$$\int_1^\infty \frac{\alpha}{[y-\overline{\propto}]^2}dy$$

We have

$= \lim_{y\to\infty}\frac{-\alpha}{y-\overline{\propto}}$ - $\lim_{y\to 1}\frac{-\alpha}{y-\overline{\propto}}$

= 1

Hence $\int_0^\infty f_{MOL-W}(x)dx = 1,$ --------------------------------------- *(3.26)*

Therefore from *(3.20)* and *(3.26),* $f_{MOL-W}(x)$density function of MOL-W distribution

is a valid density function for x>0.

Zafar and Anjum (2018) reported that a valid continuous cumulative distribution

F(x) has the following properties

(i) $\mathbf{F(0) = 0}$

(ii) $\mathbf{F(\infty) = 1}$

(I) Proving that $\boldsymbol{F_{MOL-W}(0) = 0}$

From (3.10)

$$F_{MOL-W}(x) = \frac{1-(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}{1 - \bar{\propto}\ (1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}$$

Inserting $x = 0$

$$F_{MOL-W}(0) = \frac{1-(1+\beta(0))^{-\theta} e^{-\varphi(0)^{\lambda}}}{1 - \bar{\propto}\ (1+\beta(0))^{-\theta} e^{-\varphi(0)^{\lambda}}}$$

Where $1-(1+\beta(0))^{-\theta} e^{-\varphi(0)^{\lambda}} = 0$

$F_{MOL-W}(0) = 0$ -------------------------------- *(3.27)*

(II) Proving that $F_{MOL-W}(\infty) = 1$

From (3.10)

$$F_{MOL-W}(x) = \frac{1-(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}{1 - \bar{\propto}\ (1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}$$

We have $F_{MOL-W}(\infty) = \lim_{X\to\infty} \frac{1-(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}{1-\bar{\propto}(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}$

Where $\lim_{X\to\infty} e^{-\varphi x^{\lambda}} = 0$,

$F_{MOL-W}(\infty) = 1$

It follows that $F_{MOL-W}(\infty) = 1$, -------------------------------- *(3.28)*

Hence from *(3.27)* and *(3.28)* it implies that $F_{MOL-W}(x)$ is a valid continuous cumulative distribution.

3.4.2 Survival Function of MOL-W Distribution

Kleinbaum and klein (2012) states that the survival function also known as survivor function nor reliability function is a fundamental concept which represents the probability that a subject or system will survive beyond a certain time x. statistically, it is defined as

$s(x) = P(X > x)$

$s(x) = 1 - P(X \leq x)$

$s(x) = 1 - F(x)$ --- *(3.29)*

It follows that

$$s_{MOL-W}(x) = 1 - \; F_{MOL-W}(x) \qquad \text{-------------------------- -------------------} \quad (3.30)$$

Inserting *(3.10)* into *(3.29)* this will result into

$$s_{MOL-W}(x) = \frac{\propto(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}{1-\bar{\propto}(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}} \qquad \text{-------------------------- -------------------} \quad (3.31)$$

3.4.3 Hazard Rate Function Of The MOL-W Distribution

Klein And Moschberger (2003) stated that the hazard rate function also known as hazard function or failure rate describes the instantaneous failure rate of a system at a specific time x. statistically, the hazard function is defined as

$$\mathrm{h}(x) = \frac{f(x)}{s(x)}$$

Which follows that

$$h_{MOL-W}(x) = \frac{f_{MOL-W}(x)}{s_{MOL-W}(x)} \qquad \text{------------------------- -------------------} \quad (3.32)$$

Inserting *(3.7)* and *(3.30)* into *(3.31),*

$$h_{MOL-W}(x) = \frac{\propto[\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{-(\theta+1)} e^{-\varphi x^{\lambda}}}{\left[1-\bar{\propto}(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}\right]^{2}} \div \frac{\propto(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}{1-\bar{\propto}(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}}$$

Thus,

$$h_{MOL-W}(x) = \frac{\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)}{\left[1-\bar{\propto}(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}\right](1+\beta x)} \qquad \text{------------------------- -------------} \quad (3.33)$$

3.4.4 Quantile function and median function of MOL-W Distribution

If q∈(0,1), then the quantile function of the MOL-W distribution can be derived from the solution

$$F_{MOL-W}(x_q) = q \qquad \text{------------------------- --------------------------} \quad (3.34)$$

Inserting *(3.10)* into *(3.32),*

$$\frac{1-(1+\beta x_q)^{-\theta} e^{-\varphi x_q{}^{\lambda}}}{1-\bar{\propto}\,(1+\beta x_q)^{-\theta} e^{-\varphi x_q{}^{\lambda}}} = q$$

Simplifying, it follows that,

$$\theta log(1+\beta x_q)+\varphi {x_q}^{\lambda} = -log(1-\overline{\propto}\, q)+log(1-q)$$

$$\theta log(1+\beta x_q)+\varphi {x_q}^{\lambda} + log(1-\overline{\propto}\, q)-log(1-q)=0 \quad \text{---------------} \quad (3.35)$$

Inserting $q = 0.5$ into *(3.34)* the median values are obtained for some randomly selected parameter sets.

Thus,

$$\theta log(1+\beta x_{med})+\varphi {x_{med}}^{\lambda} + log(1-0.5\,\overline{\propto})-log(0.5)=0$$

3.4.5 Moments and Mean of MOL-W Distribution

(i) Raw moment of MOL-W Distribution

Let x be a continuous random variable with density function f(X) then the r^th^ moment about the origin of x is defined by

$$\mu_r^1 = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx \quad \text{--} \quad (3.36)$$

Applying the series representation of the density function in (3. 18), the r^th^ moment about the origin of the MOL-W distribution is obtained as

$$\mu_r^1 = \int_0^{\infty} x^r \left(\propto \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega x^{j} e^{-\mu x^{\lambda}} + \propto \varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega x^{j+\lambda-1} e^{-\mu x^{\lambda}} + \propto \beta\varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega x^{j+\lambda} e^{-\mu x^{\lambda}} \right) dx$$

$$\mu_r^1 = \propto \theta\beta \int_0^{\infty} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega x^{r+j} e^{-\mu x^{\lambda}}\, dx + \propto \varphi\lambda \int_0^{\infty} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega x^{r+j+\lambda-1} e^{-\mu x^{\lambda}}\, dx + \propto \beta\varphi\lambda \int_0^{\infty} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega x^{r+j+\lambda} e^{-\mu x^{\lambda}}\, dx$$

$$\mu_r^1 = \propto \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \int_0^{\infty} x^{r+j} e^{-\mu x^{\lambda}} dx + \propto \varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \int_0^{\infty} x^{r+j+\lambda-1} e^{-\mu x^{\lambda}} dx +$$

$$\propto \beta\varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \int_0^{\infty} x^{r+j+\lambda} e^{-\mu x^{\lambda}} dx \qquad \text{-- } (3.37)$$

From Osagie and Osemwenkhae (2020), it is reported that

$$\int_0^{\infty} x^a e^{-bx^c} dx = \frac{\Gamma(a+1)/c}{cb^{(a+1)/c}}$$

Thus,

$$\int_0^{\infty} x^{r+j} e^{-\mu x^{\lambda}} dx = \frac{\Gamma(r+j+1)/\lambda}{\lambda\mu^{(r+j+1)/\lambda}} \qquad \text{-- } (3.38)$$

$$\int_0^{\infty} x^{r+j+\lambda-1} e^{-\mu x^{\lambda}} dx = \frac{\Gamma(r+j+\lambda)/\lambda}{\lambda\mu^{(r+j+\lambda)/\lambda}} \qquad \text{-- } (3.39)$$

$$\int_0^{\infty} x^{r+j+\lambda} e^{-\mu x^{\lambda}} dx = \frac{\Gamma(r+j+\lambda+1)/\lambda}{\lambda\mu^{(r+j+\lambda+1)/\lambda}} \qquad \text{-- } (3.40)$$

Substituting *(3.37)*, *(3.38)* and *(3.39)* into *(3.36)*

$$\mu_r^1 = \propto \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma(r+j+1)/\lambda}{\lambda\mu^{(r+j+1)/\lambda}} + \propto \varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma(r+j+\lambda)/\lambda}{\lambda\mu^{(r+j+\lambda)/\lambda}} +$$

$$\propto \beta\varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma(r+j+\lambda+1)/\lambda}{\lambda\mu^{(r+j+\lambda+1)/\lambda}} \qquad \text{-- } (3.41)$$

The first four r^th^ moments of the MOL-W distribution around the origin of x in terms of infinite series are obtained from *(3.40)* as

$$\mu_1^1 = \propto \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma(j+2)/\lambda}{\lambda\mu^{(j+2)/\lambda}} + \propto \varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma(j+\lambda+1)/\lambda}{\lambda\mu^{(j+\lambda+1)/\lambda}} +$$

$$\propto \beta\varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma(j+\lambda+2)/\lambda}{\lambda\mu^{(j+\lambda+2)/\lambda}} \qquad \text{-- } (3.42)$$

$$\mu_2^1 = \propto \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma(j+3)/\lambda}{\lambda\mu^{(j+3)/\lambda}} + \propto \varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma(j+\lambda+2)/\lambda}{\lambda\mu^{(j+\lambda+2)/\lambda}} +$$

$$\propto \beta\varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+\lambda+3)/\lambda}{\lambda\mu^{(j+\lambda+3)/\lambda}} \quad \text{---} \quad (3.43)$$

$$\mu_3^1 = \propto \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+4)/\lambda}{\lambda\mu^{(j+4)/\lambda}} + \propto \varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+\lambda+3)/\lambda}{\lambda\mu^{(j+\lambda+3)/\lambda}} +$$

$$\propto \beta\varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+\lambda+4)/\lambda}{\lambda\mu^{(j+\lambda+4)/\lambda}} \quad \text{---} \quad (3.44)$$

$$\mu_4^1 = \propto \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+5)/\lambda}{\lambda\mu^{(j+5)/\lambda}} + \propto \varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+\lambda+4)/\lambda}{\lambda\mu^{(j+\lambda+4)/\lambda}} +$$

$$\propto \beta\varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+\lambda+5)/\lambda}{\lambda\mu^{(j+\lambda+5)/\lambda}} \quad \text{---} \quad (3.45)$$

(iii) *Mean of the MOL-W distribution*

The mean of the MOL-W distribution can be obtained from the first moment about the origin of x which is given as in (3.41).

$$\text{Mean} = \propto \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+2)/\lambda}{\lambda\mu^{(j+2)/\lambda}} + \propto \varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+\lambda+1)/\lambda}{\lambda\mu^{(j+\lambda+1)/\lambda}} +$$

$$\propto \beta\varphi\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega \frac{\Gamma[(j+\lambda+2)/\lambda}{\lambda\mu^{(j+\lambda+2)/\lambda}} \quad \text{---} \quad (3.46)$$

3.4.6 Variance, Skewness and kurtosis of the MOL-W distribution

The variance (σ^2), measure of Skewness (S_k) and kurtosis(K_s) of the MOL-W distribution can be derived by substituting the values of the rth moment into the expression

variance (σ^2) $= \mu_2^1 - (\mu_1^1)^2$

Skewness $(S_k) = \frac{\mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3}{(\mu_2^1 - (\mu_1^1)^2)^{3/2}}$

Kurtosis (K_s) = $= \frac{\mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_2^1(\mu_1^1)^2 - 3(\mu_1^1)^4}{(\mu_2^1 - (\mu_1^1)^2)^2}$

3.5 MAXIMUM LIKELIHOOD ESTIMATION OF MOL-W DISTRIBUTION

In this section, we present the maximum likelihood estimate (MLE) of the parameters of MOL-W distribution.

Let $X_1, X_2, \ldots, X_n$ be random samples from the MOL-W distribution with density function defined in (3.7), the log likelihood function is given by

$L(x, \emptyset) = \sum_{i=1}^{n} \ln[f(x)]$ where $\emptyset = (\alpha, \theta, \beta, \varphi, \lambda)$ ------------------------------------ *(3.47)*

Inserting *(3.7)* into *(3.47)*

$$L(x, \emptyset) = \sum_{i=1}^{n} \ln\left[\frac{\propto [\theta\beta + \varphi\lambda x^{\lambda-1}(1+\beta x)](1+\beta x)^{-(\theta+1)} e^{-\varphi x^{\lambda}}}{\left[1 - \bar{\propto}\,(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}\right]^2}\right]$$

Thus

$$L(x, \emptyset) = n\ln \propto + \sum_{i=1}^{n} ln(\theta\beta + \varphi\lambda {x_i}^{\lambda-1}(1+\beta x_i)) - (\theta+1)\sum_{i=1}^{n} ln(1+\beta x_i) - \sum_{i=1}^{n} \varphi x_i - 2\sum_{i=1}^{n} ln\left[1 - \bar{\propto}\,(1+\beta x)^{-\theta} e^{-\varphi x^{\lambda}}\right] \quad \text{-- } (3.48)$$

The first partial derivatives of the log-likelihood function with respect to the different parameters are given as

$$\frac{dL(x,\emptyset)}{d\propto} = \frac{n}{\propto} + \sum_{i=1}^{n} \frac{1}{(1+\beta x)^{\theta} e^{\varphi x^{\lambda}} - 1 + \propto} \quad \text{------------------------------------- } (3.49)$$

$$\frac{dL(x,\emptyset)}{d\beta} = \sum_{i=1}^{n} \frac{\theta + \varphi\lambda x^{\lambda}}{\theta\beta + \varphi\lambda x^{\lambda-1} + \beta\varphi\lambda x^{\lambda}} - (\theta+1)\sum_{i=1}^{n} \frac{x}{1+\beta x} - 2\sum_{i=1}^{n} \frac{\theta x - \propto\theta x}{(1+\beta x)^{\theta+1} e^{\varphi x^{\lambda}} - (1-\propto)(1+\beta x)}$$

------------------------------------ *(3.50)*

$$\frac{dL(x,\emptyset)}{d\theta} = \sum_{i=1}^{n} \frac{\beta}{\theta\beta + \varphi\lambda x^{\lambda-1}(1+\beta x)} - \sum_{i=1}^{n} ln(1+\beta x) - 2\sum_{i=1}^{n} \frac{(1-\propto) ln(1+\beta x)}{(1+\beta x)^{\theta} e^{\varphi x^{\lambda}} - 1 + \propto}$$

------------------------------------ *(3.51)*

$$\frac{dL(x,\emptyset)}{d\varphi} = \sum_{i=1}^{n} \frac{\lambda x^{\lambda-1} + \beta\lambda x^{\lambda}}{\theta\beta + \varphi\lambda x^{\lambda-1} + \beta\varphi\lambda x^{\lambda}} - \sum_{i=1}^{n} {x_i}^{\lambda} + 2\sum_{i=1}^{n} \frac{{x_i}^{\lambda}(1-\propto)}{(1+\beta x)^{\theta} e^{\varphi x^{\lambda}} + \propto - 1}$$

------------------------------------ *(3.52)*

$$\frac{dL(x,\emptyset)}{d\lambda} = \sum_{i=1}^{n} \frac{\varphi x^{\lambda-1}(1+\beta x) + \varphi\lambda x^{\lambda-1}(ln x)(1+\beta x)}{\theta\beta + \varphi\lambda x^{\lambda-1}(1+\beta x)} - \sum_{i=1}^{n} \varphi {x_i}^{\lambda} ln x - 2\sum_{i=1}^{n} \frac{\varphi {x_i}^{\lambda}(1-\propto) ln x}{(1+\beta x)^{\theta} e^{\varphi x^{\lambda}} + \propto - 1}$$

------------------------------------ *(3.53)*

Now putting $\frac{dL(x,\emptyset)}{d\propto}=0$, $\frac{dL(x,\emptyset)}{d\beta}=0$, $\frac{dL(x,\emptyset)}{d\theta}=0$, $\frac{dL(x,\emptyset)}{d\varphi}=0$, $\frac{dL(x,\emptyset)}{d\lambda}=0$

$$\frac{n}{\propto}+\sum_{i=1}^{n}\frac{1}{(1+\beta x)^{\theta}e^{\varphi x^{\lambda}}-1+\propto}=0$$ ------------------------------------ *(3.54)*

$$\sum_{i=1}^{n}\frac{\theta+\varphi\lambda x^{\lambda}}{\theta\beta+\varphi\lambda x^{\lambda-1}+\beta\varphi\lambda x^{\lambda}}-(\theta+1)\sum_{i=1}^{n}\frac{x}{1+\beta x}-2\sum_{i=1}^{n}\frac{\theta x-\propto\theta x}{(1+\beta x)^{\theta+1}e^{\varphi x^{\lambda}}-(1-\propto)(1+\beta x)}=0$$

------------------------------------ *(3.55)*

$$\sum_{i=1}^{n}\frac{\beta}{\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)}-\sum_{i=1}^{n}ln(1+\beta x)-2\sum_{i=1}^{n}\frac{(1-\propto)ln(1+\beta x)}{(1+\beta x)^{\theta}e^{\varphi x^{\lambda}}-1+\propto}=0$$

------------------------------------ *(3.56)*

$$\sum_{i=1}^{n}\frac{\lambda x^{\lambda-1}+\beta\lambda x^{\lambda}}{\theta\beta+\varphi\lambda x^{\lambda-1}+\beta\varphi\lambda x^{\lambda}}-\sum_{i=1}^{n}{x_i}^{\lambda}+2\sum_{i=1}^{n}\frac{{x_i}^{\lambda}(1-\propto)}{(1+\beta x)^{\theta}e^{\varphi x^{\lambda}}+\propto-1}=0$$

------------------------------------ *(3.57)*

$$\sum_{i=1}^{n}\frac{\varphi x^{\lambda-1}(1+\beta x)+\varphi\lambda x^{\lambda-1}(lnx)(1+\beta x)}{\theta\beta+\varphi\lambda x^{\lambda-1}(1+\beta x)}-\sum_{i=1}^{n}\varphi{x_i}^{\lambda}lnx-2\sum_{i=1}^{n}\frac{\varphi{x_i}^{\lambda}(1-\propto)lnx}{(1+\beta x)^{\theta}e^{\varphi x^{\lambda}}+\propto-1}=0$$

------------------------------------ *(3.58)*

Equation *(3.54), (3.55), (3.56), (3.57), (3.58)* are not in closed form, so the parameter estimation by maximum likelihood method will be solved simultaneously by numerical approach.

3.6 CRITERIA OF MODEL COMPARISON AND SELECTION

Selecting and comparing models with the MOL-W distribution involves examing several statistical and practical criteria.

Here are some essentials criteria that are often used for comparison and selection as reported by Sunday and Osemwenkhae (2020).

3.6.1. Goodness of fit test

The goodness of fit test assesses how a well a given distribution fits a sample of observed data. In applying the test on different models, it gives clarity in decision making on the best model to chose which fits a set of data in a list of models.

Here are several commonly used goodness of fit tests and methods to determine if a particular distribution provides an adequate model for the data.

The adequacy model Package in R-software is used to obtain the estimate of the distributions and statistical tests.

1. Kolmogorov-Smirnov (K-S) Test

The kolmogorov-Smirnov test is a non parametric statistical test that assesses whether a distribution best model the sample of observed data. It is useful for checking overall fit but less sensitive to differences in the tail. The K-S test compares the empirical distribution function (EDF) of observed data with the cumulative distribution function (CDF) of the model's predicted data. The K-S statistic is the maximum absolute difference between the EDF and CDF.

2. Anderson-darling test(A-D)

The Anderson darling test is a parametric statistical test that is applied to check whether a distribution best model a sample of observed data. It places more emphasis to differences on the tails of the distribution.

The A-D test measures the square differences between the empirical distribution function and cumulative distribution function giving more eight to the tails.it is suitable for detecting differences in tails often preferred over the K-S test for small sample sizes.

3. Crammer-von MIses Test (C-V)

The cramer-von mises test, developed by cramer and von-Mises (1928) is a non-parametrtic statistical test used to assess whether a distribution best model a sample of observed data. It gives equal weight to all parts of the distribution and measures the average of the squared differences between the EDF and CDF.

4. P-P plot

Probability - Probability plot is a graphical tools used to assess how well a data set conforms to a particular theoretical probability distribution. It compares the cumulative distribution function (CDF) of the sample data with the CDF of the reference distribution.

5. Q-Q Plot

Quantile -Quantile plot is another graphical tool is another graphical tool used to compare the distribution of a data-set with a theoretical distribution to ascertain if it can be modeled by the theoretical distribution.

3.6.2 Information Criteria

These are statistical tools used for model selection helping to determine which model best fit a given data set while avoiding over-fitting. They are based on the principle of trade-off between model fit and complexity and strike a balance also between the two factors model fit and complexity. Below are some key information criteria.
The adequacy model Package in R-software is used to obtain the values of the various information criteria applied.

1. Akaike information criteria (AIC)

This is a statistical metric used to compare and evaluate models by balancing goodness of fit and model complexity. It was introduced by hirotugu akaike in 1973. it can only be applied on models that support the use of maximum likelihood estimation.

2. Corrected Akaike information criteria (CAIC)

This is a modified version of AIC design to address the shortcoming of AIC when the sample size is small or when the number of parameters in the models is relatively large compared to the sample size. CAIC introduces a correction term to AIC that accounts for the finite sample size.

3. Bayesian information criteria (BIC)

This is a statistical measure used to model selection similar to the AIC but with a stronger penalty for model complexity. It evaluates models based on their likelihood and the number of parameters prioritizing simpler models when sample sizes are large. BIC is also known as schwarz Bayesian criterion.

4. Hannah-Quinn information criteria (HQIC)

This is a statistical criterion used to for model selection also similar to the AIC and BIC. Its uniqueness is that it provides a balance between goodness of fit and model complexity with a penalty term that lies between the less strict AIC and the more conservative BIC.

The various formulas applied in computing the test values of the information criteria are

$$\mathrm{AIC} = -2\mathrm{L}(\gamma) + 2\mathrm{q}$$

$$\mathrm{CAIC} = -2\ln(\gamma) + \frac{2\mathrm{qn}}{\mathrm{n} - \mathrm{q} - 1}$$

$$\mathrm{BIC} = -2\mathrm{L}(\gamma) + 2\mathrm{q}\log(\mathrm{n})$$

$$\mathrm{HQIC} = -2\ln(\gamma) + 2\mathrm{q}\log(\log(\mathrm{n})$$

Where;

$\mathrm{L}(\gamma)$ denotes the log-likelihood function evaluated at the maximum likelihood estimates

Q denotes the number of parameters

n denotes the sample size

3.6.3 Interpretation of goodness of fit and information criteria

Given the hypothesis for the goodness of fit test for testing how well a statistical models fits a set of observed data as follows

Null hypothesis: the observed data follow the specified model

Alternative hypothesis: the observed data do not follow the specified model

Since as reported by Ronald A. Fisher (1925) that a value of 0.05 serves as a convenient cutoff for level of significance and it has since become standard threshold in Statistics, it follows that;

If P-Value is low $(\mathrm{P-value} < 0.05)$, the null hypothesis is rejected, indicating that the model does not fit the data well. But if the P-Value is high

$(\mathrm{P-value} \geq 0.05)$, there is no sufficient condition to reject the null hypothesis suggesting that the model fits the data adequately.

In comparison and selection of models,

- the model with the highest P-Value or lowest test statistic in terms of K-S test, A-D test and C-V test can be taken as the most superior over other models under comparison, suggesting that the model best fit the data.
- The model with lowest value of AIC, CAIC, BIC and HQIC, suggests higher superiority over other models in comparison with it indicating that it best fit the data set than the other models.

CHAPTER FOUR

ANALYSIS AND DISCUSSION

4.1 Introduction

This chapter focuses on the presentation and analysis of the dataset collected for this study. It begins with an overview of the dataset, then distributions are presented for comparison, allowing for a clear visualization and Analysis.

4.2 Numerical Analysis

4.2.1 Presentation of Distributions for comparison

This section entails the distributions to be compared with the Marshall-Olkin Lomax-Weibull Distribution (MOLW). The distribution to be compared includes

(I) Lomax Distribution

The Lomax distribution, also known as the Pareto Type II distribution, is a continuous probability distribution often used in survival analysis, reliability engineering, actuarial science, and economics. It is a special case of the generalized Pareto distribution and is commonly applied to model heavy-tailed data, such as insurance claims and business failure rates. It is given by the Cumulative density function;

$$F(x) = 1 - (1 + \beta x)^{\theta}$$

(II) Marshall - Olkin Lomax Distribution

The Marshall–Olkin Lomax distribution is an extension of the Lomax (Pareto Type II) distribution introduced using the Marshall–Olkin transformation, which adds a new parameter to model greater flexibility in hazard rates. It is useful in reliability analysis, survival analysis, and risk management.

$$F(x) = \frac{1 - (1 + \beta x)^{\theta}}{1 - (1 - \alpha)(1 + \beta x)^{\theta}}$$

(III) The Marshall-Olkin Weibull Distribution

The Marshall–Olkin Weibull (MOW) distribution is an extension of the Weibull distribution introduced using the Marshall–Olkin transformation, which provides greater flexibility for modeling lifetime data and reliability systems. This distribution is particularly useful in survival analysis, engineering, and risk assessment.

$$F(x) = \frac{1 - e^{-\varphi x^{\lambda}}}{1 - (1 - \alpha)(e^{-\varphi x^{\lambda}})}$$

4.2.2 Presentation of Data

Data On Failure Times Of 84 Aircraft Windshield Given In Murthy Et Al. (2004)

0.0400, 1.866, 2.3850, 3.443, 0.3010, 1.876, 2.4810, 3.467, 0.309, 1.8990, 2.610, 3.4780, 0.557, 1.9110, 2.625, 3.5780, 0.943, 1.9120, 2.632, 3.5950, 1.0700, 1.914, 2.6460, 3.699, 1.1240, 1.981, 2.661, 3.7790, 1.248, 2.0100, 2.688, 3.9240, 1.2810, 2.038, 2.823, 4.035, 1.281, 2.0850, 2.890, 4.121, 1.3030, 2.089, 2.902, 4.167, 1.4320, 2.097, 2.934, 4.2400, 1.480, 2.135, 2.962, 4.2550, 1.505, 2.154, 2.9640, 4.278, 1.506, 2.190, 3.000, 4.3050, 1.568, 2.1940, 3.103, 4.376, 1.615, 2.2230, 3.114, 4.449, 1.6190, 2.224, 3.1170, 4.485, 1.652, 2.2290, 3.166, 4.570, 1.652, 2.3000, 3.344, 4.602, 1.7570, 2.324, 3.3760, 4.663

4.2.3 Presentation of Table for Comparison and Selection

Distn	θ (Std. err.)	λ (Std. err.)	$\propto$ (Std. err.)	β (Std. err.)	φ (Std. err.)
MOLW	0.0143 (0.0220)	3.3022 (0.5383)	0.4529 (0.3397)	3.0965 (6.6872)	0.0165 (0.0171)
MOLx	43.4819 (42.4958)	---	42.9073 (21.2383)	0.0363 (0.0391)	---
MOW	---	1.0698 (0.5111)	26.3250 (53.5693)	---	1.2334 (1.3197)
Lx	43.7347 (24.3479)	---	---	0.0090 (0.0050)	---

Distn	-2ll	AIC	CAIC	BIC	HQIC	W*	A*
MOLW	251.7699	261.7699	262.5392	273.9240	266.6558	0.06392	0.45694
MOLx	256.8392	262.8392	263.1392	270.1317	265.7707	0.06232	0.50214
MOW	256.4912	262.4912	262.7912	269.7836	265.4227	0.06823	0.52678
Lx	327.3043	331.3043	331.4525	336.1659	333.2586	0.1672	0.14100

4.3 DISCUSSION

Some statistical tests for comparing the different distributions are employed to determine the distributions that best fit the data sets. They include the information criteria and goodness of fit tests. The Adequacy Model package in R software is used to obtain the estimates of the distributions, AIC, CAIC, BIC, HQIC, Cramer Von Mises (W*), Anderson Darling (A*) tests.

In the values gotten from the information criteria and goodness of fit tests, the superiority of the MOL-W distribution over the other compared distributions is evident. The MOL-W distribution has the lowest value in the AIC, CAIC, and a value lower than Lx Distribution in BIC and HQIC. Also the MOL-W distribution has the lowest test value in the Anderson Darling test (A*) and lower value than MOW and Lx Distribution in the Cramer Von Mises Test (W*).

CHAPTER FIVE

5.2 CONCLUSION

The Comparison with other traditional models, applicability and flexibility of the new distribution in lifetime analysis illustrated with the aid of a real life example demonstrate the superiority of the Marshall-Olkin Lomax-Weibull distribution in fitting complex datasets compared to traditional models and enhancing its ability to model diverse hazard rate behaviours, including increasing, decreasing, bathtub, and upside-down bathtub shapes and monotonic and nonmonotonic failure data which could be obtained complex systems used in diverse scientific fields.

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