THE PARALOGISTIC-CHEN DISTRIBUTION: MODEL, PROPERTIES AND APPLICATIONS.

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UNDERTAKING

This project work was carried out by me **RACHEAL REKIYAT MUSA (PSC1909254)**

I have not plagiarized any existing work. All published works used in this project have been duly cited and referenced

RACHEAL REKIYAT MUSA **DATE**

CERTIFICATION

The project with the title: **PARALOGISTIC-CHEN DISTRIBUTION: MODEL, PROPERTIES AND APPLICATIONS** was carried out by **RACHEAL REKIYAT MUSA** with matriculation number **PSC1909254** under the supervision of **DR. S.A OSAGIE**, has satisfied the regulations governing the award of the Bachelor of sciences degree in the Department of Statistics, University of Benin, Benin City.

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Sign. & Date

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EXTERNAL EXAMINER: ______________________

Sign. & Date

DEDICATION

This study is first and foremost dedicated to God Almighty who has been my rock and inspiration throughout my journey. His divine guidance and love have seen me through every challenge, and I humbly acknowledge his sovereignty and grace in my life.

Also, this study is dedicated to my late father Mr Musa Raymond AbdulRahman, my late Mother Mrs Okposo Susanna Okeoghene, and to my family in general especially to my big sister Mrs Okposo O. Esther and my big brothers Mr Okposo Hamilton and Mr Okposo Samuel for their great support in achieving this degree.

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ABSTRACT

This study focuses on the development of continuous lifetime distribution to model real life data sets. One approach to creating new distributions is the T-X (Tranformer-Transformed) method, which involves either adding a number of parameters to an existing distribution, raising a distribution to a power or combining existing distributions.

In this study, the Paralogistic-Chen distribution is generated using the T-X (Transformer- Transformed) method of obtaining distributions. This involves a combination of the paralogistic and the Chen distributions.

Some of the properties of the Paralogistic-Chen distribution are considered in this study and the application of the distribution will be considered to show how well the distribution fits the data and the Maximum Likelihood Estimation (MLE) is used to obtain the parameters of the distribution.

CHAPTER ONE

GENERAL BACKGROUND

1.0 INTRODUCTION

Statistical distributions play a crucial role in statistical inferences and their applications in capturing real-world phenomena accurately. Over the years, a variety of parametric distributions have been utilized in the analysis of lifetime data. However, the ongoing quest for distributions that are more flexible and robust continues, driven by the need to accurately capture complex real-world situations.

In statistical modeling and analysis, the study of lifetime distributions holds significant importance, particularly in fields such as reliability engineering and survival analysis. These distributions serve as fundamental tools for understanding the behavior of systems over time, making them essential in making informed decisions and predictions. This study introduces a new lifetime distribution called the Paralogistic-Chen distribution, generated by using the Transformer-Transformed (T-X) method by Alzaareth et al. (2013). This distribution presents a valuable alternative to traditional distributions for accurately modeling lifetime data.

This chapter provides an overview of the background of the study, its aim and objectives, the scope of the study, the organization of the study, and the definition of basic terms used in this research.

1.1 BACKGROUND OF THE STUDY

Traditionally, modeling data and making statistical inferences rely on the application of classical distributions, which may not always provide an adequate fit and often approximate the true underlying statistics. The accuracy of parametric statistical inference and modeling largely depends on the goodness of fit of the probability distribution to the given dataset, once all distributional assumptions have been met.

To address this limitation, researchers have developed alternative distributions to offer greater flexibility in modeling data and statistical inference.

In recent years, advancements in statistical methodology have led to the development of new distributions, which involve the paralogistic distribution and the Chen distribution. These distributions aim to overcome the limitations of classical distributions by either combining existing distributions or introducing new parameters to enhance their flexibility and applicability.

In the context of combining existing distributions to create new ones, Alzaatreh et al. (2013) introduced the Transformed-Transformer family of distributions, abbreviated as the T-X family. This family of distributions is formed by combining various distributions to enhance their flexibility and applicability.

In this study, we will focus on the Paralogistic-Chen distribution, which is a recent addition to the T-X family. This distribution is derived by combining characteristics of the Chen distribution and the paralogistic distribution using T-X method.

1.2 AIM AND OBJECTIVES

The aim of this study is to introduce a new lifetime distribution called the Paralogistic-Chen distribution using the Transformer-Transformed (T-X) method by Alzaareth et al. (2013).

The objectives of the study are as follows:

1. To derive the statistical properties of the Paralogistic-Chen Distribution.
2. To obtain the parameter estimates of the Paralogistic-Chen Distribution using maximum likelihood estimation methods (MLE).
3. To demonstrate the application of the Paralogistic-Chen distribution.

1.3 SCOPE OF THE STUDY

This study aims to provide a comprehensive overview of the Paralogistic Chen distribution, focusing on its properties, mathematical formulation and practical applications. By showing the significance of this distribution, the new distribution contributes to the advancement of statistical theory and its practical implications in data analysis and decision-making.

1.4 ORGANIZATION OF THE STUDY

This study will be discussed in the following chapters as follows:

Chapter one serves as an introduction to the research, providing a preamble and detailed information on the background of the study. Chapter two presents into the historical context of obtaining the distribution, focusing on the T-X transformer-transformed family of distributions, as well as the paralogistic and Chen distributions, along with other similar distributions. Chapter three focuses on examining the statistical properties and derivation of the new distribution, as well as parameter estimation using the maximum likelihood estimation method. Chapter four explores the application of the Paralogistic-Chen distribution on datasets, and compares it with other distributions. Chapter five presents a discussion of the obtained results, based on the findings of the study.

1.5 DEFINITION OF BASIC TERMS

1. LIFETIME DATA

Lifetime data refers to a type of data that captures the duration or time-to-event information of an observation or entity.

2. STATISTICAL DISTRIBUTIONS

A statistical distribution, also known as a probability distribution, is a mathematical function that describes the likelihood of various outcomes or values occurring in a dataset.

3. STATISTICAL INFERENCE

Statistical inference refers to the process of drawing conclusions or making predictions about a population based on information obtained from a sample of that population.

4. MODEL

A model is a simplified representation of a real-world system, process, phenomenon. Also a model typically consists of mathematical or statistical relationships that describe the behavior of variables in a dataset.

5. PARAMETRIC DISTRIBUTION

Parametric distributions are probability distributions that are defined by a finite number of parameters.

6. SURVIVAL ANALYSIS

Survival analysis is a statistical method used to analyze time-to-event data, particularly in biomedical and social sciences. It involves studying the time until the occurrence of an event of interest, such as death, failure, recurrence of a disease, or any other event that impacts the outcome of interest.

7. RELIABILITY ENGINEERING

Reliability engineering is a multidisciplinary field of engineering that helps to identify, analyze, and mitigate potential failures or malfunctions in systems to improve their reliability, safety, and quality.

8. HAZARD FUNCTION

The hazard function, also known as the instantaneous failure rate or conditional failure rate, is a rate at which events occur at a specific point in time, given that the subject has survived up to that time.

9. PROBABILITY DENSITY FUNCTION (PDF)

The pdf is a mathematical function that describes the probability distribution of a continuous random variable say X, which will take value exactly equal to x.

10. CUMULATIVE DENSITY FUNCTION (CDF)

The cdf of a random variable X, denoted as F(x), is a function that gives the probability that X will take a value less than or equal to x, for all real numbers x.

11. SHAPE PARAMETER

This is also known as form parameter, is a parameter that defines the shape of a probability distribution in statistics.

12. SCALE PARAMETER

This is a parameter that determines the scale or spread of a probability distribution in statistics.

13. LOCATION PARAMETER

This is also known as shift parameter, is a parameter that determines the location or position of a probability distribution along the x-axis in statistics.

14. MOMENTS

Moments are numerical measures that summarize the properties of a probability distribution or a dataset. They provide insights into the central tendency, dispersion, shape, and other characteristics of the distribution or dataset.

1.6 SUMMARY

This chapter provides an overview of the research background, outlines the aim and objectives of the study, and clarifies key terminology used in the research.

CHAPTER TWO

LITERATURE REVIEW

2.0 INTRODUCTION

This chapter will give a brief history of the paralogistic distribution, Chen distribution and Transformer-Transformed (T-X) family of distributions and their applications in the areas of applied and social science among others.

The paralogistic and Chen distribution are valuable tools in statistical modeling and data analysis, offering flexibility and applicability to a wide range of real-world datasets.

The paralogistic distribution and Chen distribution can be used as lifetime distributions. These distributions are often applied in reliability analysis and survival studies.

2.1 METHODS OF GENERATING DISTRIBUTIONS

In an effort to enhance the flexibility and improve the fitting of distribution in real-world data, many methods have been developed to generate statistical distribution and researchers have been extending the generalization of distributions by introducing additional parameters to the baseline distribution. This extension is being pursued by incorporating the baseline distribution into a family of distributions.

In the early stages of research, notable methods for generating univariate continuous distribution includes methods based on differential equations developed by Pearson (1895) and methods based on quantile functions developed by Turkey (1960).

In recent decades, there has been a continued interest in developing new methods for generating distributions that are more flexible and can better fit real-world phenomena.

According to Lee et al. (2013), many recent methods in distribution development since the 1980s, has been the 'combination' approach, which involves either combining existing distributions or adding new parameters to existing ones, resulting in a wide range of new distributions that offer improved modeling capabilities. Some of the methods include:

1. Introduction of extra parameters
2. Beta generated methods
3. Skew mixture methods
4. The Transformer-Transformed method

Mudholkar and Srivastava (1993) introduced the methods for extending the two parameters Weibull distribution by adding an extra parameter. This resulted in the development of the exponential Weibull family of distributions.

Marshall and Olkin (1997) introduced a straightforward method for adding in a single parameter to a family of distributions, which has been utilized by several authors to extend well-known distributions in recent years.

Marshall and Olkin (1997) explored two special cases, specifically when X follows an exponential or Weibull distribution. This led to the development of the Marshall-Olkin extended Weibull family of distributions, which includes several models such as Marshall-Olkin Weibull, Marshall-Olkin Lomax and Marshall-Olkin Burr XII distributions, among others.

Shaw and Buckley (2009) introduced a significant method for adding an extra parameter to a family of distributions. This method has been applied to the transmuted family of distributions.

Eugene et al. (2002) introduced the beta-generated distribution method, a novel approach to creating distributions by utilizing the beta distribution with shape parameters alpha and beta as a generator. This method enables the development of new distributions, known as beta-generated distributions, which can be used to model various types of data.

Many researchers have studied the beta generated distributions examples are Beta-Weibull, (Famoye et al. 2005), Famoye et al.(2004), Cordeiro and Lemonte (2011), and Alshawarbeh et al.(2012).

Furthermore, Jones (2009) and Cordeiro and de Castro (2011) expanded the beta-generated family of distributions by utilizing the Kumaraswamy distribution as an alternative to the beta distribution. (Kumaraswamy.1980).

More studies on beta-generated distributions are beta-pareto (Akinsele et al.2008), beta-Gumbel (Nadarajah and Kotz, 2004), beta-exponential (Nadarajah and Kotzebue, 2006), beta-gamma (Kong et al. 2007).

Azzalini (1985) introduced the skew-normal family of distributions, which is a continuous probability distribution that generalized the normal distribution to account for skewness.

Ferreira and Steel (2006) developed a comprehensive framework for creating a wide range of skewed distributions by modifying symmetric distributions, providing a flexible and systematic approach to generating new distributions with asymmetric characteristics.

Studies by Eling (2012) and Adcock et al. (2015) have identified skew-normal and skew-student t distributions as the top performers among skewed distributions, as they successfully address right-skewness and high kurtosis, making them suitable for modeling complex data sets.

The generalized skew-normal distribution introduced by Genton and Loperfido (2005), which is a notable example of a distribution generated through a specific method.

Mudholkar and Hutson (2000) expanded the normal distribution by introducing the epsilon-skew normal family, which incorporates additional parameters that adjust the level skewness.

Salinas et al. (2007) introduced a comprehensive family of skewed distribution by merging the epsilon-skew normal and the skew-normal families, creating a flexible and robust framework for modeling asymmetric data.

2.2 TRANSFORMER-TRANSFORMED (T-X) METHOD

Alzaatreh et al.(2013), proposed a general method for generating family of continuous distributions, known as the Transfomer-Transformed (T-X) family.

This method involves replacing the beta probability density function (PDF) with the PDF of a continuous random variable, denoted as "r" and applying a function W[f(x)] that satisfies the following conditions to develop the T-X family.

i. $W(F(x)) \in [a,b]$,

ii. W is differentiable and monotonically non-decreasing,

iii. $W(F(x)) \to a$ as $x \to -\infty$ and $W(F(x)) \to b$ as $x \to \infty$

Where [a,b] is the support of the random variable T for $-\infty \leq a < b \leq \infty$.

The CDF of the T-X family is defined as:

$$G(x) = \int_{a}^{w[f(x)]} r(t)\ dx = R\{W[f(x)]\} \qquad (2.1)$$

Where R is the CDF of T "the transformed"

The corresponding PDF (if it exists) of the T-X family distributions is:

$$g(x) = \{\frac{d}{dx} W[f(x)]\}\ r\{W[f(x)]\} \qquad (2.2)$$

The T-X family has been further studied and applied by various researchers Include Alzaatreh et al. (2013). Some of them are Gamma-Pareto distribution by Alzaatreh et al. (2012), Weibull-Pareto distribution by Alzaatreh et al. (2013), Weibull-X family of distribution by Alzaatreh and Ghosh (2015), gamma-half normal distribution by Alzaatreh and Knight (2013), Beta Marshall-Olkin family by Alizadeh et al. (2015), Weibull-G family of distribution by Bourguignon et al. (2014).

Aljarrah et al. (2014) introduced a method for generating the T-X family of distributions using the quantile functions.

Alzaatreh et al. (2014) also introduced a member of the T-Normal family, the Weibull-N{Exponential} distribution, which is a valuable addition to the T-Normal family, offering increased flexibility and a wider range of behaviors for modeling various types of data.

Nasir et al. (2019) introduced the T-Burr family of distributions as a subset of the broader T-X{Y} family, examining its properties and behavior, and providing a valuable addition to the field of statistical distributions.

Almeida et al. (2019) and Alzaatreh (2016) proposed the T-Weibull {Y} family and Cauchy {Y} family.

Tahir et al. (2016) suggested the poisson X-family of distributions which is a subfamily of T-X family of distribution.

Shaw and Buckley (2014) introduced a combined family of T-X and the transmuted family called Quadratic Transmuted Family of Distribution.

While investigating and analyzing the characteristics of a specific member of the T-Transmuted X family, the Exponential-Transmuted distribution was introduced by Jayakumar and Grisham (2017) using the T-X family.

2.3 PARALOGISTIC DISTRIBUTION

The paralogistic distribution is a sub-model of the generalized beta family which was introduced by McDonald (1984). The CDF of the generalized beta family distribution is given by:

$$G(x) = \int_0^{F(x)} b(t)dt \tag{2.3}$$

Where b(t) is the p.d.f. of the beta random variable and F(x) is the c.d.f. of any random variable.

The cumulative density function, CDF and the probability density function, PDF of the paralogistic distribution are given respectively by;

$$F(x; \lambda, \beta) = 1 - [1 + (\beta x)^{\lambda}]^{-\lambda}, \ x, \lambda, \beta > 0 \qquad (2.4)$$

$$f(x; \lambda, \beta) = \frac{\lambda^2}{x} [1 + (\beta x)^{\lambda}]^{-(\lambda+1)} (\beta x)^{\lambda}, \ x, \lambda, \beta > 0 \qquad (2.5)$$

Where λ>0 and β>0 is the shape and scale parameters respectively.

Idemudia and Ekhosuehi (2019) introduced a new three - parameters paralogistic distribution to the existing paralogistic distribution and the cdf and pdf are given as:

$$F(x) = 1 - [1 + (\frac{x-\theta}{c})^{\alpha}]^{\alpha}; \ x, c, \alpha > 0 \qquad (2.6)$$

$$f(x) = \frac{\alpha^2}{x-\theta} [1 + (\frac{x-\theta}{c})^{\alpha}]^{-(\alpha+1)} (\frac{x-\theta}{c})^{\alpha}; \ x, c, \alpha > 0 \qquad (2.7)$$

Where c>0, α>0 and θ>0 are the scale, shape and location parameters respectively.

The New Three Parameter paralogistic distribution (NTPLD) was applied to a study by Nicholas and Padgett (2006) to compare its performance to the other related distributions in the fitting the breaking stress of carbon fibers. The distributions compared were the paralogistic, log-logistic, gamma, transformed beta and inverse paralogistic. The NTPLD was found to provide a better fit to the data compared to the other distributions with respect to their density plots, P-P plots and Q-Q plots, where

P-P plots represent the empirical distribution function evaluated at each data plot against the fitted distribution function and Q-Q plots represent the empirical quantile against the theoretical quantile.

2.4 CHEN DISTRIBUTION

The Chen distribution is a statistical distribution used for modeling lifetime data. It is a flexible distribution that can represent increasing, decreasing and bathtub curve shapes for failure rates.

Chen (2000) developed a new two-parameter lifetime distribution with bathtub shaped or increasing failure rate (IFR) function.

The pdf and cdf of chen distribution is given as:

$$g(x; \lambda, \beta) = \lambda\beta x^{\beta-1} exp\left\{x^{\beta} + \lambda\left(1 - e^{x^{\beta}}\right)\right\} \quad (2.8)$$

$$G(x;\lambda, \beta) = 1 - \exp\left\{\lambda\left(1 - e^{x^{\beta}}\right)\right\}, x > 0, \lambda, \beta > 0 \quad (2.9)$$

The Chen distribution (Chen, 2000) was developed by compounding the Weibull and Exponential distribution.

The extensions of the Chen distribution in the lifetime includes a study by Xie et al. (2002) which modifies the Chen distribution by adding the lacking scale parameter, thereby creating a three-parameter extended Weibull (EW) distribution.

Chaubey and Zhang (2015) developed an extension of the Chen distribution called the exponentiated Chen distribution using the Lehman alternatives also known as exponentiated type family (Gupta et al.1998, Nadarajah and kotz,2006)

Thach and Bris (2021) introduced the additive Chen-Weibull distribution, a innovative combination of the Weibull and Chen distributions, designed to model the reliability of series systems comprising independent components, providing a valuable tool for understanding and predicting system performance.

Abdulzeid et al. (2022) introduced an extension of the Chen distribution which they call the Modified Extended Chen (MEC) distribution. The CDF and PDF is given respectively as:

$$F(y) = \left[\rho\left(e^{x^{-b}} - 1\right) + 1\right]^{-a}, x \geq 0 \qquad (2.10)$$

$$f(y) = ab\rho x^{-b-1} e^{x^{-b}} \left[\rho\left(e^{x^{-b}} - 1\right) + 1\right]^{-a-1} \qquad (2.11)$$

Where a>0 and b>0 are the shape parameters and ρ>0 is a scale parameter.

Ramesh and Vijay (2021) studied and proposed a three parameter univariate continuous Logistic-Chen (LC) distribution which they applied to a lifetime data set. The CDF and PDF of the Logistic-Chen distribution is defined respectively as:

$$F(x) = 1 - \frac{1}{1 + \left[exp\left\{\lambda\left(e^{x^{\beta}} - 1\right)\right\} - 1\right]^{\alpha}}, (\alpha, \beta, \lambda) > 0, x > 0 \qquad (2.12)$$

$$f(x) = \frac{\alpha\beta\lambda \exp\left\{\lambda\left(e^{x^{\beta}} - 1\right) + x^{\beta}\right\}\left[\exp\left\{\lambda\left(e^{x^{\beta}} - 1\right)\right\} - 1\right]^{\alpha-1}}{\left\{1 + \left[\exp\left\{\lambda\left(e^{x^{\beta}} - 1\right)\right\} - 1\right]^{\alpha}\right\}^{2}} \qquad (2.13)$$

2.5 METHODS OF PARAMETERS ESTIMATION

Parameter estimation is the process of using sample data to estimate the parameters of a statistical model.

There are various methods for parameter estimation and the choice of method depends on the specific characteristics of the data and the underlying statistical model. Some of the methods of parameter estimation include:

1. Probability plotting
2. Rank Regression (Least square)
3. Maximum likelihood Estimation (MLE)
4. Bayesian Estimation Methods
5. Methods of moments Estimators
6. Minimum Mean Squared error (MMSE)
7. Cramér-Rao bound

In this study, we shall utilize the maximum likelihood estimation (MLE) for parameter estimation.

2.5.1 Maximum likelihood Estimation (MLE) Method

Maximum likelihood Estimation (MLE) is a widely used method for estimating the parameters of a statistical model. It has made significant contributions of both scientific and commercial domains.

Maximum likelihood Estimation is a method of estimating unknown parameters of a distribution given some observed data. The parameter values are obtained such that they maximize the likelihood described by the distribution produced by the data.

By denoting the unknown parameter(s) or any distribution generally by θ and since the probability distribution depends on θ, then we can write $f(x)$ as f(x;θ). Next, the likelihood function denoted as L (θ,x) should be obtained as it is used to determine the probability of the observed data given the parameters of the distribution.

The MLE is an essential aspect of inference theory and it is a statistical method used to obtain the most likely values of parameters by maximizing the likelihood function which is used to find the best estimate of the unknown parameters.

Procedure of MLE Method:

1. State the probability function $f(x)$
2. Find the likelihood function $L(\theta) = \prod_{i=1}^{n} f(x)$
3. Find the log likelihood function
4. Obtain the partial derivations of the log likelihood with respect to the parameters of interest and equate to zero

Mathematically, if $f(x)$ is the pdf of a continuous probability distribution, the mle are presented as follows:

Given the pdf distribution i.e f(x_i, θ) $\forall\ i = 1(1)n$

1. likelihood function say, L$(\theta) = \prod_{i=1}^{n}$ f(x_i, θ)
2. log likelihood function say, log L(θ) = log $[\prod_{i=1}^{n}$ f$(x_i, \theta)]$
3. partial derivations of the log likelihood with respect to the parameters of interest: $\frac{\partial \log \mathrm{L}(\theta)}{\partial \theta} = \frac{\partial \log [\prod_{i=1}^{n} \mathrm{f}(x_i,\theta)]}{\partial \theta}$
4. Equating to zero implies $\frac{\partial \log [\prod_{i=1}^{n} \mathrm{f}(x_i,\theta)]}{\partial \theta} = 0$
5. Then solve explicitly or numerically to obtain the parameter of interest θ.

2.6 SUMMARY

This chapter discusses the history and modifications of paralogistic and Chen distributions, including their applications by various researchers. It also discusses the applications of both the distributions in modeling real-life data by demonstrating their superior performance compared to other distributions. It also discusses the parameter estimation by the used of maximum likelihood estimation to estimate the parameters under study.

CHAPTER THREE

METHODOLOGY

3.0 INTRODUCTION

This chapter discusses the development of the Three parameters paralogistic Chen distribution through the Transformer-Transformed (T-X) method, which aims to create the lifetime distributions based on existing ones. This chapter presents the mathematical derivatives of the distribution's probability density function, survival function and hazard function along with relevant statistical properties like moments and maximum likelihood estimation methods.

3.1 T-X FAMILY OF DISTRIBUTION

The T-X (Transformer-Transformed) method is a method of developing flexible distributions or family of distributions. it is generally accepted because a generator can be any baseline distribution. Mathematically, it is defined as

$$F(x) = \int_{c}^{W[G(x)]} r(t)dt, \qquad (3.1)$$

where a is a real number, r(t) is the probability density function of the baseline distribution and W(G(x)) is the random variable X cumulative density function.

3.2 PARALOGISTIC DISTRIBUTION

The Paralogistic distribution introduced by McDonald (1984) is created from the Burr distribution by collapsing 2 of the parameters as one.

Mathematically defined as

F(x) = 1 – $\left[1+\ t^{\theta}\right]^{-\theta}, \theta > 0,$ (3.2)

Where, θ>0 is the shape parameter

3.3 CHEN DISTRIBUTION

The Chen distribution introduced by Chen (2000) is a statistical distribution used for modeling lifetime data and survival analysis whose rate increases and decreases.

G(x) = 1 - $e^{\lambda\left(1-\ e^{x^{\beta}}\right)},$ (3.3)

where λ>0 and β>0 is the shape and parameter respectively.

3.4 PARALOGISTIC-CHEN DISTRIBUTION

In this section, we will focus on the constructing the three-parameter Paralogistic-Chen distribution. The method of generating this distribution using the T-X technique is outlined below

Let G(x) define a baseline continuous distribution, then

W[G(x)] = - Log (1 - G(x)), therefore

R(t) = $\int_{c}^{-\ Log\ (1\ -\ G(x))} r(t)dt,$ (3.4)

Where [- Log (1 - G(x))] is the upper bound of the distribution, and r(t) is the pdf of the base distribution.

Let the cdf of the paralogistic distribution be defined as

$$F(t) = 1 - \left(1 + t^{\theta}\right)^{-\theta},$$

Then;

$$F(x) = 1 - \left[1 + \left\{-\log\left(1 - G(x)\right)\right\}^{\theta}\right]^{-\theta} \tag{3.5}$$

If the cdf of the Chen distribution is defined as

$$G(x) = 1 - e^{\lambda\left(1 - e^{x^{\beta}}\right)} \tag{3.6}$$

Simplying equation (3.6)

$$1 - G(x) = e^{\lambda\left(1 - e^{x^{\beta}}\right)},$$

which is substituted into equation (3.5) to obtain

$$F(x) = 1 - \left[1 + \left\{-Log_{e}\left(e^{\lambda\left(1 - e^{x^{\beta}}\right)}\right)\right\}^{\theta}\right]^{-\theta}$$

$$= 1 - \left[1 + \left\{-\lambda\left(1 - e^{x^{\beta}}\right)\right\}^{\theta}\right]^{-\theta}$$

$$= 1 - \left[1 + \left\{-\lambda + \lambda e^{x^{\beta}}\right\}^{\theta}\right]^{-\theta}$$

$$F(x) = 1 - \left[1 + \left\{\lambda\left(e^{x^{\beta}} - 1\right)\right\}^{\theta}\right]^{-\theta} \tag{3.7}$$

Which define the CDF of the Paralogistic-Chen distribution.

3.4.1 THE PROBABILITY DENSITY FUNCTION (PDF)

We can obtain the PDF by differentiating the CDF in equation (3.7) with respect to x. We will denote the PDF as f(x)

$$f(x) = \frac{dF(x)}{dx}$$

$$f(x) = \frac{d\left[1 - \left[1 + \left\{\lambda\left(e^{x\beta} - 1\right)\right\}^{\theta}\right]^{-\theta}\right]}{dx}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad (3.8)$$

$$u = 1 + \left\{\lambda\left(e^{x\beta} - 1\right)\right\}^{\theta}$$

$$u = 1 + \left\{\lambda e^{x\beta} + \lambda\right\}^{\theta}$$

$$\frac{du}{dx} = 0 + \theta\left\{\lambda e^{x\beta} + \lambda\right\}^{\theta-1} \lambda\beta x^{\beta-1} e^{x\beta}$$

$$= \theta\lambda\beta x^{\beta-1} e^{x\beta}\left\{\lambda e^{x\beta} + \lambda\right\}^{\theta-1}$$

$$= \theta\lambda\beta x^{\beta-1} e^{x\beta}\left\{\lambda(e^{x\beta} + 1)\right\}^{\theta-1}$$

$$\frac{dy}{du} = 0 + \theta\left[1 + \left\{\lambda e^{x\beta} + \lambda\right\}^{\theta}\right]^{-\theta-1}$$

$$= \theta\left[1 + \left\{\lambda e^{x\beta} + \lambda\right\}^{\theta}\right]^{-\theta-1}$$

Hence,

$$f(x) = \theta^2 \lambda \beta x^{\beta-1} e^{x^\beta} \left\{ \lambda (e^{x^\beta} + 1) \right\}^{\theta-1} \left[1 + \left\{ \lambda (e^{x^\beta} + 1) \right\}^{\theta} \right]^{-\theta-1} \tag{3.9}$$

The PDF in equation (3.9) can be re-written in another form

Recall $(1 + az)^n = \sum_{i=0}^{\infty} \binom{n}{i} \mathrm{x}^i a^i$ and $(1 + az)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} (-1)^{\mathrm{i}} z^i a^i$

This implies

$$\left[1 + \left\{ \lambda (e^{x^\beta} + 1) \right\}^{\theta} \right]^{-\theta-1} = \left[1 + \left\{ \lambda (e^{x^\beta} + 1) \right\}^{\theta} \right]^{-(\theta+1)}$$

$$= \sum_{i=0}^{\infty} \binom{\theta+1-1}{i} (-1)^i \left(\left\{ \lambda (e^{x^\beta} + 1) \right\}^{\theta} \right)^i$$

$$= \sum_{i=0}^{\infty} \binom{\theta+i}{i} (-1)^i \lambda^i (e^{x^\beta} + 1)^{\theta i}$$

Hence,

$$f(x) = \theta^2 \lambda \beta x^{\beta-1} e^{x^\beta} \left\{ \lambda^{\theta-1} (e^{x^\beta} + 1)^{\theta-1} \right\} \sum_{i=0}^{\infty} \binom{\theta+i}{i} (-1)^i \lambda^i (e^{x^\beta} + 1)^{\theta i}$$

$$= \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} e^{x^\beta} \sum_{i=0}^{\infty} \binom{\theta+i}{i} (-1)^i (e^{x^\beta} + 1)^{\theta(i+1)-1}$$

$$= \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} e^{x^\beta} \sum_{i=0}^{\infty} \binom{\theta+i}{i} (-1)^i \left[e^{x^\beta} \left(1 - e^{-x^\beta} \right) \right]^{\theta(i+1)-1}$$

$$= \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} e^{x^\beta} \sum_{i=0}^{\infty} \binom{\theta+i}{i} (-1)^i e^{x^{\beta[\theta(i+1)-1]}} \left(1 - e^{-x^\beta} \right)^{\theta(i+1)-1}$$

$$= \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} \sum_{i=0}^{\infty} \binom{\theta+i}{i} (-1)^i e^{x^{\beta[\theta(i+1)]}} \left(1 - e^{-x^\beta} \right)^{\theta(i+1)-1}$$

Recall that

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

$$f(x) = \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{\theta+i}{i} (-1)^i \frac{[\theta(i+1)]^j}{j!} x^{j\beta} (1 - e^{-x^\beta})^{\theta(i+1)-1}$$

Note: $(1 - e^{-x^\beta})^{\theta(i+1)-1} = \sum_{k=0}^{\infty} \binom{\theta(i+1)-1}{k} (-1)^{\mathrm{k}} e^{-kx^\beta}$

Then,

$$f(x) = \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\theta(i+1)-1}{k} \binom{\theta+i}{i} (-1)^{i+k} \frac{[\theta(i+1)]^j}{j!} x^{j\beta} e^{-kx^\beta} \quad (3.1)$$

3.4.2 THE SURVIVAL FUNCTION S(x)

Given the CDF of the distribution from equation (3.7) as

$F(x) = 1 - \left[1 + \left\{\lambda\left(e^{x^\beta} - 1\right)\right\}^\theta\right]^{-\theta}$, the survival function S(x) is defined as;

$S(x) = 1 - F(x)$

$$= 1 - [1 - \left[1 + \left\{\lambda\left(e^{x^\beta} - 1\right)\right\}^\theta\right]^{-\theta}]$$

$$= 1 - 1 - \left[1 + \left\{\lambda\left(e^{x^\beta} - 1\right)\right\}^\theta\right]^{-\theta}$$

$$= \left[1 + \left\{\lambda\left(e^{x^\beta} - 1\right)\right\}^\theta\right]^{-\theta} \quad (3.12)$$

3.4.3 THE HAZARD FUNCTION h(x)

The hazard function h(x) is derived from the PDF and survival function. Mathematically

$$h(x) = \frac{f(x)}{S(x)}$$

$$f(x) = \theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda(e^{x^\beta} + 1)\right\}^{\theta-1}[1 + \left\{\lambda(e^{x^\beta} + 1)\right\}^{\theta}]^{-\theta-1}$$

$$S(x) = \left[1 + \left\{\lambda\left(e^{x^\beta} - 1\right)\right\}^{\theta}\right]^{-\theta}$$

$$h(x) = \frac{\theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda(e^{x^\beta} + 1)\right\}^{\theta-1}[1+\left\{\lambda(e^{x^\beta} + 1)\right\}^{\theta}]^{-\theta-1}}{\left[1+\left\{\lambda\left(e^{x^\beta} - 1\right)\right\}^{\theta}\right]^{-\theta}}$$

For simplicity, let p = $1 + \left\{\lambda(e^{x^\beta} + 1)\right\}^{\theta}$

$$h(x) = \frac{\theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda(e^{x^\beta} + 1)\right\}^{\theta-1}[p]^{-\theta-1}}{[p]^{-\theta}}$$

$$h(x) = \frac{\theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda(e^{x^\beta} + 1)\right\}^{\theta-1}p^{-\theta}\,p^{-1}}{[p]^{-\theta}}$$

Therefore:

$$h(x) = \theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda(e^{x^\beta} + 1)\right\}^{\theta-1}P^{-1}$$

Hence,

h(x) = $\theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda(e^{x^\beta}+1)\right\}^{\theta-1}[1+\left\{\lambda(e^{x^\beta}+1)\right\}^{\theta}]^{-1}$ (3.13)

3.4.4 THE QUANTILE FUNCTION

The Quantile function of the distribution can be obtained by solving for x using the CDF

F(x) = u or F (x_u) = u, 0<u<1

Therefore,

$$u = 1 - \left[1+\left\{\lambda\left(e^{x^\beta}-1\right)\right\}^{\theta}\right]^{-\theta}$$

$$1 - u = \left[1+\left\{\lambda\left(e^{x^\beta}-1\right)\right\}^{\theta}\right]^{-\theta}$$

$$(1-u)^{-\frac{1}{\theta}} = 1+\left\{\lambda\left(e^{x^\beta}-1\right)\right\}^{\theta}$$

$$(1-u)^{-\frac{1}{\theta}} - 1 = \left\{\lambda\left(e^{x^\beta}-1\right)\right\}^{\theta}$$

$$[(1-u)^{-\frac{1}{\theta}} - 1]^{\frac{1}{\theta}} = \lambda\left(e^{x^\beta}-1\right)$$

$$\frac{[(1-u)^{-\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda} = \left(e^{x^\beta}-1\right)$$

$$\left\{\left[\frac{[(1-u)^{-\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda}\right]+1\right\} = e^{x^\beta}$$

$$\ln\left\{\left[\frac{[(1-u)^{-\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda}\right]+\ 1\right\} = ln\left(e^{x^{\beta}}\right)$$

$$\ln\left\{\left[\frac{[(1-u)^{-\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda}\right]+\ 1\right\} =\ {x_u}^{\beta}$$

$$x_u = \left[ln\left\{\left[\frac{[(1-u)^{-\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda}\right]+\ 1\right\}^{\frac{1}{\beta}}\right], \lambda>0 \qquad (3.14)$$

Equation (3.14) can be used for random number generation for selected values of θ, β, λ.

3.4.5 THE MEDIAN USING THE QUANTILE FUNCTION

To obtain the median using quantile function given in equation (3.14), we make u=0.5

$$x_u = [ln\left\{\left[\frac{[(1-u)^{-\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda}\right]+\ 1\right\}^{\frac{1}{\beta}}]$$

Using u=0.5

$$x_{0.5} = [ln\left\{\left[\frac{[(1-0.5)^{-\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda}\right]+\ 1\right\}^{\frac{1}{\beta}}]$$

$$x_{0.5} = [ln\left\{\left[\frac{[(0.5)^{-\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda}\right]+\ 1\right\}^{\frac{1}{\beta}}]$$

$$x_{0.5} = [ln\left\{\left[\frac{[\left(\frac{1}{2}\right)^{-\frac{1}{\theta}} - 1]^{\frac{1}{\theta}}}{\lambda}\right] + 1\right\}^{\frac{1}{\beta}}]$$

$$x_{0.5} = [ln\left\{\left[\frac{[(2)^{\frac{1}{\theta}}-1]^{\frac{1}{\theta}}}{\lambda}\right] + 1\right\}^{\frac{1}{\beta}}], \lambda>0 \tag{3.15}$$

3.4.6 MOMENTS OF THE DISTRIBUTION (MOMENT ABOUT THE ORIGIN)

The r^{th} moment denoted as μ^r is defined as

$$\mathrm{E}\,(x^r) = \int_0^\infty x^r f(x)dx\,,$$

Therefore,

$$\mu^r = \mathrm{E}\,(x^r)$$

$$\mu^r = \int_0^\infty x^r\, \theta^2\lambda^{\theta+i}\beta x^{\beta-1} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}x^{j\beta}e^{-kx^\beta}dx$$

$$= \theta^2\lambda^{\theta+i}\beta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\int_0^\infty x^{r+\beta-1+j\beta}\, e^{-kx^\beta}dx$$

$$= \theta^2\lambda^{\theta+i}\beta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\int_0^\infty x^{r+\beta(j+1)-1}\, e^{-kx^\beta}dx$$

Remarks

Let $\int_0^\infty x^m e^{-\beta x^n}\, dx = \frac{\Gamma(\gamma)}{n\beta^\gamma}$

Where $\gamma = \frac{m+1}{n}$, which implies

m = $r + \beta(j+1) - 1, \beta = k \, and \, n = \beta$

Therefore,

$$\int_0^\infty x^{r+\beta(j+1)-1} \, e^{-kx^\beta} dx = \frac{\Gamma\left(\frac{[r+\beta(j+1)-1]+\ 1}{\beta}\right)}{\beta k^{\left(\frac{[r+\beta(j+1)-1]+\ 1}{\beta}\right)}}$$

$$= \frac{\Gamma\left(\frac{[r+\beta(j+1)]}{\beta}\right)}{\beta k^{\left(\frac{[r+\beta(j+1)]}{\beta}\right)}}$$

$$\mu^r = \theta^2 \lambda^{\theta+i} \beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\theta(i+1)-1}{k} \binom{\theta+i}{i} (-1)^{i+k} \frac{[\theta(i+1)]^j}{j!} \frac{\Gamma\left(\frac{[r+\beta(j+1)]}{\beta}\right)}{\beta k^{\left(\frac{[r+\beta(j+1)]}{\beta}\right)}}$$

$$= \theta^2 \lambda^{\theta+i} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\theta(i+1)-1}{k} \binom{\theta+i}{i} (-1)^{i+k} \frac{[\theta(i+1)]^j}{j!} \frac{\Gamma\left(\frac{[r+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[r+\beta(j+1)]}{\beta}\right)}} \qquad (3.16)$$

3.4.7 MEAN OF THE DISTRIBUTION

We will obtain the mean of the distribution by setting r = 1 in equation (3.16)

$$\mu = \theta^2 \lambda^{\theta+i} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\theta(i+1)-1}{k} \binom{\theta+i}{i} (-1)^{i+k} \frac{[\theta(i+1)]^j}{j!} \frac{\Gamma\left(\frac{[1+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[1+\beta(j+1)]}{\beta}\right)}} \qquad (3.17)$$

3.4.8 VARIANCE OF THE DISTRIBUTION

Recall that Var(X) = $\sigma^2 = E(X^2) - [E(X)]^2$

By setting r=2, we obtain

$$E(X^2) = \theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\frac{\Gamma\left(\frac{[r+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[r+\beta(j+1)]}{\beta}\right)}}$$

$$=\theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\frac{\Gamma\left(\frac{[2+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[2+\beta(j+1)]}{\beta}\right)}}$$

And

$$E(X) = \theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\frac{\Gamma\left(\frac{[1+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[1+\beta(j+1)]}{\beta}\right)}}$$

Hence,

$$\text{Var(X)}= \left[\theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\frac{\Gamma\left(\frac{[2+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[2+\beta(j+1)]}{\beta}\right)}}\right] -$$

$$\left[\theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\frac{\Gamma\left(\frac{[1+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[1+\beta(j+1)]}{\beta}\right)}}\right]^2 \qquad (3.18)$$

3.4.9 CENTRAL MOMENTS OF THE DISTRIBUTION

The Central Moment denoted as E$(x-\mu)^r$ at point is defined mathematically as

E$(x-\mu)^r = \int_0^\infty (x-\mu)^r f(x)dx$

Recall that

$$f(x) = \theta^2\lambda^{\theta+i}\beta x^{\beta-1}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}x^{j\beta}e^{-kx^\beta}$$

Hence,

$$\text{E}(x-\mu)^r = \int_0^\infty (x-\mu)^r\,\theta^2\lambda^{\theta+i}\beta x^{\beta-1}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}x^{j\beta}e^{-kx^\beta}dx$$

$$= \theta^2\lambda^{\theta+i}\beta\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\int_0^\infty (x-\mu)^r\,x^{\beta-1+j\beta}e^{-kx^\beta}dx$$

$$=\theta^2\lambda^{\theta+i}\beta\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\int_0^\infty (x-\mu)^r\,x^{\beta(j+1)-1)}e^{-kx^\beta}dx$$

Recall that

$$(a+b)^n = \sum_p^n \binom{n}{p} a^p b^{n-p} \text{ or } (a+b)^n = \sum_p^n \binom{n}{p} b^p a^{n-p}$$

Hence,

$$(x-\mu)^r = \sum_p^r \binom{r}{p} x^p(-\mu)^{r-p} \text{ or } (x-\mu)^r = \sum_p^r \binom{r}{p} (-\mu)^p x^{r-p}$$

$$\text{E}(x-\mu)^r = \theta^2\lambda^{\theta+i}\beta\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\int_0^\infty\sum_p^r\binom{r}{p}x^p(-\mu)^{r-p}\,x^{\beta(j+1)-1)}e^{-kx^\beta}dx$$

$$\mathrm{E}(x-\mu)^r = \theta^2\lambda^{\theta+i}\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p}^{r} \binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}\binom{r}{p}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}(-\mu)^{r-p}\int_0^{\infty} x^{p+\beta(j+1)-1}\, e^{-kx^{\beta}}dx$$

Recall that

$$\int_0^{\infty} x^m\, e^{-\beta x^n} dx = \frac{\Gamma(\gamma)}{n\beta^{\gamma}}$$

Where $\gamma = \frac{m+1}{n}$, which implies

m = $p+\beta(j+1)-1, \beta = k \; and \; n = \beta$

Therefore,

$$\int_0^{\infty} x^{p+\beta(j+1)-1}\, e^{-kx^{\beta}} dx = \frac{\Gamma\left(\frac{[p+\beta(j+1)-1]+1}{\beta}\right)}{\beta k^{\left(\frac{[p+\beta(j+1)-1]+1}{\beta}\right)}}$$

$$= \frac{\Gamma\left(\frac{[p+\beta(j+1)]}{\beta}\right)}{\beta k^{\left(\frac{[p+\beta(j+1)]}{\beta}\right)}}$$

Therefore,

$$\mathrm{E} \qquad (x-\mu)^r = \theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p}^{r}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}\binom{r}{p}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}(-\mu)^{r-p}\frac{\Gamma\left(\frac{[p+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[p+\beta(j+1)]}{\beta}\right)}}$$

$$=\theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p}^{r}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}\binom{r}{p}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}(-1)^{r-p}(\mu)^{r-p}\frac{\Gamma\left(\frac{[p+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[p+\beta(j+1)]}{\beta}\right)}}$$

$$=\theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p}^{r}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}\binom{r}{p}\frac{[\theta(i+1)]^j}{j!}(-1)^{(i+k)+(r-p)}(\mu)^{r-p}\frac{\Gamma\left(\frac{[p+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[p+\beta(j+1)]}{\beta}\right)}}$$

∀ p=1,2,3,…,r (3.19)

3.4.10 MOMENT GENERATING FUNCTION (MGF) OF THE DISTRIBUTION

The moment generating function (MGF) is given as

$M_x(t) = E\,(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$

Where $f(x) = \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\theta(i+1)-1}{k} \binom{\theta+i}{i} (-1)^{i+k} \frac{[\theta(i+1)]^j}{j!} x^{j\beta} e^{-kx^\beta}$

This implies,

$$M_x(t) = \int_0^\infty e^{tx}\, \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\theta(i+1)-1}{k} \binom{\theta+i}{i} (-1)^{i+k} \frac{[\theta(i+1)]^j}{j!} x^{j\beta} e^{-kx^\beta} dx$$

$$= \theta^2 \lambda^{\theta+i} \beta x^{\beta-1} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\theta(i+1)-1}{k} \binom{\theta+i}{i} (-1)^{i+k} \frac{[\theta(i+1)]^j}{j!} \int_0^\infty e^{tx}\, x^{\beta(j+1)-1} e^{-kx^\beta} dx$$

Using exponential series, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \quad -\infty < x < \infty$

This implies that

$$e^{tx} = 1 + (tx) + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots = \sum_m^\infty \frac{x^m t^m}{m!}$$

Hence,

$M_x(t) = \theta^2 \lambda^{\theta+i} \beta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_m^\infty \frac{t^m}{m!} \binom{\theta(i+1)-1}{k} \binom{\theta+i}{i} (-1)^{i+k} \frac{[\theta(i+1)]^j}{j!} \int_0^\infty x^m x^{\beta(j+1)-1}\, e^{-kx^\beta} dx$

Recall that

$$\int_0^\infty x^{m+\beta(j+1)-1}\, e^{-kx^\beta} dx = \frac{\Gamma\left(\frac{[m+\beta(j+1)-1]+\,1}{\beta}\right)}{\beta k^{\left(\frac{[m+\beta(j+1)-1]+\,1}{\beta}\right)}}$$

$$= \frac{\Gamma\left(\frac{[m+\beta(j+1)]}{\beta}\right)}{\beta k^{\left(\frac{[m+\beta(j+1)]}{\beta}\right)}}$$

$$M_x(t) = \theta^2\lambda^{\theta+i}\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m}^{\infty} \frac{t^m}{m!}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\frac{\Gamma\left(\frac{[m+\beta(j+1)]}{\beta}\right)}{\beta k^{\left(\frac{[m+\beta(j+1)]}{\beta}\right)}}$$

$$= \theta^2\lambda^{\theta+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m}^{\infty}\frac{t^m}{m!}\binom{\theta(i+1)-1}{k}\binom{\theta+i}{i}(-1)^{i+k}\frac{[\theta(i+1)]^j}{j!}\frac{\Gamma\left(\frac{[m+\beta(j+1)]}{\beta}\right)}{k^{\left(\frac{[m+\beta(j+1)]}{\beta}\right)}} \quad (3.20)$$

3.4.11 MAXIMUM LIKELIHOOD ESTIMATION OF THE DISTRIBUTION (MLE)

Maximum likelihood Estimation (MLE) is a widely used method for estimating the parameters of a statistical model. It is a procedure used to determine the values of the unknown parameters.

The Maximum Likelihood Estimation is obtained as follows:

Let $f(x) = \theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda\left(e^{x^\beta}+1\right)\right\}^{\theta-1}\left[1+\left\{\lambda\left(e^{x^\beta}+1\right)\right\}^{\theta}\right]^{-(\theta+1)}$

The likelihood function L = $\prod_{i=1}^{n}\left[\theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda\left(e^{x^\beta}+1\right)\right\}^{\theta-1}\left[1+\left\{\lambda\left(e^{x^\beta}+1\right)\right\}^{\theta}\right]^{-(\theta+1)}\right],$

Then the Log-Likelihood function is given as;

$$\log(\mathrm{L}) = \log\left(\prod_{i=1}^{n}\left[\theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{ \lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta-1}\left[1 + \left\{\lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta}\right]^{-(\theta+1)}\right]\right)$$

$$=\sum_{i=1}^{n} log\left[\theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{ \lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta-1}\left[1 + \left\{\lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta}\right]^{-(\theta+1)}\right]$$

$$=\sum_{i=1}^{n}\left[log\theta^2 + log\lambda + log\beta + logx^{\beta-1} + loge^{x^\beta}\left\{ \lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta-1} + log\left[1 + \left\{\lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta}\right]^{-(\theta+1)}\right]$$

$$=\sum_{i=1}^{n}\left[log\theta^2 + log\lambda + log\beta + logx^{\beta-1} + loge^{x_i{}^\beta}+log\left\{ \lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta-1} + log\left[1 + \left\{\lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta}\right]^{-(\theta+1)}\right]$$

$$=\sum_{i=1}^{n}\left[2log\theta + log\lambda + log\beta + (\beta-1)logx_i + x_i{}^\beta + (\theta-1)log\left\{ \lambda\left(e^{x^\beta} + 1\right)\right\} - (\theta+1)log\left[1 + \left\{\lambda\left(e^{x^\beta} + 1\right)\right\}^{\theta}\right]\right]$$

$$=\sum_{i=1}^{n}2log\theta + \sum_{i=1}^{n}log\lambda + \sum_{i=1}^{n}log\beta + \sum_{i=1}^{n}x_i{}^\beta + \sum_{i=1}^{n}(\beta-1)logx_i + \sum_{i=1}^{n}(\theta-1)log\left\{ \lambda\left(e^{x_i{}^\beta} + 1\right)\right\} - \sum_{i=1}^{n}(\theta+1)log\left[1 + \left\{\lambda\left(e^{x_i{}^\beta} + 1\right)\right\}^{\theta}\right]$$

$$=\sum_{i=1}^{n}2log\theta + \sum_{i=1}^{n}log\lambda + \sum_{i=1}^{n}log\beta + \sum_{i=1}^{n}x_i{}^\beta + (\beta-1)\sum_{i=1}^{n}logx_i + (\theta-1)\sum_{i=1}^{n}log\left\{ \lambda\left(e^{x_i{}^\beta} + 1\right)\right\}-(\theta+1)\sum_{i=1}^{n}log\left[1 + \left\{\lambda\left(e^{x_i{}^\beta} + 1\right)\right\}^{\theta}\right]$$

Let P= log (L), therefore

$$P=2nlog\theta+nlog\lambda+nlog\beta+\sum_{i=1}^{n}x_i{}^\beta+(\beta-1)\sum_{i=1}^{n}logx_i+(\theta-1)\sum_{i=1}^{n}log\left\{ \lambda\left(e^{x_i{}^\beta} + 1\right)\right\}-(\theta+1)\sum_{i=1}^{n}log\left[1 + \left\{\lambda\left(e^{x_i{}^\beta} + 1\right)\right\}^{\theta}\right] \quad (3.21)$$

The derivative of equation (3.21) gives

$$\frac{\partial P}{\partial\theta}=\frac{2n}{\theta} + \sum_{i=1}^{n}log[\mathrm{e}^{-1+x_i^\beta}\lambda] - (1+\theta)\sum_{i=1}^{n}\frac{(\mathrm{e}^{-1+x_i^\beta}\lambda)^\theta log[\mathrm{e}^{-1+x_i^\beta}\lambda]}{1+(\mathrm{e}^{-1+x_i^\beta}\lambda)^\theta} - \sum_{i=1}^{n}log[1+(\mathrm{e}^{-1+x_i^\beta}\lambda)^\theta]$$

$$\frac{\partial P}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}log[x_i] + \sum_{i=1}^{n}log[x_i]x_i^\beta + (-1+\theta)\sum_{i=1}^{n}log[x_i]x_i^\beta - (1+\theta)\sum_{i=1}^{n}\frac{\mathrm{e}^{-1+x_i^\beta}\theta\lambda(\mathrm{e}^{-1+x_i^\beta}\lambda)^{-1+\theta}log[x_i]x_i^\beta}{1+(\mathrm{e}^{-1+x_i^\beta}\lambda)^\theta}$$

$$\frac{\partial P}{\partial\lambda} = \frac{n}{\lambda} + \frac{n(\theta-1)}{\lambda} - (\theta+1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-1+x_i^\beta}\theta(\mathrm{e}^{-1+x_i^\beta}\lambda)^{-1+\theta}}{1+(\mathrm{e}^{-1+x_i^\beta}\lambda)^\theta}$$

The estimates $\widehat{\theta, \beta,}\ \hat{\lambda},$ is obtained by equating $\frac{\partial P}{\partial \theta} = 0,\ \frac{\partial P}{\partial \beta} = 0,\ \frac{\partial P}{\partial \lambda} = 0$ and solve the system of nonlinear equations.

This is achieved using any iterative method with the aid of statistical software packages.

3.5 SUMMARY

Some statistical properties and parameters estimation of the Paralogistic-Chen distribution has been considered in this chapter.

CHAPTER FOUR

ANALYSIS AND INTERPRETATION OF DATA

4.0 INTRODUCTION

This chapter focuses on the practical application of the Paralogistic-Chen distribution on two datasets, comparing its performance with other distributions.

The chapter will also discuss some concepts and methods used in achieving some of the objectives of the study. These include the goodness of fit tests and the criteria for discrepancy, such as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) and others.

4.1 GOODNESS OF FIT TESTS

Consider X1, X2,..., Xn as a random sample extracted from a distribution. A goodness-of-fit test serves as a statistical approach to assess whether a given random sample conforms to a predefined distribution. In this context, three of the goodness of fit tests will be discussed: they include the Kolmogorov-Smirnov (K-S) test, the Cramer-von Mises test and the Anderson-Darling test.

These tests offer means to evaluate the degree of agreement between the observed sample data and the specified distribution, providing valuable insights into the adequacy of the chosen distributional assumption.

4.1.1 KOLMOGOROV-SMIRNOV (K-S)

The Kolmogorov-Smirnov test, often referred to as the K-S test, is a non-parametric statistical test used to compare the distribution of a sample to a reference probability distribution or to compare the distributions of two samples. It is used for testing if a sample comes from a population with a specific distribution.

The Kolmogorov-Smirnov test is conducted by comparing the empirical distribution function of the sample to the cumulative density function of the distribution, essentially measuring the distance between them.

Mathematically, it is denoted by D_n and it is defined as

$$D_n = Supx|F_n - F(x)|$$

Where $Supx$ = Supreme of set of distances

F_n= empirical density function

F(x) = cumulative density function of the specified distribution

4.1.2 CRAMÉR-VON MISES TEST

The Cramér–von Mises criterion W^* is a criterion used for judging the goodness of fit of a cumulative distribution function F^* compared to a given empirical distribution function F_n, or for comparing two empirical distributions. It is also used as a part of other algorithms, such as minimum distance estimation.

It is defined as:

$$W^* = \int_{-\infty}^{\infty} [F_n(x) - F^*(x)]^2 \; dF^*(x)$$

Where $F_n(x)$ is the empirical distribution function. It should be noted that the distribution with the smallest test statistic is considered more better than the others.

4.1.3 ANDERSON-DARLING TEST

The Anderson-Darling test is used to test if a sample of data came from a population with a specific distribution. It makes use of the specific distribution in calculating critical values.

The Anderson-Darling test statistic is defined as:

$$A^2 =- N - S$$

Where S = $\sum_{i=1}^{n} \frac{(2i-1)}{N} [InF(Y_i) + In(1 - F(Y_N + 1 - i))]$

4.2 CRITERIA FOR DISCREPANCY

Criteria for discrepancy, also known as Information Criteria, are methods used to compare statistical models when they are not nested. That is, in situations where adding parameters to a model can improve its fit, these criteria are employed to evaluate and select the most appropriate model among competing alternatives.

Common information criteria include the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Deviance Information Criterion (DIC) and others. These criteria assist in selecting the most appropriate model by balancing goodness-of-fit with model complexity.

Some of the discrepancy criteria methods used include:

1. AKAIKE INFORMATION CRITERION (AIC)

The Akaike Information Criterion (AIC) is a statistical method used for model selection, particularly when comparing non-nested models. It estimates the relative quality of these models based on a balance between goodness-of-fit and model complexity. AIC is an estimator of prediction error and relative quality of statistical models for a given set of data. It compares the quality of each model relative to other models, providing a means for model selection.

The AIC is calculated using the formula:

$$AIC = -2\ln(L) + 2k$$

Where:

L is the maximum likelihood of the model and k is the number of parameters in the model.

2. HANNAN-QUINN INFORMATION CRITERION (HQIC)

The Hannan-Quinn Information Criterion (HQIC) is a statistical method used for model selection, similar to the Akaike Information Criterion (AIC). Its purpose is to identify the best-fitting model among a set of distributions while also adjusting the model complexity.

Mathematically, HQIC is given by the formula

$$\mathrm{HQIC} = -2L_{max} + 2k\ln(\ln(n)),$$

Where,

L_{max} is the log-likelihood, k is the number of parameters, and n is the number of observations.

3. BAYESIAN INFORMATION CRITERION (BIC)

Bayesian Information Criterion (BIC), also known as the Schwarz criterion (SBC or SBIC), is a statistical method used for model selection, similar to AIC and HQIC. Its purpose is to identify the best model among a set of distributions by balancing how well a model fits the data and its complexity.

The BIC is calculated as

$$\mathrm{BIC} = k\ln(n) - 2\ln(L(\theta))$$

Where,

k is the number of parameters, n is the sample size, and L(θ) is the likelihood of the model given the data at maximum likelihood values of θ.

It should be noted that a lower BIC values indicate a better model fit and simplicity.

4. CONSISTENT AKAIKE INFORMATION CRITERION (CAIC)

The Consistent Akaike Information Criterion (CAIC) is a statistical measure use to address the issue of bias in model selection. It balances model fit and complexity, but it adjusts the penalty for model complexity based on the sample size. This adjustment helps provide more accurate estimates of model performance, particularly in situations with small sample sizes, making the CAIC a valuable tool for selecting the most appropriate model among a set of distributions.

The formula for the corrected Akaike Information Criterion (CAIC) is:

$$\text{CAIC} = \text{AIC} + \frac{2k(k+1)}{n-k-1}$$

Where:

- L is the maximum likelihood of the model.
- k is the number of parameters in the model.
- n is the sample size.

4.3 APPLICATION USING LIFETIME DATA

In this chapter, we will analyze the data using two lifetime distributions: the Paralogistic- Chen distribution (ParChD) and the Chen distribution (CHD).

The probability density function (pdf) for the distributions are given respectively:

1. Paralogistic-Chen distribution:

$$f(\text{x}; \lambda, \beta,\theta) = \theta^2\lambda\beta x^{\beta-1}e^{x^\beta}\left\{\lambda(e^{x^\beta}+1)\right\}^{\theta-1}\left[1+\left\{\lambda(e^{x^\beta}+1)\right\}^{\theta}\right]^{-\theta-1}$$

2. Chen distribution: $f(\text{x}; \lambda, \beta) = \lambda\beta x^{\beta-1}exp\left\{x^\beta + \lambda\left(1-e^{x^\beta}\right)\right\}$

Data Set 1

The first data set used in this analysis comes from Lawless (1982) and represents the results of endurance tests on deep groove ball bearings. Specifically, the data consists of the number of million revolutions each of the 23 ball bearings withstood before failing, which were recorded during the life tests and they are:

17.88,28.92,33.00,41.52,42.12,45.60,48.80,51.84,51.96,54.12,55.56,67.80,68.44,68.64, 68.88,84.12,93.12,98.64,105.12,105.84,127.92,128.04,173.40

Data Set 2

The second dataset consists of the daily number of Covid-19 infected persons for 73 days in Nigeria between 20th October and 31st December, 2020. The Covid-19 data set was collected from the national center for diseases control (ncdc) at http://covid19.nedc.gov.ng/

72,37,138,77,48,62,119,113,147,150,170,162,111,72,137,155,180,223,59,300,94,152, 180,212,156,112,152,157,152,236,146,143,246,155,56,168,198,169,246,110,82,145,281,122,343,324,310,318,390,550,474,675,796,617,418,201,758,930,1145,806,920,501, 356,999,1133,1041,784,829,838,397,749,1016,1031

Table 4.1 parameter estimates of the model parameters and maximum of the log-likelihood function for data set 1

Distribution	λ (std error)	β (std error)	θ (std error)	**-LL**
ParChD (λ, β, θ)	0.0137 (0.0192)	0.3180 (0.0938)	2.2044 (1.3512)	113.9521
CH (λ, β)	0.0045 (0.0020)	0.3802 (0.0160)	________	115.2848

Table 4.2 The discrepancy criteria for Data Set 1

Distribution	AIC	CAIC	BIC	HQIC	W^*	A^*	KS (P-Value)
ParChD (λ, β, θ)	233.9043	235.1675	237.3108	234.761	0.0638	0.3483	0.1666 (0.4941)
CH (λ, β)	234.5695	235.1695	236.8405	235.1407	0.1012	0.5934	0.1709 (0.4618)

Table 4.3 parameter estimates of the model parameters and maximum of the log-likelihood function for data set 2

Distribution	λ (std error)	β (std error)	θ (std error)	**-LL**
ParChD (λ, β, θ)	0.0042 (0.0022)	0.3042 (0.0278)	1.0586 (0.2161)	502.607
CH (x; λ, β)	0.00997 (0.00302)	0.25416 (0.00867)	______	505.9506

Table 4.4 The discrepancy criteria for Data Set 2

Distribution	AIC	CAIC	BIC	HQIC	W^{*}	A^{*}	KS (P-Value)
ParChD (λ, β, θ)	1011.214	1011.562	1018.085	1013.952	0.4038	2.3630	0.1349 (0.1405)
CH (x; λ, β)	1015.901	1016.073	1020.482	1017.727	0.6232	3.4837	0.1584 (0.05139)

4.4 REMARKS

The Paralogistic-Chen distribution showed better performance than the standard Chen distribution.

4.5 SUMMARY

The analysis in this chapter reveals that the Paralogistic-Chen distribution is the most effective model compared with the standard Chen distribution, yielding superior results when applied to the lifetime data sets.

CHAPTER FIVE

DISCUSSION AND CONCLUSION

5.0 INTRODUCTION

In this study, we have successfully developed a new lifetime distribution and applied this distribution to various data sets, comparing its performance to existing distribution. It is crucial to summarize our findings with some discussions and draw meaningful conclusions to finalize our investigation.

5.1 DISCUSSIONS

The analysis of two data sets from chapter 4 yielded results presented in tables 4.1, 4.2, 4.3, and 4.4. Comparing the Paralogistic-Chen distribution with other distribution like the Chen distribution, the Paralogistic-Chen distribution showed a higher p-value and a lower K-S value. This indicates that the Paralogistic-Chen distribution outperforms the other distributions, demonstrating its flexibility and superior fit compared to the alternatives.

5.2 CONCLUSION

The results obtained from the analysis indicated that the newly proposed Paralogistic-Chen distribution has shown superior performance compared to the Chen distribution based on information criterion tests. The usefulness and importance of these lifetime distributions extend to various scientific fields, including economics, engineering, medical, environmental, hydrology, and social science, as they can handle more complex data sets. Hence, it can be concluded that the Paralogistic-Chen distribution should be considered as a preferred model as it fit better than the other distributions when dealing with future cases that require modeling of lifetime data.

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