

**SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS AND ITS  
APPLICATIONS**

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**FEBRUARY, 2025**

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**A PROJECT WORK PRESENTED TO THE DEPARTMENT OF  
MATHEMATICS UNIVERSITY OF BENIN, BENIN CITY IN PARTIAL  
FULFILLMENT OF THE REQUIREMENT FOR THE AWARD OF  
BACHELOR OF SCIENCE DEGREE (B.SC) DEGREE IN INDUSTRIAL  
MATHEMATICS**

**FEBRUARY, 2025**

## UNDERTAKING

This project work was carried out by me **ESE EHONWA** with matriculation number **PSC2008282**. I have not plagiarized any existing work. All published work used in this project work have been cited and referenced appropriately.

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**ESE EHONWA**

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Date

## CERTIFICATION

This is to certify that this project work was carried out **ESE EHONWA**, with matriculation number **PSC2008282** under my supervision of in partial fulfilment of the requirement for the award of the Degree of Bachelor of Science in Industrial Mathematics in the Department of Mathematics, Faculty of Physical Science, University of Benin, Benin City.

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Date

## **DEDICATION**

This project work is dedicated to God Almighty for His guidance, mercies, direction and provision throughout the course of this study and to my family for their unwavering love, support, and encouragement throughout my academic journey. Their belief in me has been my greatest motivation.

## ACKNOWLEDGEMENT

I would like to express my sincere gratitude to God Almighty for His grace all through my university journey. To my project supervisor; Dr.(Mrs.) Omoregie, I am deeply grateful for your patience and guidance throughout my research work. Your mentorship was instrumental in helping me navigate the complexities of my research and I am thankful for your encouragement.

Also, I extend my heartfelt appreciation to other lecturers and the HOD PROF. D. OKUONGHAE, in the Department of Mathematics who have in one way or the other been impactful to me. To my parents; Mr. and Mrs. Williams Ehonwa, and my siblings; Aisosa, Ede, Etinosa and Osawese Ehonwa, I am grateful for their unwavering support, belief in my academic pursuits, encouragements and their sacrifices. To my Uncle who I fondly call my big daddy, Mr. Odion Ehonwa, who has been pivotal in my academic pursuits. I am immensely thankful for your financial support, advice, sacrifices and unwavering support in my academic pursuits.

I would also like to acknowledge the contributions of Rev. Fr. John Adesotu, and to my friends turned family Temi, Grace, Blossom, Divine who have supported me and made the journey worthwhile, your support has been crucial to the completion of this project. Thank you all for your help and encouragement.

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## ABSTRACT

Second-order partial differential equations (PDEs) are fundamental in mathematical physics, engineering, and applied sciences. These equations involve second-order derivatives of an unknown function with respect to multiple independent variables. They are broadly classified into three types: elliptic, parabolic, and hyperbolic, based on their characteristic behaviour. Notable examples include the Laplace equation, the heat equation, and the wave equation, each governing essential physical phenomena such as steady-state distributions, diffusion processes, and wave propagation, respectively.

Solutions to second-order PDEs often require analytical or numerical techniques, including separation of variables, Green's functions, Fourier and Laplace transforms, and finite difference methods. Boundary and initial conditions play a crucial role in determining well-posed solutions. Recent advancements in computational methods, such as finite element analysis and deep learning-based PDE solvers, have significantly improved the ability to model complex systems.

## CHAPTER ONE

### 2.0 INTRODUCTION

#### 1.3 BACKGROUND OF STUDY

“Partial differential equations became and remain the heart of mathematics” These were the words of Morris Kline in 1972. Like ordinary differential equations, PDEs are equation to be solved in which the unknown element is a function, but in PDEs the function is one of several variables and of course the known information relates the function and its partial derivatives with respect to several variables.

In many ways, differential equations are the heart of analysis and calculus, two of the most important branches of mathematics for over 300 years. Differential equations are an integral part or a goal of many undergraduate calculus courses. As an important mathematical tool for the physical sciences, differential equation has no equal. So it is widely accepted that differential equations are important in both applied and pure mathematics.

The story begins with the developers of calculus like Sir Isaac Newton, Gottfried William Leibniz and Fermat. In the 17<sup>th</sup> century as soon as these brilliant mathematicians had sufficient understanding and notation for derivatives, derivatives began to appear in equations. However they soon found out that solution for these equations were not so easy. The integral (anti-derivative) and its theoretical role in the First fundamental theorem of calculus which states that. “if  $F$  is continuous on the closed interval  $[a,b]$  and  $F$  is the anti derivative (indefinite integral) on  $F$  an  $[a,b]$  then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

This offered direct help only when the variables were separated in very special circumstances. Brezis and Browder (1998).

Then at the beginning of the 18<sup>th</sup> century came the like of Jacob and Johan Bernoulli, Halley, Riccati, Leonard Euler, Jean d'Alembert and Louis Joseph Lagrange. These men began work on the application of differential equation in astronomy and physical science. Jacob Bernoulli worked on planetary motion using the principles of gravity and momentum developed by Newton especially the development of catenary and the use of polar coordinates. Halley went on to motion of comet that today bears his name. Brook Taylor used series to solved differential equation, his development of finite difference began a new branch of mathematics closely related to the development differential equations. Brezis and Browder, (1998).

From the Encyclopaedia Britanica (vol.13), the study of PDEs started with the study of mechanics (ie the theory of the mechanics of continuous bodies) and physics. PDES gradually drew their lifeblood from this area as well as from heat conduction, fluid mechanics, electromagnetism, quantum mechanics and other parts of physics. From the middle of the 18<sup>th</sup> century contributions came from Leonhard Euler's and Jean d'Alembert's work on wave motion, Pierre-Simon Laplace on potential theory, Jean-Baptists-Joseph Fourier on heat conduction and Friedrich Gauss on potential and magnetic theory. The theory of PDES can always go hand in hand with development in mathematical physics and has drawn much of its strength from the association as seen even in the theoretical works of pure mathematics of distinction such as Augustine-Louis Cauchy, Bernhard Riemann and Sofya Kovalevsky.

Euler though had the benefit of earlier work but the key to his understanding was his knowledge and perception of functions (exponential, logarithmic. Trigonometric and other elementary function) which he used to develop general procedures for solutions of many kinds of equation.

He found out that these function were the keys to understanding differential equation and developing method for their solutions, studying their roles, structures, properties and definition. His technique of conjecturing and finding undetermined coefficient were major steps in the development of this subject. Later in 1739 developed the method of variation of parameter. Zautykov (1955).

We can say that he payed the foundation for others after him to both refine his work and forge entirely new ideas inaccessible to Euler's 18<sup>th</sup> century perspective. From the companion Encyclopedia of the history and philosophy of mathematical science, Jean le Rond D'Alembert was one of them with his work on continuum mechanics where he formulated his first PDE in 1743 though unable to solve it. Three years later, he won a prize at the Berlin Academy of sciences for his work in which he solved a number of PDEs. His most influential work being the solution of the problem on vibrating strings in 1747. Then Lagrange followed closely in Euter's foot step, developing more theory and extending result in mechanics, especially equations of motion (3-body problem) and potential energy. In 1788, he also introduced general equation of motion for dynamic system, today known as Laranges equations.

In 1799, Simon Pierre Laplace introduced the ideas of Laplacian of a function. Then Legendre's work on different equations was motivated by projectile motion and for the first time taking into account new factors, Such as air resistance and initial velocities. Next in line Jean-Baptist-Joseph Fourier with mathematical research made contributions in the study and computation of heat diffusions and the solution of differential equations. Much of that work appears in Fourier's "The Analytical theory of heat" in 1822. In which he made extensive use of the series that bear his name. This result was an important tool for the study of Oscillations. Brezis and Browder (1998).

From the 19<sup>th</sup> century, Carl Friedrich Gauss used differential equations to improve theories on planetary orbit and gravitation, while Augustin-Louis Cauchy applied differential equations to model propagation of waves on the surface of a liquid of which the results are currently a hydrodynamics. He also invented classic in method of characteristics, which is important in the analysis and solution of many PDEs. Being the first to fully define the ideas of convergence and absolute convergence of infinite series and also initiated the rigorous analysis of calculus and differential equation. He is also the first to develop systematic theory for complex numbers and to develop the Fourier transforms to provide algebraic solution to differential equations.

Poisson in his earlier work in 1811, applies differential equations to physics and mechanics including elasticity and vibrations, most of his work being in the analysis and solutions of differential equations. Friedrich Wilhelm Bessel applied his knowledge in differential equations to astronomy. His work on Bessel function was done to analyze planetary perturbations but was later used to solve differential equations. Joseph Liouville was the first to solve boundary value problem which was later refined by Fredholm and Hilbert in the 20<sup>th</sup> century.

Later on from the middle of the 19<sup>th</sup> century, Jacobi, developed the theory of determinants and transformations into a powerful tool for evaluating multiple integrals and solving differential equations. The structure of the Jacobian was developed in 1841. Then in 1876 Lipschitz developed existence theorems for solution of first order differential equations. Then Bernhard Riemann, who obtained his doctorate under the supervision of Gauss and working with physicist Wilhelm Weber, contributed to result in dynamics and physics. Next, Sofya Kovalevsky the greatest woman mathematician prior to the 20<sup>th</sup> century, who overcame discrimination over her gender, had the opportunity to work/study with Weierstrass. Earlier in her research completed

three papers on PDEs. She primarily worked on theory of PDEs and a central result on the existence of solution still bears her name. Brezis and Browder, (1998).

Now in the 20<sup>th</sup> Century, Carl Runge developed numerical methods for the solving different equations that arose in his study of atomic spectra. Then Richard Courant and Garret Birkhoff pioneered successfully implanting numerical methods for differential equations on computers to provide fast, efficient solutions, for large scale, complicated systems on complex geometries. Brezis and Browder (1998).

These among other contributed directly and indirectly to the development of PDEs.

#### **1.4 OBJECTIVES OF STUDY**

- a) To define basic concepts of partial differentiation
- b) To identify types of functions used in partial differentiations.
- c) To explain theorems relating to partial differential equations.
- d) To bring out methods to solving variety of partial differential equations.
- e) To focus on some types of second order partial differential equations.
- f) To apply these equations to physical problems.

#### **1.3 SCOPE OF STUDY**

The scope of this study is designed to cover definitions of differentiation, partial derivatives and function under it, chain rule and Taylor's theorems (one variable and more than one variables). It also covers methods for solving differential equations, some second order partial differential equations (wave equation, Heat equation and Laplace equation) with their respective methods of finding solutions and their applications to problems affecting our world most especially in physics.

## 1.4 SIGNIFICANCE OF STUDY

The project will be of importance:

- i. To undergraduates in mathematics, physics and engineering, to know how to apply concepts used here in their various fields.
- ii. To mathematicians and applied mathematics in their research in mathematical physics or other areas in of mathematics.
- iii. To social scientists, concepts used here could be applied to their field.
- iv. To lecturers in making preparatory notes for their lectures in mathematical methods of partial differential equations.

This can be achieved through research into the topic in question and simplifying them understanding of all that read it, to even non-mathematicians. This is done through the use of examples with brief notes to explain to the last detail what is needed to be known about “partial differential equations”

## 1.5 BASIC DEFINITIONS

### 1. Partial Differential Equation (PDE)

A **partial differential equation** is an equation that involves partial derivatives of an unknown function with respect to two or more independent variables. It is expressed as:

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, \dots) = 0$$

where  $u = u(x, y)$  or  $u = u(x, y, t)$ , and the subscripts represent partial derivatives.

### 2. Second-Order Partial Differential Equation

A PDE is classified as **second-order** if the highest derivative appearing in the equation is of second order. The general form of a second-order PDE in two variables is:

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y, u, u_x, u_y) = 0$$

where  $A, B, C,$  and  $D$  are functions of  $x, y,$  and possibly  $u$  and its first derivatives.

### 3. Classification of Second-Order PDEs

A second-order PDE is classified based on the **discriminant**  $B^2 - 4AC$ :

- **Elliptic PDE:** If  $B^2 - 4AC < 0$ , the equation is elliptic.
  - Example: **Laplace's equation**  $\rightarrow u_{xx} + u_{yy} = 0$
- **Parabolic PDE:** If  $B^2 - 4AC = 0$ , the equation is parabolic.
  - Example: **Heat equation**  $\rightarrow u_t = ku_{xx}$
- **Hyperbolic PDE:** If  $B^2 - 4AC > 0$ , the equation is hyperbolic.
  - Example: **Wave equation**  $\rightarrow u_{tt} = c^2u_{xx}$

### 4. Homogeneous and Non-Homogeneous PDEs

- A **homogeneous PDE** has no independent term (right-hand side is zero):

$$Au_{xx} + Bu_{xy} + Cu_{yy} = 0$$

- A **non-homogeneous PDE** has an independent function  $f(x, y)$  on the right-hand side:

$$Au_{xx} + Bu_{xy} + Cu_{yy} = f(x, y)$$

### 5. Boundary and Initial Conditions

- **Boundary Conditions (BCs):** Additional constraints given on the boundary of the domain where the PDE is solved.
  - **Dirichlet Condition:** Specifies the function's value on the boundary  $\rightarrow u(a, b) = g(x, y)$
  - **Neumann Condition:** Specifies the normal derivative on the boundary  $\rightarrow \frac{\partial u}{\partial n} = h(x, y)$

- **Mixed Condition:** A combination of Dirichlet and Neumann conditions.
- **Initial Conditions (ICs):** Given when the PDE involves time  $t$ , specifying the function's value at  $t = 0$ .

## 6. Laplace's Equation (Elliptic PDE)

A fundamental second-order PDE given by:

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

This equation appears in steady-state heat conduction, electrostatics, and fluid dynamics.

## 7. Heat Equation (Parabolic PDE)

Represents the diffusion of heat over time:

$$u_t = ku_{xx}$$

where  $k$  is the thermal diffusivity.

## 8. Wave Equation (Hyperbolic PDE)

Models the propagation of waves in a medium:

$$u_{tt} = c^2 u_{xx}$$

where  $c$  is the wave speed.

## 9. Method of Separation of Variables

A technique for solving PDEs by assuming the solution can be written as the product of single-variable functions:

$$u(x, y) = X(x)Y(y)$$

This method reduces a PDE into two ordinary differential equations (ODEs).

## **CHAPTER TWO**

### **LITERATURE REVIEW**

#### **2.1. INTRODUCTION**

Second order Partial Differential Equations (PDEs) are equations involving an unknown function of multiple variables and its partial derivatives up to the second order. They play a central role in mathematics physics, engineering and applied sciences as they describe various physical phenomena, such as heat conduction, wave propagation and fluid flow.

In this chapter, we shall be carrying out a literature review concerning second order differential equations in which individuals have worked on in the past and in recent years.

#### **2.2. MAJOR CONTRIBUTIONS MADE BY RESEARCHERS IN THE FIELD OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS**

The field of second order partial differential equations (PDEs) has been significant contributions from mathematics and researchers over the centuries. These contributions have shaped the theory, methods, and applications of PDEs in diverse areas like physics, engineering, and applied mathematics. Below are some major contributions and milestones in this field

Luis Caffarelli, Louis Nirenberg, and Joel Spruck(1980s) In a series of papers during, they addressed the Dirichlet problem for nonlinear second-order elliptic equations, significantly advancing the understanding of fully nonlinear PDES. Their work primarily focused on fully nonlinear elliptic equations, particularly Monge-Ampère type equations and curvature equations that arise in differential geometry and geometric analysis.

Lawrence C. Evans His book "Partial Differential Equations," first published in 1998 with a second edition in 2010, offers a comprehensive survey of modern techniques in the theoretical study of PDEs, emphasizing nonlinear equations.

Mohammed S. Mechee, Fudziah Ismail, Zurni Omar, Zulaiha Ali Othman (2015) in their research in the Study of General Second-Order Partial Differential Equations using Homotopy Perturbation

Method. These authors demonstrated the efficiency and accuracy of HPM in obtaining approximate solutions that align well with exact solutions. The research paper explores the application of the homotopy perturbation method (HPM) to solve generalized linear second-order.

Giuseppe Da Prato and Jerzy Zabczyk. "Second Order Partial Differential Equations in Hilbert Spaces" (2002) This work delves into the analysis of second-order PDEs within the framework of Hilbert spaces, providing insights into the theoretical underpinnings and applications of these equations in functional analysis.

Robert L. Herman (2022) in his work introduces several generic second-order linear PDEs and explores how these equations naturally lead to the study of boundary value problems for ordinary differential equations. It emphasizes the occurrence of these generic differential equations in one to three spatial dimensions and discusses their linear nature.

Andrea Pezzi Geometric and Functional Considerations" (2017). Pezzi's paper addresses second-order linear PDEs with constant and non-constant coefficients in two variables that admit real characteristics. The study focuses on understanding these PDEs from both geometric and functional perspectives, providing insights into their structure and solutions.

Jean-Baptiste Joseph Fourier (1768-1830). Developed the Fourier series and methods for solving the heat equation, laying the groundwork for PDE analysis in physics. His work on the heat equation was one of the first systematic treatments of second-order parabolic PDEs.

Abarbanel, S. and Ditkowski, A. (2000) in their research paper, studied the temporal behavior and rate of convergence of error bounds of finite difference

approximation to partial differential equations. They determined the dependence of the error bounds on mesh size and time. For this purpose, hyperbolic and parabolic partial differential equations are used.

Islam, M.R. and Alias, N. (2010) used the finite difference method to discretize a parabolic partial differential equation. They presented a mathematical simulation model using one dimensional parabolic equation. This model is regarding to moisture and temperature behavior of tropical herbs during dehydration. Here, Jacobi, Gauss Siedel and Red black Gauss seidal iterative methods are studied. It has proved that dehydration model is capable to simulate mass and temperature distribution through numerical methods approach. This mathematical simulation is time consuming and capable to reduce the risk of real experiments in actual process.

## CHAPTER THREE

### METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

#### 3.1 DIRECT INTEGRATION

This method is the easiest used to solve the simplest form of partial differential equations.

Example 3.1.1

If given that  $\delta^2 u / \delta y^2 = 0$ , solve for  $U(x,y)$

Solution

To obtain  $U(x,y)$ , we integrate firstly w. r. t.  $y$

$$\int \delta^2 u / \delta y^2 = \delta u / \delta y = f(x)$$

Then integrate again w. r. t.  $y$

$$U(x, y) = yf(x) + g(x)$$

#### 3.2 INITIAL VALUE AND BOUNDARY VALUE PROBLEMS

As with any differential equations the arbitrary constants or functions in any particular case are determined from additional information given concerning the variables of the equation which are called initial value and boundary value conditions.

Example 3.2.1

Solve  $\delta^2 u / \delta y^2 = 12x^2(t + 1)$  given that  $x = 0, u = \cos 2t$ , and  $\delta u / \delta x = \sin t$

Solution

$$\delta^2 u / \delta x^2 = 12x^2(t + 1)$$

Integrating both sides w.r.t x

$$\int \delta^2 u / \delta x^2 dx = \int 12x^2(t + 1)$$

$$\Rightarrow \int \frac{\delta u}{\delta x} = 4x^2(t + 1) + f(t) \quad (1)$$

Where the arbitrary function  $f(t)$  takes the place of the normal arbitrary integration.

Integrate again w.r.t x

$$U(x, y) = x^4(t + 1) + xf(t) + g(t) \quad (2)$$

Where  $g(t)$  is the second arbitrary function. To find the two arbitrary functions  $f(t)$  and  $g(t)$ , we apply the given initial conditions that at  $x = 0$ ,  $u = \cos 2t$ , and  $\delta u / \delta x = \sin t$ .

Substituting  $x = 0$ , and  $\delta u / \delta x = \sin t$  in (1)

$$\sin t = 4(0)^3(t + 1) + f(t)$$

$$\sin t = 0 + f(t)$$

$$f(t) = \sin t$$

Substituting  $x = 0$  and  $u = \cos 2t$  in equation (2)

$$\cos 2t = 0^4(t + 1) + 0 \sin 0 + g(t)$$

$$\cos 2t = g(t)$$

$$f(t) = \sin t, g(t) = \cos 2t$$

$$\therefore U(x, y) = x^4(t + 1) + x \sin t + \cos 2t$$

## BOUNDARY VALUE PROBLEM

Historically, the name boundary value problem was attributed to only problems for which the P.D.E was of the elliptic variety. The first boundary value problems to be studied in detail involved the potential equation (Laplace equation) and are now referred to as boundary value problems of the first and second kinds.

The Dirichlet problem is the boundary value problem of the first kind. If  $R$  denoted a region in the plane and  $C$  its boundary curve, then, this problem is characterized by

$$\nabla^2 u = 0 \quad \text{in } R$$

$$u = f \quad \text{on } C$$

Where we specify the value of  $u$  at each point of the boundary. An example is to find the steady-state temperature distribution in a region  $R$  given that the temperature is known every on the boundary of  $R$ .

And the Newman problem is the boundary value problem of the second kind it is characterized by

$$\nabla^2 u = 0 \quad \text{in } R$$

$$\frac{\delta u}{\delta \eta} = f \quad \text{on } C$$

The derivative  $\frac{\delta u}{\delta \eta}$  is called the normal derivative of  $u$  and is positive in the direction of the outward normal to the boundary curve  $C$ . in steady-state temperature problems, the normal derivatives specifies the heat flow across the boundary of  $R$ .

There is the boundary value problem of the third kind, which is Robins problem, also called the mix portion of C is used. But we cannot give separate treatment of it at this level. In most potential problems, the region R is a bounded region and the problem is then called an interior problem. An exterior problem arises when the region R is unbounded but contains a hole.

### 3.3 SEPARATION OF VARIABLES

Normally in the solutions of differential equations especially of the first order separation of variable is dealt with by taking like term together. Nut for a second order P.D.E, we seek a solution of  $U(x, y, z, t)$  (expressed in Cartesian coordinates) and it has the product form

$$U(x, y, z, t) = X(x)F(y)Z(z)T(t)s$$

This form of solution is said to be separate in x,y,z and t, this method is mostly used in the solution of the wave equation of which we shall go into detail in the next chapter.

Let us consider this example

#### Example 3.3.2

Find the solution of the equation

$$\delta^2 u / \delta x^2 = \frac{1}{C} \delta^2 u / \delta t^2 \quad (1)$$

Which is periodic in x and t and satisfies the following boundary and initial conditions.

$$U(0, t) = U(l, t) = 0; t \geq 0 \quad (2)$$

$$U(0, t) = f(x), 0 \leq t \leq 1 \quad (3)$$

$$\left(\frac{\delta U}{\delta t}\right)_{t=0} = g(x), \quad 0 \leq t \leq 1 \quad (4)$$

For now we are only going to concern ourselves with equation (1) and leave the rest for the next chapter. We want to concentrate on using separation of variables to break the P.D.E to a number of ordinary differential equations then solving is made easier.

### Solution

Let us assume the solution of the form

$$U(x, t) = X(x)T(t)$$

Where  $X(x)$  is a function of  $x$  and  $T(t)$  is a function of  $t$ . we now differentiate and substitute in equation (1) above, we have

$$\frac{\delta u}{\delta x} = X'(x)T(t); \quad \delta^2 u / \delta x^2 = X''(x)T(t)$$

$$\frac{\delta u}{\delta t} = X(x)T'(t); \quad \delta^2 u / \delta t^2 = X(x)T''(t)$$

Substituting in (1), we have

$$X''T = \frac{1}{C^2}XT''$$

Then we divide through by  $XT$

$$\frac{X''}{X} = \frac{1}{C^2} \cdot \frac{T''}{T} = \lambda$$

We now choose our eigenvalue,  $\lambda$ , to evaluate them to, we will use  $\lambda = -\omega^2$ , we now have,

$$\frac{X''}{X} = \frac{-\omega^2}{C^2} \quad (a)$$

$$\frac{T''}{T} = -\omega^2 \quad (b)$$

Equation (a) and (b) above now gives us two ordinary differential equations to solve. They are.

$$X'' + \frac{\omega^2}{C^2}X = 0, T'' + \omega^2T = 0$$

Where we have the following solutions

$$X(x) = A\cos \omega/c x + B\sin \omega/c x$$

$$T(t) = C\cos \omega t + D\sin \omega t$$

The solution using this method in some cases could involve the use of Fourier series, although the above example treated a second order P.D.E (wave equation), the steps to take are as follows.

- a. By applying the method of separating variables or product method we shall obtain two O.D.Es.
- b. We shall determine the solutions of these two equations that satisfy the boundary conditions.
- c. Using fourier series to compose the solution in order to get a solution of the equation that satisfies the conditions in step 2. (This will be explained further in section 3.6)

Note: this method was developed by Daniel Bernoulli in 1741.

### 3.4 SUPER POSITION PRINCIPLE

We are going to state a principle which will be used in solving various differential equations though we may not show any example but it is involved one way or another in the solution that do come up in the equation that we are going to treat in chapter 4.

The super position principle states that if  $u_1, u_2, \dots, u_k$  are solutions of a homogeneous linear P.D.E. in some domain R then the linear combination

$$U = C_1U_1 + C_2U_2 + \dots + C_kU_k$$

Where,  $C_1, C_2, \dots, C_k$  are constants is also a solution in R.

### 3.5 D'ALEMBERT SOLUTION

This method of solution was developed by Jean LeRond D'Alembert in 1747 to be applied to a large class of equations, but was not so. This method is mostly applied to hyperbolic equations like the wave equation.

Let us start by considering the linear second order P.D.E. of z variables of the form

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + HU = 0 \quad (1)$$

Where A, B, C, D, E and H are coefficients of the equation and functions of independent variables x and y.

(1) can also be written as

$$AU_{xx} + BU_{xy} + CU_{yy} + F(x, y, u, p, q) = 0 \quad (1)$$

Where  $P = U_x$  and  $q = U_y$

And

$$F(x, y, u, p, q) = DU_x + EU_y + HU$$

To solve the above equation we reduce it to a simpler form called the canonical form where we shall then solve by direct integration we use the principal part also known as Euler's equation i.e.

$$AU_{xx} + BU_{xy} + CU_{yy} = 0$$

We then use the determinant  $B^2 = 4AC$  to classify the equations on whether they are elliptic, hyperbolic and parabolic.

- i. If  $B^2 = 4AC > 0$ : the equation is hyperbolic and the roots are real.
- ii. If  $B^2 = 4AC < 0$  the equation is elliptic and the roots are complex
- iii. If  $B^2 = 4AC = 0$  the equation is parabolic and the roots are equal.

Then we introduce two new independent variables for example, for the equation

$$U_{xx} - \frac{1}{c^2}U_u = 0$$

We introduce  $V = x + ct$  and  $Z = x - ct$ ; V and Z are functions of x and t in the equation and are use to form a new P.D.E or transform the former P.D.E into a new one with U being a function of V and Z, then after the transformation and substitution of the new equation into the former one, we now reduce the new equation to its canonical form (simplest form) like

$$4 \frac{\delta^2 u}{\delta v \delta z} = 0$$

Which will now be,

$$\frac{\delta^2 u}{\delta v \delta z} = 0$$

Integrate w.r.t.  $z$ , we have,

$$\frac{\delta u}{\delta v} = h(v)$$

Where  $h(v)$  is the arbitrary function of  $v$ , integrate again w.r.t.  $v$ , we have

$$\begin{aligned} U &= \int h(v) dv + g(z) \\ &= h(v) + g(z) \end{aligned}$$

Where  $g(z)$  is the arbitrary function of  $x$ .

We now substitute back with  $v = x + ct$  and  $z = x - ct$

$$U(x, y) = h(x + ct) + g(x - ct)$$

Further examples will be used in chapter four to show this method can be applied to the solution of partial differential equations.

## 3.6 FOURIER SERIES, INTEGRALS AND TRANSFORMATION

### 3.6.1 FOURIER SERIES

In using Fourier series to solve a P.D.E we shall follow the step stated in section 3.4 which are;

1. By applying the so-called method of separating variables or product method, we shall obtain two ordinary differential equation
2. Then we shall determine solutions of these two equations that satisfy the boundary conditions.

3. We finally use Fourier series to compose the solutions in order to get the solution of the equation that satisfies the condition in step 2.

We will use the above steps to find solutions to the PDEs, we will discuss in chapter four, most especially in the wave and heat equations but this method is not generally like other.

### 3.6.2 FOURIER INTEGRAL

In this case we use Fourier Integrals instead of Fourier series in the function in question is not assumed to be periodic using the same method as in the preceding section. This we shall also show in chapter four.

The representation of the function  $f(x)$  by a Fourier integral is

$$f(x) = \int_{-\infty}^{\infty} [A(x) \cos sx + B(x) \sin sx] dx \quad (1)$$

$$A(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos st dt \quad (2)$$

$$B(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin st dt \quad (3)$$

This is derived from the Fourier integral representation.

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos[s(t-x)] dt dx \quad (4)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) [\cos st \cos sx + \sin st \sin sx] dt dx \quad (5)$$

Which we shall use in the next section in the definition of a Fourier transform

We can also deduce that

$$f(x) = \int_{-\infty}^{\infty} A(x) \cos sx \, dx \quad (6)$$

If  $f(x)$  is an even function, where

$$A(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos sx \, dx = A(s) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos sx \, dx$$

$$B(s) = 0$$

Similarly, if  $f(x)$  is an odd function, where  $A(s)=0$

$$B(s) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin sx \, dx$$

$$F(s) = \frac{2}{\pi} \int_0^{\infty} B(s) \sin sx \, dx$$

(6) is called the Fourier cosines integral representation of  $f(x)$  and (7) is the Fourier sine integral representation of  $f(x)$ .

### 3.6.3 FOURIER TRANSFORMS

Like Laplace transforms, Fourier transforms are used in the solutions of Differential equations without the need of solving separately using boundary conditions, that is, solving for homogeneous and particular solutions.

The fourier transform of a function  $f(t)$  is defined by

$\mathcal{F}\{f(t); s\} = F(s)$  and defined as follows

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-st} f(t) dt$$

While the inverse transform denoted by  $\mathcal{F}\{f(t); t\} = f(t)$  and defined as follows;

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-st} F(s) dt = f(t)$$

Which is derived from equation (4) in section 3.6.2

From equation (6) in section 3.6.2 we can obtain the Fourier cosine transform denoted by

$\mathcal{F}\{f(t); s\} = F_e(s)$  and defined as

$$F_e(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos st dt, \text{ where } s > 0$$

And the inverse cosine transform

$$F_e^{-1}(s) = f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_e(s) \cos st dt = f(t), \text{ where } t > 0$$

The above Fourier integral representation occurs if  $f$  is an even function.

But in the case of the function  $f$  being an odd function we derive from equation (7) in 3.6.2 the

following Fourier integral representation, that is Fourier sine transform denoted by  $\mathcal{F}\{f(t); s\} =$

$F_e(s)$  defined by

$$F_e(s) = F(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin st dt, \text{ where } s > 0$$

While the inverse sine transform is thus

$$F_e^{-1}(s) = f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_e(s) \sin st \, dt = f(t), \text{ where } t > 0$$

### 3.7 LAPLACE TRANSFORM

Laplace transforms are used to solve partial differential equations as well as ordinary differential equations as well as ordinary differential equations. The steps required to solve partial differential equations are almost similar to solving ordinary differential equations. To achieve this we take the following steps;

1. Take the Laplace transform with respect to one of the two variables, usually  $t$ , (this is because it mostly represents time and  $t$  is always greater than zero, i.e.  $t > 0$ ). This gives an ordinary differential equation for the transform of the unknown function. This is so since derivatives of this function with respect to the other variable slips into the transformed equation. The later also incorporates the given boundary and initial conditions.
2. Solving that ordinary differential equation, obtain the transform of the unknown function.
3. Taking the inverse transform, obtain the solution of the given problem.

The above steps could also be used for Fourier transform. Though there are similarities between the transform method and the Fourier integral method, but the most notable distinction is that the transform method eliminates the need to deal with eigenvalue problems of Fourier representation.

We shall treat this in chapter four in the applications to wave and heat equations.