Mathematical Model on Harvesting Strategies in Itebukunmi Fishing Ground of Nigeria

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A THESIS WRITTEN IN THE DEPARTMENT OF MATHEMATICS AND SUBMITTED TO THE COLLEGE OF POSTGRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF DEGREE OF DOCTOR OF PHILOSOPHY IN INDUSTRIAL MATHEMATICS OF THE UNIVERSITY OF BENIN, BENIN CITY, EDO STATE, NIGERIA.

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CERTIFICATION

This is to certify that this thesis was written by Edamisan AMUSEGHAN with Matriculation Number PG/PSC1818852 in the Department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, Nigeria under the supervision of Prof. K. O. MUKA and Prof. C. I. Nkeki.

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CERTIFICATION OF THESIS ON PLAGIARISM

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DEDICATION

This thesis work is dedicated to the glory of God Almighty, the creator of heaven and earth.

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ABSTRACT

Itebukunmi is a riverine community in Ondo State Nigeria, known for its high traffic in fishing activities. The economic importance of fishing activities in Itebukunmi to the Nigerian economy necessistate the need to study the harvesting strategy of fishing in Itebukunmi waters and where necessary determine scientifically regulatory policy that will ensure sustained growth in population.

The Mathematical model of three species of fishes in Itebukunmi couple with human activites was derived using systems of ordinary differential equations. The qualitative analysis of the model such as the local, global, stability and bioeconomic analysis were done using linearization approach and bifurcation analysis.

The result of the quantitative analysis showed that : as the control of the harvesting rate of cat fish increases, the population of cat fish and African knife fish, in Itebukunmi river increases, while the Ophiocephalus fish population decreases. Also, as the control of the harvesting rate of Ophiocephalus fish increases, the population of Ophiocephalus fish and African knife fish, in Itebukunmi river increases, while the cat fish population decreases. Furthermore, as the control of the harvesting rate of African knife fish increases, the population of African knife fish , in Itebukunmi river increases, while the cat fish and Ophiocephalus fish population decreases

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CHAPTER ONE

Introduction

1.1 Background to the study

Fishing is one of the major occupation of the Ijaw Apoi, Ijaw Arogbo and the Ilaje people that occupy the riverine area of Ondo state, Nigeria. Itebukunmi people are a sub-group of the Ilaje ethnic group in Ilaje Local Government Area of Ondo state, Nigeria. They are traditionally coastal and riverine people. They are among the Ilaje tribe who settled along the Atlantic coast line of the present Ondo state, Nigeria. Itebukunmi lies close to other Ilaje towns or villages such as Igbo-Egunrin and Mahintedo. It lies in the Latitude of about 6.44463°N and longitude 4.59604°E The Itebukunmi river (as a good fishing ground), comprised a lot of species such as Tilapia, Fresh water cat fish, Clarias cat fish, Ophiocephalus, Heterotis, gymnachus niloticus, Hepseptus, Momyrops, Alestes etc. Decline of disappearance of some species once common on the river are Herring, Hake, Electric fish etc, as noted by local fishers-suggesting local extirpations even if not recognised as formally extinct. This particular study focuses on cat fish, Ophiocephalus fish and gymnachus niloticus species. Furthermore, fishery has a lot of benefits to humans, such as it serves as food, creates income and job opportunities. In general, fishery has great impact on the socio-economic and infrastructural development of any country. According to the United Nations (2023), marine fisheries directly or indirectly employ more than two hundred million (200,000,000) people worldwide, and the livelihoods of more than three billion people depend on marine and coastal biodiversity.

However, as much as forty percent (40%) of the world’s oceans are heavily affected by human activities, including pollution, overfishing, and coastal development, (United State Census Bureau Report (2022). These activities may result in depleted fisheries and loss of coastal habitats, posing a major threat to marine biodiversity and the food supply of millions of people.

Of the six hundred (600) marine fish stocks monitored by the United Nations (U.N), Food and Agriculture Organization (FAO), more than seventy percent (70%) of the world's fish species are fully exploited or depleted, (United State Census Bureau Report (2022).

The contribution of fisheries and aquaculture to food security, nutrition, and livelihoods, both now and in the future, depends on many factors including economic, environmental, governance, policy, and social justice issues, (United State Census Bureau Report (2022). Managing fish populations, sustainably requires commitment and cooperation at all levels, including from individuals, local communities, government, and institutions across the globe. Sustainable fishing practices ensure that fish populations remain healthy and productive, environmental impacts are minimized, and that those who depend on fishing can maintain their livelihoods.

A popular fishing activity widely reported in Nigeria is the Argungu fishing festival, Taylor (2021). The Argungu fishing festival is an annual four days festival in Kebbi state, in the North-West Nigeria. The festival began in the year 1934. The festival has brought huge progress to the development of Argungu town in the area of the fishing festival, the state and Nigeria in particular. Argungu fishing festival of Nigeria is the Africa's biggest fishing celebration (Taylor, 2021). It is an annual event that takes place between late February and March to mark the end of the farming season and the start of the fishing season.

Also in Nigeria, especially the southern Nigerian waterways also have activities that are not widely reported. Although, different species are known in each water bodies. These species includes:

List of Common Fish Species Found in Nigerian Waterways

(i) Tilapia
(ii) Fresh water Cat fish
(iii) Clarias Cat fish
(iv) Hemichromics
(v) Heterotis
(vi) Gymnachus niloticus
(vii) Aurus
(viii) Hepseptus
(ix) Electric fish
(x) Megalops, Tappon
(xi) Mullet
(xii) Alestes
(xiii) Momyrops
(xiv) Shine nose
(xv) Mudskipper
(xvi) Pike Ophiocephalus Turbot
(xvii) Moray eel
(xviii) Crab Eel
(xix) Cray fish
(xx) Shrimps
(xxi) Million fish
(xxii) Lobster
(xxiii) Periwinkle
(xxiv) Shellfish
(xxv) Manati
(xxvi) Cat fish
(xxvii) OPhiocephalus fish

Source: ADP (2004) Ondo State Agricultural Development Project.

Figure 1.1: Pictures of some fish species, local canoe used for fishing, some of the river terminals, and traps used for catching fishes about Itebukunmi river are shown(See Appendix Figure A1-A10 for details)

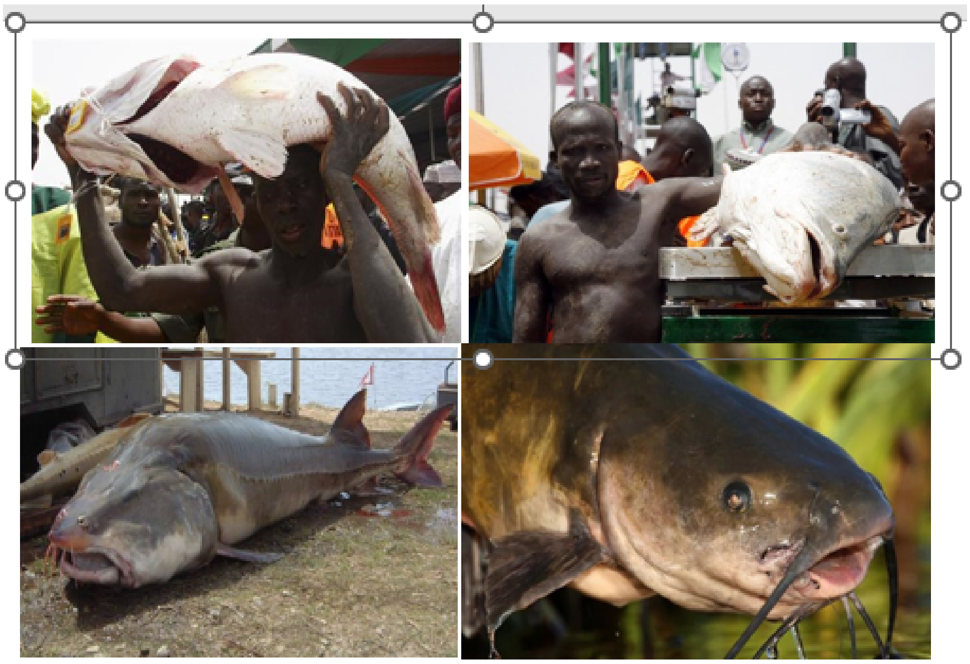


Figure 1.2: Some big fishes caught at Argungu Festival in Nigeria as photographed and reported (Taylor, 2021).

One of the reasons why big fishes grow so large in Argungu river is that the festival rules include a prohibition on fishing in the key river stretch for most of the year, so fish stock are protected and allowed to grow larger than they otherwise might under constant fishing pressure.This deliberate conservation means: large fish accumulate in the designated river stretch. Only during the festival can they be caught. Hence local fisheries managers aim to balance tradition with sustainability. Hence, there is need for proper management so as to guide against extinction of fish species. This is achievable by seeking for best harvesting strategies that would ultimately ensure sustainability.

Table 1.1: Ecological Relationship (Kralles, 2018)

Type of Interaction	**Oragnism 1**	**Organism 2**
Mutualism	+	+
Commensalism	+	0
Competition	-	-
Predation (includes Parasitism)	+	-

Key:

- **“+” = benefits organism**
- **“-” = harms organism**

- **“0” = no effect**

1.2 Basic Terminology in Fishery

Fishery (fisheries): Fishing activities; the catching, processing and marketing of fish or other seafood.

Fishing ground: A place where fish or other seafood are caught.

Harvesting: The removal or gathering of (mature) fish species.

Prey: A living thing that is eaten by another living thing.

Predation: According to Jawad (2018), an organism (predator) that feeds on another organism (prey) for food. In this interaction the existence of prey enhance the predator, while the latter might threaten the presence of the prey. Typical example of the former are bats eating insects and snakes eating mice.

Functional Responses: According to Abrams and Ginzburg (2000), functional responses are the comsumption rate of an individual predator on the prey density. Functional responses are generally categorized, such as prey density-dependent $f(N)$, ratio-dependent $f(\frac{N}{P})$ and prey-predator density dependent $f(N, P)$ and where N is the prey density, P is the predator density. Holling type I and II responses are positive constants that describes the effect of capture rate and time used on feeding rate of the predator on prey

1.3 Statement of the Problem

The Itebukunmi fishing ground in Ilaje local government area of Ondo state, Nigeria is an important source of livelihood and food for local communities. However, increasing fishing pressure and reliance on traditional, non-scientific harvesting practices have led to declining fish stock and reduced catches. The lack of a location-specific mathematical framework makes it difficult to evaluate the effects of the different harvesting strategies and ensure sustainable fish exploitation. Therefore, there is a need for a mathematical model to analyze fish population dynamics and harvesting strategies in the Itebukunmi fishing ground, with the aim of supporting sustainable fisheries management and long-term resource conservation, Hence the research questions that arises are:

Fishery influences the socio-economic value of any nation. How does the functional responses for each prey species affect/changes the rate at which the population goes to equilibrium and also the equilibrium values?

How would including fisheries management strategies in formulation of mathematical model for prey-predator modeling?

What would this do to the rate at which the population goes to equilibrium? What would the equilibrium values be?

Is there any parameter(s) to be included or checked/controlled to help the management control harvesting rate?

Developing a Mathematical model by a management is different from an Academic proposing a mathematical model. Hence, a model in this respect should be developed from management point of view. A model that is plausible and realistic.

To control human activities in a fishing ground, this would require a good mathematical model that incorporate a wide range of prey-predator factors to be able to come up with a reasonable analysis of the dynamics of the species involved in a study. Hence, Some highlights of the areas that have not been explored in the discussed literature are hereby presented: Naturally, the response of a prey fish to a predator fish differs. Hence prey fish should have a different functional response. The intensity of human activities to extract the fish, human effort (E) should be regulated through mathematical function (to be included in the model). A control variable of every fishery management is the fishing effort, which is defined as a measure of the intensity of fishing operations. In general, fishing effort is regulated by quotas, trip limits and gear restrictions, (Edels and wang, 2016) Parameters, to increase growth rate as to increase harvesting rate were not captured in literature to the best of our knowledge. Mathematical models exist for aquatic life in Lake Victoria, and other rivers as seen in the literature.

Itebukunmi fishing ground which plays vital role in the economic growth in Ondo state in Nigeria has no research activities done that could help boost and maintain its aquatic ecosystem to the best of my knowledge.

1.4 Aim and Objectives

Aim:

The aim of this study is to formulate Mathematical models on harvesting strategies in two-prey one-predator system in Itebukunmi riverine area of Ilaje Local Government Area, Ondo State, Nigeria.

The objectives are to:

1. determine an optimal management control in fishing activities in Itebukunmi
2. determine effect of the parameters therein how they affect the dynamics of a two-prey one-predator system of Itebukunmi fishing ground in Ilaje Local Government Area, Ondo State.
3. provide insights in policy formulations and proffer control strategies for policy decision makers in Itebukunmi fishing ground.

1.5 Organization of the thesis

The thesis is organized in five chapters. Chapter one gives the background to the study, statement of the problem, motivation and objectives of the study. Chapter two presents the Literature review, focusing on two prey - one predator systems. Chapter three deals with formulating and qualitative analysis of the model while Chapter four deals with numerical simulation of the model. Chapter five outlines findings, contribution to knowledge, conclusion and recommendations.

CHAPTER TWO

Literature Review

2.1 Introduction

Ecological balance in populations of predictors and prey has profound impact in the overall existence of human habitat. Therefore, the need to study how different species grows, compete for food and human interventions or interruptions in the ecological balance is vital for management of natural resources. On most important in the study of aquatic life of fishes and the predictors and human involvements. Among the most foundational contributions to mathematical ecology is the work of Lotka (1925) and Volterra (1927), whose independently derived models laid the groundwork for the formal study of prey-predator systems. Alfred James Lotka, writing from a physical biology perspective, and Vito Volterra, responding to a real-world ecological puzzle posed by the proportional shifts in predatory and prey fish catches in the Adriatic Sea following the First World War, each arrived at a remarkably similar system of nonlinear ordinary differential equations. Their combined effort gave birth to what is now widely known as the Lotka-Volterra model, perhaps the most celebrated model in mathematical ecology.

The biological problem that motivated Volterra's formulation was both practical and intellectually compelling. After the war, Italian fisheries biologists noticed that the proportion of predatory fish in catches from the upper Adriatic Sea had increased noticeably, while prey fish proportions had declined. Volterra sought to explain this paradox using mathematical reasoning. His model, which describes the temporal evolution of prey and predator populations through coupled differential equations, showed that the equilibrium between the two populations is inherently periodic, and that a disruption such as a wartime halt to fishing naturally leads to a temporary increase in predator populations. This insight was groundbreaking.

The classical Lotka-Volterra two-species prey-predator model is expressed as:

$$\frac{dx}{dt} = x(a - by) \tag{2.1.1}$$

$$\frac{dy}{dt} = y(-c + dx) \tag{2.1.2}$$

where x and y represent the population densities of the prey and predator species respectively, and a, b, c, d are positive constants describing intrinsic growth rates, interaction coefficients, and conversion efficiencies. The model reveals cyclic population oscillations, a defining feature of many prey-predator systems found in nature.

Since this foundational formulation, the mathematical modelling of prey-predator systems has grown enormously in scope and sophistication. Researchers have progressively relaxed the simplifying assumptions of the Lotka-Volterra model — such as the absence of carrying capacity, the linearity of the functional response, and the neglect of human intervention — to develop models that are more biologically realistic and applicable to specific ecosystems. These extensions have introduced concepts such as logistic prey growth, density-dependent competition, Holling-type functional responses, time delays, age structure, environmental stochasticity, and harvesting, all of which enrich our understanding of ecological dynamics.

It is within this broader intellectual tradition that the present study situates itself. Itebukunmi fishing ground, located within the Ilaje Local Government Area of Ondo State in Nigeria, is a riverine ecosystem of significant ecological and economic importance. The community depends heavily on fishing as a primary source of livelihood, yet mathematical modelling of fish population dynamics in this specific environment remains largely absent from the literature. This chapter therefore reviews existing mathematical models for prey-predator systems with harvesting, drawing on the key contributions that have shaped the field, and identifies the gaps that the present work addresses.

2.2 Mathematical Models for Prey-Predator Systems with Harvesting

The integration of harvesting into prey-predator models marks a significant transition from purely ecological inquiry to applied fisheries mathematics. Harvesting, in this context, refers to the deliberate removal of fish or other organisms by human agents, typically for commercial, subsistence, or recreational purposes. When modelled appropriately, harvesting introduces an additional loss term into the population dynamics equations, and the study of its effects on system stability, equilibria, and long-term sustainability becomes of paramount importance.

A broad generalization of the Lotka-Volterra framework was proposed by Freedman (1980), who presented a more flexible prey-predator model given by the system:

$$\frac{dx}{dt} = x \cdot g(x) - y \cdot f(x) \tag{2.2.1}$$

$$\frac{dy}{dt} = y[-e + p(x)]... \tag{2.2.2}$$

where$x$and y denote the population densities of the prey and predator, respectively. The function $g(x)$ represents the per capita growth rate of the prey in the absence of predators, $f(x)$ is the functional response of the predator to prey density, and $p(x)$ is the numerical response of the predator. These functions are assumed to be continuous, differentiable, and to satisfy several biologically meaningful conditions: $g(x)$ is a decreasing function with a zero at the carrying capacity k; $f(0) = 0$ with $f'(x) > 0$; and$p(0) = 0$ with $p(x)$ monotonically increasing and bounded above.

This generalization was important because it allowed researchers to explore a much wider family of prey-predator models by specifying different forms of the functional response, rather than committing to the linear predation assumed in the Lotka-Volterra model.

2.2.1 One Prey-One Predator Model with Harvesting

An important early contribution to the study of harvesting in a prey-predator context was made by Kar (2003), who examined a delayed prey-predator model featuring Holling Type II functional response and harvesting applied exclusively to the prey. The rationale for this particular choice of functional response was biological: unlike the linear (Holling Type I) response, which implies unlimited feeding capacity, the Holling Type II response saturates as prey density increases, reflecting the time a predator requires to handle and consume each captured prey item.

Kar's model without delay may be written compactly as:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{cxy}{(m+x)} - qEx \tag{2.2.3}$$

$$\frac{dy}{dt} = y\left[-d + \frac{fx}{(m+x)}\right] \tag{2.2.4}$$

where r is the intrinsic growth rate of the prey, k is the carrying capacity, c is the capture rate, m is the half-saturation constant, f is the conversion efficiency, d is the predator's natural mortality rate, and qEx is the harvesting term based on the catch-per-unit-effort (CPUE) hypothesis, with q the catchability coefficient and E the harvesting effort.

The CPUE hypothesis, which postulates that the rate of catch is proportional to both the level of fishing effort and the stock density, is a standard assumption in fisheries mathematics. It provides a tractable way to link the economics of fishing operations to the population dynamics of the exploited stock. A real-world analogy often cited in the literature is the Antarctic krill-whale community: krill represent a primary food source for whales, and heavy harvesting of krill directly threatens the whale population by reducing prey availability (Kar, 2004).

Among Kar's principal findings was that increasing harvesting effort on the prey leads to a decline in the predator's equilibrium population, a result that underscores the indirect but powerful influence of prey harvesting on higher trophic levels. The delay term in the model ensured biological realism by restricting harvesting to mature individuals, thereby capturing the age-structured nature of real fishery populations. This study also provided an early demonstration that harvesting could function as a stabilizing or destabilizing force depending on the prevailing parameter regime.

2.2.2 Two-Prey One-Predator Models with Harvesting

More complex ecological realities often involve multiple prey species interacting simultaneously with a common predator. The analysis of such systems requires the extension of single-prey models to multi-species frameworks. Green (2004) studied a one-predator, two-prey system in which the prey species did not compete with each other, and predation followed the density gradient of the prey populations. A key finding from this work was that when a predator allocates its foraging effort between two prey species in proportion to their relative densities, the predator tends to stabilize the overall system dynamics. The study also explored limit cycles and chaotic behavior, revealing the richness of dynamical outcomes possible in multi-species models. However, the model did not consider competition between the prey species, nor did it incorporate any form of harvesting.

A more comprehensive treatment was offered by Kar and Chaudhuri (2004), who formulated a two-prey, one-predator fisheries model in which both prey species compete with each other for a shared resource, and the predator feeds linearly on both. Both prey populations are subject to continuous harvesting by human fishers. The governing system of equations is:

$$\frac{dx_1}{dt} = x_1\left[\lambda_1\left(1-\frac{x_1}{k_1}\right) - \alpha_{12}x_2 - \alpha_{13}x_3\right] - q_1Ex_1 \tag{2.2.5}$$

$$\frac{dx_2}{dt} = x_2\left[\lambda_2\left(1-\frac{x_2}{k_2}\right) - \alpha_{21}x_1 - \alpha_{23}x_3\right] - q_2Ex_2 \tag{2.2.6}$$

$$\frac{dx_3}{dt} = x_3\left[\alpha_{31}x_1 + \alpha_{32}x_2 - x_3\right] \tag{2.2.7}$$

where x_1, x_2 are prey densities and x_3 is the predator density; λ_1, λ_2 are biotic potentials; k_1, k_2 are environmental carrying capacities; α_{12}, α_{21} are interspecific competition coefficients; α_{13}, α_{23} are predation coefficients; α_{31}, α_{32} are conversion parameters; and q_1, q_2 are catchability coefficients for the respective prey species, with E being the common harvesting effort. Notably, the predator population in this model is not subject to any harvesting. This model was analyzed with particular attention to the bioeconomic equilibrium and the optimal harvesting policy. The bioeconomic analysis, drawing on the concepts developed by Clark (1979), seeks to identify the harvesting effort at which the net economic revenue from the fishery is maximized, subject to the biological constraints imposed by the population dynamics. The concept of bioeconomic equilibrium bridges the gap between ecological modelling and economic theory, making it a central concern in applied fisheries mathematics. Kar and Chaudhuri (2004) further studied a simpler two-species system that serves as a building block for the fuller model:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right) - \frac{cxy}{(m+x)} - qEx \tag{2.2.8}$$

$$\frac{dy}{dt} = y\left(-d + \frac{fx}{(m+x)}\right) \tag{2.2.9}$$

This formulation, incorporating Holling Type II functional response and prey harvesting through the CPUE mechanism, demonstrated how careful harvesting policy could maintain a sustainable fishery while avoiding the collapse of either the prey or the predator population. A related but distinct approach was taken by Kar and Chandan (2009), who constructed a bioeconomic model of a two-prey, one-predator system with a more nuanced treatment of the predator's functional response. In their model, the predator's feeding rate increases linearly with the first prey species but nonlinearly with the second, reflecting a biologically realistic scenario in which the predator has a preference for one prey type over the other. Both prey species are subject to separate harvesting efforts:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - a_1xy - b_1xz - q_1E_1x \tag{2.2.10}$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{l}\right) - a_2xy - \frac{b_2yz}{(m+y)} - q_2E_2y \tag{2.2.11}$$

$$\frac{dz}{dt} = a_1b_1xz + \frac{a_2b_2yz}{(m+y)} - cz \tag{2.2.12}$$

where x, y are the two prey populations and z is the predator population;r, s are biotic potentials;k, l are carrying capacities;a_1, a_2 are interspecific competition coefficients; b_1, b_2 are predation coefficients;$\alpha_1, \alpha_2 < 1$ are conversion factors for the two prey species;$E_1, E_2$are harvesting efforts; m is the half-saturation constant; and c is the natural mortality rate of the predator. The harvesting termsq_1E_1x andq_2E_2y are both based on the CPUE hypothesis.

This model was particularly noteworthy for its economic dimension: by assigning constant prices per unit biomass and constant costs per unit effort to each prey fishery, the study formulated an objective function aimed at maximizing net economic rent while simultaneously maintaining the predator population at a biologically viable level. The dual objective — economic optimization and ecological conservation — is central to the philosophy of sustainable fisheries management.

2.2.3 Resource Patchiness, Mutualism, and Logistic Growth

Not all contributions to this field focused exclusively on harvesting. Several important studies examined the structural properties of prey-predator systems and the conditions under which stable coexistence is possible. Vlastmil and Eisner (2006) investigated a one-consumer, two-resource system in which the resources were spatially distributed across two habitat patches. Their analysis revealed that when resources grow exponentially and handling times are negligible, apparent competition consistently drives the weaker resource to extinction. However, when logistic growth was incorporated into the model, species permanence — the sustained coexistence of all species — was guaranteed. This finding underscored the ecological importance of carrying capacity as a stabilizing force in multi-species systems. In a complementary direction, Fay and Greeff (1999, 2006) developed a model to study the predator-prey dynamics of lions, wildebeests, and zebras in the Kruger National Park of South Africa. Their work demonstrated that by carefully incorporating biologically plausible features such as logistic growth with mutualism among wildebeests and zebras, seasonal calving patterns, and predator removal through culling, it was possible to obtain a model that closely matches available empirical data. This study remains an instructive example of how mathematical abstraction and biological realism can be productively combined in ecological modelling.

2.2.4 Delayed Harvesting and Age-Structured Models

Time delays represent another dimension of biological realism that has received considerable attention in the prey-predator modelling literature. Toaha and Hassan (2008) developed a deterministic continuous prey-predator model extending the classical Lotka-Volterra framework by incorporating time delays and constant rates of harvesting for both species. Time delay in this context captures the biological phenomenon of gestation, wherein there is a finite lag between the consumption of prey and the resulting growth in the predator population. Their analysis showed that time delays could destabilize otherwise stable equilibrium points, potentially inducing oscillatory behavior or even chaos in the population dynamics. However, the model did not focus specifically on the functional response of the predator or the species-specific feeding patterns, which limits its direct applicability to structured fisheries contexts. Nonetheless, the study highlighted that the timing of harvesting — not just its magnitude — can have profound consequences for the stability of fish stocks.

2.2.5 Multi-Species Model with Holling Type II Response: The Lake Victoria Case

One of the most policy-relevant contributions reviewed in this thesis is the work of Raymond, Hugo, and Kung’aro (2019), which presented a bioeconomic prey-predator fisheries model featuring Holling Type II functional responses and harvesting of all three species in the system.

The case study was drawn from Lake Victoria in East Africa, focusing on the ecological relationship between Nile perch (the predator), cichlid fish, and tilapia (both prey species). This ecosystem has been heavily disrupted by human activity, including the introduction of Nile perch in the 1950s and subsequent overfishing. The model takes the following form:

$$\frac{dx_1}{dt} = x_1\lambda_1\left(1 - \frac{x_1}{k_1}\right) - \alpha_{12}x_1x_2 - \frac{\alpha_{13}x_1x_3}{(1+\beta x_1)} - q_1E_1x_1 \tag{2.2.13}$$

$$\frac{dx_2}{dt} = x_2\lambda_2\left(1 - \frac{x_2}{k_2}\right) - \alpha_{21}x_1x_2 - \frac{\alpha_{23}x_2x_3}{(1+\gamma x_2)} - q_2E_2x_2 \tag{2.2.14}$$

$$\frac{dx_3}{dt} = -\omega x_3 + \alpha_{31}\cdot\frac{\alpha_{13}x_1x_3}{(1+\beta x_1)} + \alpha_{32}\cdot\frac{\alpha_{23}x_2x_3}{(1+\gamma x_2)} - q_3E_3x_3 \tag{2.2.15}$$

where $x_1(t), x_2(t)$represent the populations of cichlid and tilapia fish (the prey), and $x_3(t)$ represents the population of Nile perch (the predator) at time t. The parameters λ_1, λ_2 are per capita intrinsic growth rates;k_1, k_2 are environmental carrying capacities; α_{12}, α_{21} are interspecific competition coefficients; α_{13}, α_{23} are predation coefficients; α_{31}, α_{32} are conversion parameters;$\beta, \gamma$are half-saturation parameters for the Holling Type II terms; ω is the natural mortality rate of the Nile perch; and q_1, q_2, q_3 are catchability coefficients with corresponding harvesting efforts E_1, E_2, E_3. The model was built on four biologically grounded assumptions: (i) cichlid and tilapia have unlimited food supply within the lake; (ii) Nile perch depend entirely on these two species as food sources; (iii) interspecific competition between the two prey species is exploitative; and (iv) in the absence of the predator, prey populations grow logistically up to the lake's carrying capacity. The findings of Raymond et al. (2019) illustrated the complex interplay between harvesting rates and population dynamics, demonstrating that disproportionate harvesting of any one species could cascade through the ecosystem and destabilize the others. The study was also notable for its bioeconomic analysis, which examined conditions under which sustainable harvesting could be maintained. Nevertheless, the model has a significant limitation that is directly relevant to the present study: the harvesting terms — represented simply as

E_1, E_2, E_3

— do not explicitly incorporate conservation or regulatory mechanisms. The harvesting parameters are treated as constants rather than as variables subject to management control. This makes it difficult for a policymaker to use the model directly as a decision-support tool for regulating fishing effort. Additionally, Raymond et al. (2019) used Lake Victoria as the case study, a context quite different from a small riverine community like Itebukunmi. There are no equivalent mathematical studies for Itebukunmi fishing ground in the available literature.

2.2.6 Other Notable Contributions

Yunfei, Rong, and Yongzhen (2013) presented a prey-predator model with harvesting of two species, demonstrating through numerical simulations that as long as the prey population does not go extinct, both prey and predator can coexist abundantly within a reserve zone. Their study reinforced the importance of spatial conservation measures such as no-take zones as a complement to harvest regulation.

Ganguli and Kar (2017) formulated a one-prey, one-predator model focusing on resource-based predation. Their analysis showed that extinction of prey was driven primarily by predation pressure rather than resource limitation — a finding with direct implications for fisheries management: targeting predator control, rather than purely boosting prey resources, may be an effective conservation strategy in certain ecosystems.

Elmojtaba, Alsawaii, and Al-moqbali (2020) contributed an optimal control analysis of a prey-predator model with harvesting and variable carrying capacity, adding further nuance to the understanding of how environmental variability interacts with human exploitation. Their incorporation of a variable carrying capacity is particularly relevant for tropical riverine systems like Itebukunmi, where seasonal flooding, rainfall variability, and habitat degradation can significantly alter the effective carrying capacity of fish populations.

More recently, Chatterjee and Pal (2023) examined optimal harvesting policy through tax imposition as a regulatory instrument, demonstrating mathematically how fiscal tools can achieve conservation objectives without requiring direct biological intervention. Yoshioka (2024) extended the analysis further by considering heterogeneity in biological resources and uncertain stock assessments, reflecting the growing recognition that real fisheries are never perfectly observed or controlled.

2.3 Summary

This chapter has provided a structured review of the major mathematical models developed for prey-predator systems with harvesting, with emphasis on two-prey, one-predator formulations relevant to the present study. The review traced the intellectual lineage from the classical Lotka-Volterra model through increasingly realistic generalizations incorporating logistic growth, Holling-type functional responses, interspecific competition, time delays, spatial structure, and harvesting.

Several important themes emerge from this review. First, harvesting is far more than a simple removal mechanism: its rate, timing, selectivity, and regulatory context can fundamentally alter the stability of the entire ecosystem. Second, the functional response — the relationship between prey density and predator consumption rate — plays a central role in determining whether systems stabilize at positive equilibria or collapse. The choice between Holling Type I, II, or ratio-dependent responses is not merely technical; it reflects genuine biological differences in how predators encounter and process prey.

Third, the majority of existing models treat harvesting effort as a fixed parameter rather than as a managed variable subject to regulatory control. This is a significant gap for conservation-oriented fisheries management, where the whole point is to determine the optimal level and form of harvesting regulation. The introduction of explicit regulatory parameters — such as the conservation rates ρ_1, ρ_2, and ρ_3 incorporated in the present model — directly addresses this gap.

Fourth, and most directly relevant to the present study, there exists no mathematical model specifically designed for the Itebukunmi fishing ground. While models have been developed for Lake Victoria, the Antarctic krill-whale system, the Adriatic Sea, and Kruger National Park, the Itebukunmi riverine fishery in Ondo State, Nigeria has received no equivalent formal mathematical treatment. Given the socioeconomic importance of this fishing ground to the Ilaje people and the broader Ondo State economy, this is a significant gap that the present work seeks to fill.

The open research questions identified in this review — particularly the need to incorporate species-specific functional responses, explicit management control variables, reproductive boosting strategies, and locally calibrated parameters — form the basis of the mathematical model formulated and analyzed in Chapter Three.

CHAPTER THREE

Model Formulation and Analysis

3.1 Introduction

Mathematical model in two-prey, one - predator system in Itebukunmi riverine area of Ilaje Local Government Area, Ondo State of Nigeria, is presented in this chapter. The prey-predator model, in which the *Gymnarchus niloticus* acts as the predator while the cat fish and the *Ophiocephalus* fish act at the preys, is considered to investigate how to enhance the sustainability of the fishes in Itebukunmi, a fishing ground area in Ilaje Local Government Area of Ondo State, Nigeria, without compromising the biological, economic and social objectives for the benefit of present and future generations.

Figure 3.1: Catfish source:photo ID:14943869

Figure 3.2: Ophiocephalus (source:stock-photo-snakehead-or-channa-ophiocephalus-argus-asian-fish-2204040921)

Figure 3.3: Gymnarchus niloticus source:Fishbase, 2019

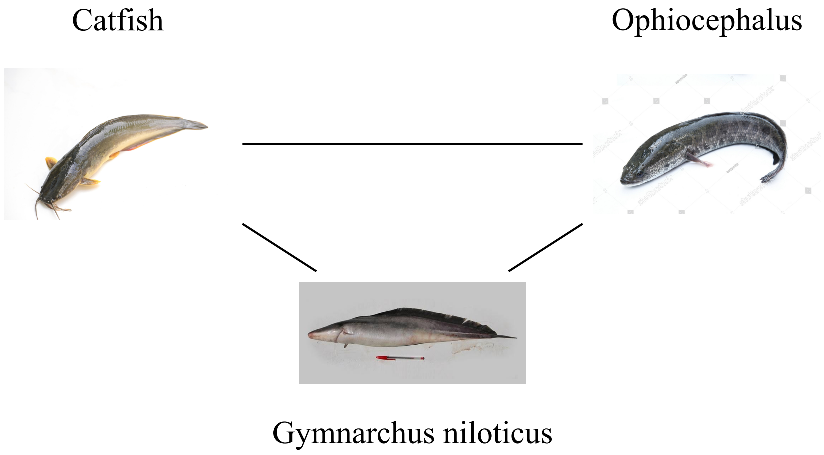


Figure 3.4: A clear picture of the fishes involved in the model formulation

3.2 The Model Description

The interactions between and among the two prey and one predator are illustrated in the Figure 3.5 The descriptions for the variables and parameters of the model are contained in Tables 3.1 and 3.2

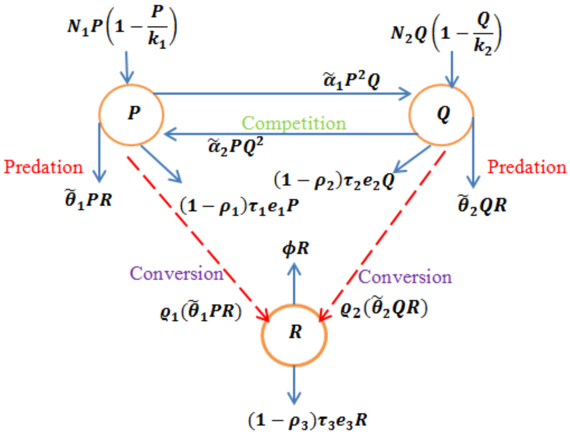


Figure 3.5: Ecological compartmental model showing the relationship between and among the species

Table 3.1: Nomenclatures for the model variables

Model variables	Descriptions
$P(t)$	Population of cat fish at time t
$Q(t)$	Population of ophiocephalus fish at time t
$R(t)$	Population of African knifefish (Gymnarchus niloticus) at time t

Table 3.2: Nomenclatures for the model parameters

Parameters	Nomenclatures
N_1	Growth rate of P
N_2	Growth rate of Q
$\tilde{\alpha_1}$	Reduction rate for P due to intra and inter competition
$\tilde{\alpha_2}$	Reduction rate for Q due to intra and inter competition
$\tilde{\theta_1}$	Death rate for P due to predation by R
$\tilde{\theta_2}$	Death rate for Q due to predation by R
k_1, k_2	Environmental carrying capacities for P and Q respectively
ϕ	Decay rate for R due to scarceness of P and Q
τ_1, τ_2, τ_3	Catchability rates for P, Q and R respectively
ϱ_1, ϱ_2	Conversion rates for P and Q respectively
e_1, e_2, e_3	Human harvesting effort rates on P, Q and R respectively
ρ_1, ρ_2, ρ_3	Regulatory rates to conserve P, Q and R respectively

The interaction between P and Q is competitive while the interactions between P and R also Q and R are predatory. Therefore, the proposed model is governed by the following system of first-order nonlinear ordinary differential equations based on the model flow diagram in figure 3.5.

$$\frac{dP}{dT} = N_1 P \left(1 - \frac{P}{k_1}\right) - \tilde{\alpha_1} P^2 Q - \tilde{\theta_1} PR - (1-\rho_1)\tau_1 e_1 P, \tag{3.2.1}$$

$$\frac{dQ}{dT} = N_2 Q \left(1 - \frac{Q}{k_2}\right) - \tilde{\alpha_2} P Q^2 - \tilde{\theta_2} QR - (1-\rho_2)\tau_2 e_2 Q, \tag{3.2.2}$$

$$\frac{dR}{dT} = -\phi R + \varrho_1(\tilde{\theta_1} PR) + \varrho_2(\tilde{\theta_2} QR) - (1-\rho_3)\tau_3 e_3 R. \tag{3.2.3}$$

With initial conditions $P(0) = P_0 > 0, Q(0) = Q_0 > 0, R(0) = R_0 > 0$. Besides, $\tau_1 < \tau_2 < \tau_3$, for the catchability for the three fishes. Also, assume $e_1 < e_2 < e_3$ because of the differences in the economic importance of the fishes.

From the model equations (3.2.1-3.2.3), there exists relationship and interaction between the three species in the ecosystem.

The model is also built around the following fundamental assumptions of ecological studies.
(i)The only favorite food supplies to the predators (the African knifefish) are the cat fish and the ophiocephalus fish.
(ii)None of the fishes can survive alone in the river (Panayotova and Hallare, 2021). So as to

enhance ecosystem stability.

For ease of computation, the system (3.2.1)-(3.2.3) is rescaled to reduce the number of parameters as follows Thota (2020)
take $n_1 = N_1, n_2 = N_2, p = \dfrac{P}{k_1}, q = \dfrac{Q}{k_2}, r = \dfrac{R}{\varpi}, t = T \quad \Rightarrow P = pk_1, Q = qk_2, R = r\varpi$
and with relative derivatives $\dfrac{dp}{dt} = \dfrac{1}{k_1}\dfrac{dP}{dT}, \dfrac{dq}{dt} = \dfrac{1}{k_2}\dfrac{dQ}{dT}, \dfrac{dr}{dt} = \dfrac{1}{\varpi}\dfrac{dR}{dT}$, then by appropriate substitution, the system of model (3.2.1)-(3.2.3) becomes

$$
\begin{aligned}
k_1\frac{dp}{dt} &= n_1k_1p(1-p) - \tilde{\alpha}_1(k_1p)^2(k_2q) - \tilde{\theta}_1(k_1p)(\varpi r) - (1-\rho_1)\tau_1e_1(k_1p), && (3.2.4)\\
k_2\frac{dq}{dt} &= n_2k_2q(1-q) - \tilde{\alpha}_2(k_1p)(k_2q)^2 - \tilde{\theta}_2(k_2q)(\varpi r) - (1-\rho_2)\tau_2e_2(k_2q), && (3.2.5)\\
\varpi\frac{dr}{dt} &= -\phi(\varpi r) + \varrho_1(\tilde{\theta}_1k_1p\varpi r) + \varrho_2(\tilde{\theta}_2k_2q\varpi r) - (1-\rho_3)\tau_3e_3(\varpi r). && (3.2.6)
\end{aligned}
$$

Dividing (3.2.4), (3.2.5) and (3.2.6) by k_1, k_2 and ϖ respectively and absorbing scale factors into new parameters, i.e., $\alpha_1 = \tilde{\alpha}_1k_1k_2, \theta_1 = \tilde{\theta}_1\varpi, \alpha_2 = \tilde{\alpha}_2k_1k_2, \theta_2 = \tilde{\theta}_2\varpi, m_1 = \varrho_1(\tilde{\theta}_1k_1), m_2 = \varrho_2(\tilde{\theta}_2k_2)$ then

$$
\begin{aligned}
\frac{dp}{dt} &= n_1p(1-p) - \alpha_1p^2q - \theta_1pr - (1-\rho_1)\tau_1e_1p, && (3.2.7)\\
\frac{dq}{dt} &= n_2q(1-q) - \alpha_2pq^2 - \theta_2qr - (1-\rho_2)\tau_2e_2q, && (3.2.8)\\
\frac{dr}{dt} &= -\phi r + m_1pr + m_2qr - (1-\rho_3)\tau_3e_3r, && (3.2.9)
\end{aligned}
$$

with initial conditions $p(0) = p_0 > 0, q(0) = q_0 > 0, r(0) = r_0 > 0$. Besides, $\tau_1 < \tau_2 < \tau_3$, $e_1 < e_2 < e_3$.

3.3 Positivity, Existence and Uniqueness of Solution

Following the approach in Dennis (2012),Thota (2020) and Ayoade et al.(2019), the solutions of the model are expected to be positive, exist and unique since it monitors real-life population. The following theorem is used to verify the positivity of model's solutions

Theorem 3.1 (Thota (2020)). *The proposed model (3.2.7)-(3.2.9), is solution set* $\{p(t), q(t), r(t)\}$ *combined with the initial condition is non-negative for all* $t > 0$.

Proof. Following the suggestion in Thota (2020), the first equation in the system (3.2.7)-(3.2.9) is evaluated to verify the positivity of the model (3.2.7)-(3.2.9) equation:

$$\begin{aligned}
\frac{dp}{dt} &= n_1p(1-p) - \alpha_1 p^2 q - \theta_1 pr - (1-\rho_1)\tau_1 e_1 p, \\
\frac{dp}{dt} &\geq -[(1-\rho_1)\tau_1 e_1]p, \\
\frac{dp}{p} &\geq -[(1-\rho_1)\tau_1 e_1]dt, \\
\int \frac{dp}{p} &\geq -\int [(1-\rho_1)\tau_1 e_1]dt, \\
\ln p &\geq -[(1-\rho_1)\tau_1 e_1]t + c, \\
p(t) &\geq e^{-[(1-\rho_1)\tau_1 e_1]t+c}, \\
p(t) &\geq Je^{-[(1-\rho_1)\tau_1 e_1]t},
\end{aligned}$$

where $J = e^c$. At initial time, $t = 0$,

$$\text{Implies that } p(0) \geq J$$

$$p(t) \geq p(0)e^{-[(1-\rho_1)\tau_1 e_1]t}$$

and it follows that

$$\begin{aligned}
q(t) &\geq q(0)e^{-[(1-\rho_2)\tau_2 e_2]t}, \\
r(t) &\geq r(0)e^{-[\phi+(1-\rho_3)\tau_3 e_3]t}.
\end{aligned}$$

Since $e^L > 0$ for all real values of L then it follows that it is sufficient to conclude that the solutions $\{p(t), q(t), r(t)\}$ for the model are positive for all $t > 0$. □

Definition 3.1. *Let D be a non-empty open and connected region in* $\mathbb{R}^{n+1}$ *and let* $\xi : D \to \mathbb{R}^n$*,* $(t, x) \to \xi(t, x), x = (p, q, r), \quad t \in (0, t) \subseteq \mathbb{R}^+$*,*

$$\xi(t, x) = \begin{pmatrix} \xi_1(t, p) \\ \xi_2(t, q) \\ \xi_3(t, r) \end{pmatrix}, \quad x = (p, q, r)$$

where ξ is continuous, and

$$\frac{\partial \xi}{\partial x} = \begin{pmatrix} \frac{\partial \xi_i}{\partial p} \\ \frac{\partial \xi_i}{\partial q} \\ \frac{\partial \xi_i}{\partial r} \end{pmatrix}, \quad i = 1, 2, 3$$

$\xi_i = (\xi_1, \xi_2, \xi_3)$ *, where* $\xi_1 = \xi_1(t, p), \quad \xi_2 = \xi_2(t, q)$*, and* $\xi_3 = \xi_3(t, r)$*.*

Definition 3.2. *Let Y be a Banack space and T a transformation from a subset X of Y into Y. Then, T is called a contraction on X if there exists $\beta \in \mathbb{R}^+$ such that $0 < \beta < 1$ and $||T_x - T_y|| \leq \beta ||x - y||_Y \; \forall \; x, y \in X$.*

Theorem 3.2. ***Contraction Mapping***

Suppose D is a closed subset of a Banach space Y, then any contraction mapping T of D onto itself has a unique fixed point. That is, if $T : D \to D$ and

$$||T_x - T_y||_Y \leq \beta ||x - y||_Y,$$

then there exists $x \in D$ such that

$$T_x = x.$$

Then x is said to be the fixed point of T.

Theorem 3.3. ***(Existence and Uniqueness)***

Let ξ be a function defined and continuous in some region D of $\mathbb{R}^{n+1}$. If in addition, ξ possess continuous first order partial derivative $\frac{\partial \xi}{\partial x_i}$, $i = 1, 2, 3$ in D. Then, for every point $(0, \bar{\omega}) \in D$, $\bar{\omega} = (p_0, q_0, r_0)$, there exists a unique solution

$$x = (A(t), B(t), C(t)) = Z(t)$$

of the system of initial value problem

$$\frac{dx}{dt} = \xi(t, x), \quad x(0) = \bar{\omega} \tag{3.3.1}$$

define in some neighbourhood of $(0, \bar{\omega})$ *such that* $x(0) = (A(0), B(0), C(0)) = \bar{\omega}$.

Proof: Assume that D is an open, connected subset of $\mathbb{R}^{n+1}$ and $(0, \bar{\omega}) \in D$, then there exists positive real constants a, b_1, b_2 and b_3 such that the closed and bounded set

$$S = \{(t, x) : |t - v| = |t| \leq a, \quad ||x - \bar{\omega}|| \leq b, \qquad for$$
$$b = (b_1, b_2, b_3), \quad such \quad that \quad ||p - p_0|| \leq b_1, \quad ||q - q_0|| \leq b_2, \quad ||r - r_0|| \leq b_3, \quad (t, x) \in D\}$$

is contained entirely in D. Since ξ and ξ_{x_i}, $i = 1, 2, 3$ are continuous in D and S is contained in D, we have ξ and ξ_{x_i} are also contained in S for $i = 1, 2, 3$.
Hence, S is closed and bounded, so we can find constants $M > 0$ and $N > 0$ such that

$$||\xi(t, x)|| \leq M, \tag{3.3.2}$$
$$||\xi_{x_i}(t, x)|| \leq N, \qquad i = 1, 2, 3 \quad \forall \quad (t, x) \in S. \tag{3.3.3}$$

If $M = (M_1, M_2, M_3) > (0, 0, 0)$ and $N = (N_1, N_2, N_3) > (0, 0, 0)$, then from (3.3.2), we have

$$||\xi_1(t, p)|| \leq M_1, \qquad ||\xi_2(t, q)|| \leq M_2, \qquad ||\xi_3(t, r)|| \leq M_3, \qquad and$$
$$||\xi_p(t, p)|| \leq N_1, \qquad ||\xi_q(t, q)|| \leq N_2, \qquad ||\xi_r(t, r)|| \leq N_3.$$

By the Mean-Value theorem extension and (3.3.2), we have

$$||\xi(t, x) - \xi(t, y)|| \leq 3N||x - y||, \tag{3.3.4}$$

for (t, x) and $(t, y) \in S$. Let S_K be a subset of S defined as follows for K chosen such that

$$K \leq a \qquad (i)$$
$$K \leq \frac{b}{M} \qquad (ii)$$
$$K < \frac{1}{3N} \qquad (iii)$$

Thus, $S_K = \{(t,x) : |t| \leq K \quad and \quad ||x - \bar{\omega}|| \leq b$ which implies that $||p - p_0|| \leq b_1, \quad ||q - q_0|| \leq b_2, \quad ||r - r_0|| \leq b_3\} \ \forall \quad (0, \bar{\omega}) \in D.$ From this, we define a function space

$$\psi_K = \{x(t) = (p(t), q(t), r(t)) : \quad (t, x(t)) \in S_K \quad and \quad x(t) \quad is \quad continuous \quad in \quad |t| \leq k\}.$$

This space is a subspace of the Banach space $\psi([-K, K], \mathbb{R}^3)$ which is contained in S_K.

Now, for $x \in \psi_r$, we define a transformation T by

$$z(t) = T_{x(t)} = \bar{\omega} + \int_0^t \xi(s, x(s))ds \qquad \forall \qquad |t| \leq K$$

Domain $(T) \subseteq \psi_K$. Hence,

$$\begin{aligned} ||z(t) - \bar{\omega}|| &= || \int_0^t \xi(s, x(s)ds|| \\ &\leq \int_0^t ||\xi(s, x(s))||ds \\ &\leq M|t| \leq M_K \leq M \cdot \frac{b}{M} = b. \end{aligned}$$

That is,

$$||z(t) - \bar{\omega}|| \leq b,$$

which implies that

$$z(t) = T_{x(t)} \in \psi_K, \tag{3.3.5}$$

hence T maps ψ_K to ψ_K.

Also, $z(0) = \bar{\omega}$.

Next, we show that T is a contraction for $x, y \in \psi_K$.

$$\begin{aligned} ||T_x - T_y||_{\psi_K} &= || \int_0^t \left[\xi\left(s, x(s)\right) - \xi\left(s, y(s)\right)\right] ds|| \\ &\leq \int_0^t ||\xi\left(s, x(s)\right) - \xi(s, y(s))||ds \end{aligned}$$

Thus, by (3.3.4), we have

$$\begin{aligned} ||T_x - T_y|| &\leq 3N \int_0^t ||x(s) - y(s)||ds \\ &\leq 3N \max_{t\in[-k,k]} \{||x(t) - y(t)||\} \int_0^t ds \\ &\leq 3N||x(t) - y(t)||_{\psi_K} |t - 0| \\ &\leq 3KN||x(t) - y(t)||_{\psi_k}. \end{aligned}$$

Hence,

$$||T_x - T_y||_{\psi_K} \leq \beta||x(t) - y(t)||_{\psi_K}, \tag{3.3.6}$$

where

$$\beta = 3KN, \quad \beta > 0, \qquad \beta = 3KN < 3 \cdot \left(\frac{1}{3N}\right) N = 1$$

Therefore, $\beta < 1$

We now conclude that $0 < \beta < 1$. It then follows that T is a contraction mapping on ψ_K. Therefore, by contraction mapping theorem, T has a fixed point in ψ_K.

Next, we find the fixed point by defining a sequence of successive approximation as follows:

$$z_0(t) = \bar{\omega},$$

$$T_{z_n(t)} = z_{n+1}(t) = \bar{\omega} + \int_0^t \xi(s, z_n(s))ds, \tag{3.3.7}$$

$$|t - 0| \leq K, \quad n \in \mathbb{N}.$$

By definition, $z_j(t)$ for each $j = 1, 2, \cdots, n$ and $z_j(t) \in \psi_K$.

Let $m, n \in \mathbb{N}$ such that $n > m$. Then

$$\begin{aligned}||z_{n+1}(t)-z_{m+1}(t)||_{\psi_K} &= ||Tz_n(t)-Tz_m(t)||_{\psi_K}\\ &\leq \beta||z_n(t)-z_m(t)||_{\psi_K}\\ &= \beta||Tz_{n-1}(t)-Tz_{m-1}(t)||_{\psi_K}\\ &\leq \beta(\beta||z_{n-1}(t)-z_{m-1}(t)||_{\psi_K})\\ &= \beta^2||z_{n-1}(t)-z_{m-1}(t)||_{\psi_K}\\ &= \beta^2||Tz_{n-2}(t)-Tz_{m-2}(t)||_{\psi_K}\\ &\leq \beta^3||z_{n-2}(t)-z_{m-2}(t)||_{\psi_K}\\ &\vdots\\ &\leq \beta^{m+1}||z_{n-m}(t)-z_0(t)||_{\psi_K}.\end{aligned}$$

We observe that

$$\begin{aligned}z_{n-m}(t) &= z_{n-m}(t)-z_{n-m-1}(t)+z_{n-m-1}(t)-z_{n-m-2}(t)+z_{n-m-2}+\cdots\\ &\quad +z_3(t)-z_2(t)+z_2(t)-z_1(t)+z_1(t)-z_0(t)+z_0(t).\end{aligned}$$

Hence, terms would cancel out

$$z_{n-m}(t)-z_0(t)=z_{n-m}(t)-z_{n-m-1}+\cdots+z_2(t)-z_1(t)+z_1(t)-z_0(t).$$

Taking the norm of both sides, to have

$$\begin{aligned}&||z_{n-m}(t)-z_0(t)||_{\psi_K} \leq ||z_{n-m}(t)-z_{n-m-1}(t)||_{\psi_K}+\cdots+||z_2(t)-z_1(t)||_{\psi_K}+||z_1(t)-z_0(t)||_{\psi_K}\\ &\leq \beta^{n-m-1}||z_1(t)-z_0(t)||_{\psi_K}+\cdots+\beta^2||z_1(t)-z_0(t)||_{\psi_K}+\beta||z_1(t)-z_0(t)||_{\psi_K}+||z_1(t)-z_0(t)||_{\psi_K}\\ &\leq (\beta^{n-m-1}+\beta^{n-m-2}+\cdots+\beta^2+\beta+1)||z_1(t)-z_0(t)||_{\psi_K}\\ &\leq \frac{1}{1-\beta}||z_1(t)-z_0(t)||_{\psi_K}\\ &\leq \frac{b}{1-\beta},\end{aligned}$$

Since $z_1(t) \in \psi_K$ and $||z_1(t) - z_0(t)|| \leq b$. Therefore, we have

$$\begin{aligned} ||z_{n+1}(t) - z_{m+1}(t)||_{\psi_K} &\leq \beta^{m+1}||z_{n-m}(t) - z_0(t)||_{\psi_K} \\ &\leq \beta^{m+1} \cdot \frac{b}{1-\beta} = \frac{b}{1-\beta}\beta^{m+1}. \end{aligned}$$

Now, as $m \to \infty$, $n \to \infty$, and $\frac{\beta^{m+1}}{1-\beta} \to 0$.
So that $||z_{n+1}(t) - z_{m+1}(t)||_{\psi_K} \to 0$ as $m \to \infty$. Hence, the sequence of iteration $\{z_n(t)\}$ is a Cauchy sequences. Since ψ_K is a Banach space and $\{z_n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, it must converge to ψ_K to a unique limit.
Let z be the unique limit that is,

$$z(t) = \lim_{n\to\infty} z_n(t), \quad for \quad |t| \leq K,$$

we now have $(t, z(t)) \in \psi_K$, and

$$z(0) = \lim_{n\to\infty} z_n(0) = \bar{\omega}.$$

For any $n \in \mathbb{N}$,

$$||Tz(t) - Tz_n(t)||_{\psi_K} \leq \beta||z(t) - z_n(t)||_{\psi_K}$$

Taking the limit as $n \to \infty$, we have

$$||z(t) - z_n(t)||_{\psi_K} \to 0,$$

that ultimately yields,

$$||Tz(t) - Tz_n(t)||_{\psi_k} \to 0$$

It then follows that

$$Tz(t) = \lim_{n\to\infty} Tz_n(t) = \lim_{n\to\infty} z_{n+1}(t) = z(t).$$

So $z(t)$ is a fixed point of the mapping T, with $z(0) = \bar{\omega}$ and

$$z(t) = Tz(t) = \bar{\omega} + \int_0^t \xi(s, x(s))ds.$$

Hence, z is a solution of the initial value problem (3.3.1).

Next, we show that the solution of our ecological model (3.2.7) - (3.2.9) has a unique solution.

Suppose there exist another solution $Q(t)$ such that $Q(0) = \bar{\omega}$ and is defined and continuous on $|t| \leq K_1$, for some $K_1 > 0$. Let $z = \min\{K, K_1\}$, then $(t, Q(t)) \in \psi_k$, for $|t-0| \leq Q$ and $(t, z(t)) \in \psi_K$, for $|t-0| \leq z$. So we have z and Q as solution on the interval $|t-0| \leq z$. By definition, z and $Q \in \psi_z$. so

$$\begin{aligned} ||z(t) - Q(t)||_{\psi_z} &= ||Tz(t) - TQ(t)||_{\psi_z} \\ &\leq \beta ||z(t) - Q(t)||_{\psi_z}. \end{aligned}$$

It therefore follows that

$$||Tz(t) - TQ(t)||_{\psi_z} \geq (1-\beta)||z(t) - Q(t)||_{\psi_z}$$

That is,

$$||z(t) - Q(t)||_{\psi_z}(1-\beta) \leq 0, \quad 0 < \beta < 1.$$

But, $1 - \alpha \geq 0$.

So that

$$||z(t) - Q(t)||_{\psi_z} \leq 0$$

But norm is a positive function, therefore the only possibility is that $||z(t) - q(t)||_{\psi_z} = 0$. It implies $z(t) - Q(t) = 0$ and $z(t) = Q(t)$ for $|t| \leq z$. Hence, the uniqueness of solution.

3.4 MODEL ANALYSIS

The proposed ecological model, is studied theoretically as follows:

3.4.1 Equilibria

When the system is at steady state, the rate of change of each variable with respect to time becomes zero, equations (3.2.7) - (3.2.9) becomes

$$\begin{aligned} n_1p(1-p)-\alpha_1p^2q-\theta_1pr-(1-\rho_1)\tau_1e_1p=0,\\ n_2q(1-q)-\alpha_2pq^2-\theta_2qr-(1-\rho_2)\tau_2e_2q=0,\\ -\phi r+m_1pr+m_2qr-(1-\rho_3)\tau_3e_3r=0, \end{aligned} \tag{3.4.1}$$

Following Panayotova and Hallare (2021), five equilibrium points are obtained. These equilibrium points correspond to when:

- i none of the three ecological variables are in existence;
- ii only cat fish is nonexistence;
- iii only ophiocephalus fish is nonexistence;
- iv only African knife fish is nonexistence;
- v all the three ecological variables are in existence.

Case 1: *none of the three ecological variables are in existence* $\mathcal{E}_1(0,0,0)$
From (3.4.1), substitute zero for each variable then the solution is the point

$$\mathcal{E}_1=(0,0,0). \tag{3.4.2}$$

Case 2: *only cat fish is nonexistence* $\mathcal{E}_2=(0,q,r)$
From (3.4.1), substitute zero for p then (3.4.1) becomes

$$\begin{aligned} (q)[n_2(1-q)-\theta_2r-(1-\rho_2)\tau_2e_2]=0,\\ (r)[-\phi+m_2q-(1-\rho_3)\tau_3e_3]=0, \end{aligned}$$

$$q=0,\ \ r=0\Rightarrow$$

$$n_2(1-q)-\theta_2 r-(1-\rho_2)\tau_2 e_2=0,$$

$$-\phi+m_2 q-(1-\rho_3)\tau_3 e_3=0,$$

$\therefore r^* = \dfrac{n_2[m_2-\phi-(1-\rho_2)\tau_2 e_2]-m_2(1-\rho_2)\tau_2 e_2}{m_2\theta_2}$ and
$q^* = \dfrac{\phi+(1-\rho_3)\tau_3 e_3}{m_2}$.

Hence, the solution is the point $\mathcal{E}_2=(0,0,0)$, or

$$\mathcal{E}_2=\left(0,\frac{\phi+(1-\rho_3)\tau_3 e_3}{m_2},\frac{n_2[m_2-\phi-(1-\rho_2)\tau_2 e_2]-m_2(1-\rho_2)\tau_2 e_2}{m_2\theta_2}\right). \tag{3.4.3}$$

Case 3: *only ophiocephalus fish is nonexistence* $\mathcal{E}_3=(p,0,r)$

From (3.4.1), substitute zero for q then (3.4.1) becomes

$$(p)[n_1(1-p)-\theta_1 r-(1-\rho_1)\tau_1 e_1]=0,$$

$$(r)[-\phi+m_1 p-(1-\rho_3)\tau_3 e_3]=0,$$

$$p=0,\ \ r=0\Rightarrow,$$

$$n_1(1-p)-\theta_1 r-(1-\rho_1)\tau_1 e_1=0,$$

$$-\phi+m_1 p-(1-\rho_3)\tau_3 e_3=0,$$

$\therefore r = \dfrac{n_1[m_1-\phi-(1-\rho_3)\tau_3 e_3]-m_1(1-\rho_1)\tau_1 e_1}{m_1\theta_1}$ and
$p = \dfrac{\phi+(1-\rho_3)\tau_3 e_3}{m_1}$.

Hence, the solution is the point

$$\mathcal{E}_3=(0,0,0)$$

or

$$\mathcal{E}_3=\left(\frac{\phi+(1-\rho_3)\tau_3 e_3}{m_1},0,\frac{n_1[m_1-\phi-(1-\rho_3)\tau_3 e_3]-m_1(1-\rho_1)\tau_1 e_1}{m_1\theta_1}\right) \tag{3.4.4}$$

Case 4: *only African knife fish is nonexistence* $\mathcal{E}_4=(p,q,0)$

From (3.4.1), substitute zero for r then (3.4.1) becomes

$$(p)[n_1(1-p)-\alpha_1 pq-(1-\rho_1)\tau_1 e_1]=0,$$

$$(q)[n_2(1-q)-\alpha_2 pq-(1-\rho_2)\tau_2 e_2]=0,$$

$$p=0,\ \ q=0\Rightarrow,$$

$$n_1(1-p)-\alpha_1 pq-(1-\rho_1)\tau_1 e_1=0,$$

$$n_2(1-q)-\alpha_2 pq-(1-\rho_2)\tau_2 e_2=0,$$

$\therefore$

$$p=\frac{n_2(1-q)-(1-\rho_2)\tau_2 e_2}{\alpha_2 q}\ \text{ and }\ q=\frac{n_1(1-p)-(1-\rho_1)\tau_1 e_1}{\alpha_1 p}.$$

Hence, the solution is the point

$$\mathcal{E}_4=(0,0,0)$$

or

$$\mathcal{E}_4=\left(\frac{n_2(1-q)-(1-\rho_2)\tau_2 e_2}{\alpha_2 q},\frac{n_1(1-p)-(1-\rho_1)\tau_1 e_1}{\alpha_1 p},0\right). \tag{3.4.5}$$

Case 5: *all the three ecological variables are in existence* $\mathcal{E}_5=(p,q,r)$

From (3.4.1), consider the equations for p and q then (3.4.1) becomes

$$(p)[n_1(1-p)-\alpha_1 pq-\theta_1 r-(1-\rho_1)\tau_1 e_1]=0,$$

$$(q)[n_2(1-q)-\alpha_2 pq-\theta_2 r-(1-\rho_2)\tau_2 e_2]=0,$$

$$p=0,\ \ q=0\Rightarrow,$$

$$n_1(1-p)-\alpha_1 pq-\theta_1 r-(1-\rho_1)\tau_1 e_1=0,$$

$$n_2(1-q)-\alpha_2 pq-\theta_2 r-(1-\rho_2)\tau_2 e_2=0,$$

$\therefore$

$$p=\frac{n_2(1-q)-\theta_2 r-(1-\rho_2)\tau_2 e_2}{\alpha_2 q}\ \text{ and }\ q=\frac{n_1(1-p)-\theta_1 r-(1-\rho_1)\tau_1 e_1}{\alpha_1 p}.$$

Adding up the equations for p and q then

$$n_1(1-p)+n_2(1-q)-pq(\alpha_1+\alpha_2)-r(\theta_1+\theta_2)-(1-\rho_1)\tau_1e_1-(1-\rho_2)\tau_2e_2=0 \Rightarrow$$

$$r=\frac{n_1(1-p)+n_2(1-q)-pq(\alpha_1+\alpha_2)-(1-\rho_1)\tau_1e_1-(1-\rho_2)\tau_2e_2}{(\theta_1+\theta_2)}$$

Hence, the solution is the point

$$\mathcal{E}_5=(0,0,0)$$

or

$$\mathcal{E}_5=\left(\frac{n_2(1-q)-\theta_2r-(1-\rho_2)\tau_2e_2}{\alpha_2q},\frac{n_1(1-p)-\theta_1r-(1-\rho_1)\tau_1e_1}{\alpha_1p},\frac{A_1}{A_2}\right), \quad (3.4.6)$$

where $A_1=n_1(1-p)+n_2(1-q)-pq(\alpha_1+\alpha_2)-(1-\rho_1)\tau_1e_1-(1-\rho_2)\tau_2e_2$ and $A_2=\theta_1+\theta_2$.

Hence, the equilibrium points are:

$$\begin{aligned}
\mathcal{E}_1&=(0,0,0),\\
\mathcal{E}_2&=\left(0,\frac{\phi+(1-\rho_3)\tau_3e_3}{m_2},\frac{n_2[m_2-\phi-(1-\rho_2)\tau_2e_2]-m_2(1-\rho_2)\tau_2e_2}{m_2\theta_2}\right),\\
\mathcal{E}_3&=\left(\frac{\phi+(1-\rho_3)\tau_3e_3}{m_1},0,\frac{n_1[m_1-\phi-(1-\rho_3)\tau_3e_3]-m_1(1-\rho_1)\tau_1e_1}{m_1\theta_1}\right),\\
\mathcal{E}_4&=\left(\frac{n_2(1-q)-(1-\rho_2)\tau_2e_2}{\alpha_2q},\frac{n_1(1-p)-(1-\rho_1)\tau_1e_1}{\alpha_1p},0\right),\\
\mathcal{E}_5&=\left(\frac{n_2(1-q)-\theta_2r-(1-\rho_2)\tau_2e_2}{\alpha_2q},\frac{n_1(1-p)-\theta_1r-(1-\rho_1)\tau_1e_1}{\alpha_1p},\frac{A_1}{A_2}\right).
\end{aligned} \quad (3.4.7)$$

3.4.2 Stability Analysis of the Equilibrium Points

Local stability and global stability are the two stability analyses of the equilibrium points that are often discussed in ecological models (Thota, 2020).

The present study shall adopt the linearisation approach together with the Routh-Hurwitz stability criterion as well as Bendixson-Dulac and Poincare-Bendixson theorems with suitable Lyapunov functions.

3.4.3 Local Stability Analysis of the Equilibrium Points

The Routh-Hurwitz stability criteria are outlined in (Okyere, 2012) and the proofs are available in (Gantmacher, 1964). To classify the local stability of the equilibrium points $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$ and $\mathcal{E}_5$, the model is linearized about each of the points and the conclusion is reached from the well known stability theorem outlined in Theorem 3.4 Okyere,(2012).

Theorem 3.4 (Okyere (2012))**.** *An equilibrium point $\mathcal{E}$ is asymptomatically stable if all the eigenvalues of $J(\mathcal{E})$ have negative real parts. If at least one of the eigenvalues of $J(\mathcal{E})$ has a positive real part, then the equilibrium point is unstable.*

Linearizing the model, the Jocobian matrix J of the system is derived in (3.4.8)

$$J = \begin{pmatrix} f_1' & -\alpha_1 p^2 & -\theta_1 p \\ -\alpha_2 q^2 & f_2' & -\theta_2 q \\ m_1 r & m_2 r & A_3 \end{pmatrix}, \tag{3.4.8}$$

where $f_1' = n_1(1-2p) - 2\alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1$, $f_2' = n_2(1-2q) - 2\alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2$ and $A_3 = -\phi + m_1 p + m_2 q - (1-\rho_3)\tau_3 e_3$.

Case 1: *None of the three ecological variables are in existence*

Evaluating J at the equilibrium point $\mathcal{E}_1 = (0,0,0)$,

$\therefore$

$$J(\mathcal{E}_1) = J(0,0,0) = \begin{pmatrix} n_1 - (1-\rho_1)\tau_1 e_1 & 0 & 0 \\ 0 & n_2 - (1-\rho_2)\tau_2 e_2 & 0 \\ 0 & 0 & -f_3' \end{pmatrix}, \tag{3.4.9}$$

where $f_3' = [\phi + (1-\rho_3)\tau_3 e_3]$. The eigenvalues of $J(\mathcal{E}_1)$ are $\lambda_1 = n_1 - (1-\rho_1)\tau_1 e_1, \lambda_2 = n_2 - (1-\rho_2)\tau_2 e_2$ and $\lambda_3 = -[\phi + (1-\rho_3)\tau_3 e_3]$. An equilibrium point is stable if all the eigenvalues of the Jacobian matrix evaluated at the equilibrium point are negative otherwise the equilibrium point is unstable. In our study, it is only unstable equilibrium point that is desirable because we are interested in the continuous co-existence of the three organisms in the Itebukunmi river. However, all the eigenvalues may be negative if $n_1 < (1-\rho_1)\tau_1 e_1$ and $n_2 < (1-\rho_2)\tau_2 e_2$. Under this condition, the equilibrium point $\mathcal{E}_1 = (0,0,0)$ becomes stable and all the organisms are eradicated from the river. This is completely unacceptable and could be guided against, if the populations of the cat fish and the ophiocephalus fish which are to be hunted are highly regulated. That is, $\rho_1 \to 1$ and $\rho_2 \to 1$ may instigate $n_1 > (1-\rho_1)\tau_1 e_1$, $n_2 > (1-\rho_2)\tau_2 e_2$ and consequently bring about unstable equilibrium which implies continuous co-existence of the creatures in the river.

Case 2: *Only cat fish is non-existence*

Suppose $\mathcal{E}_2$ exists then at the equilibrium

$$\mathcal{E}_2 = \left(0, \frac{\phi + (1-\rho_3)\tau_3 e_3}{m_2}, \frac{n_2[m_2 - \phi - (1-\rho_2)\tau_2 e_2] - m_2(1-\rho_2)\tau_2 e_2}{m_2\theta_2}\right)$$

$$J(\mathcal{E}_2) = \begin{pmatrix} n_1 - \theta_1 J_1 - (1-\rho_1)\tau_1 e_1 & 0 & 0 \\ -\alpha_2 \left\{\dfrac{\phi + (1-\rho_3)\tau_3 e_3}{m_2}\right\}^2 & J_2 & -\theta_2 \left\{\dfrac{\phi + (1-\rho_3)\tau_3 e_3}{m_2}\right\} \\ m_1 J_1 & m_2 J_1 & 0 \end{pmatrix}, \qquad (3.4.10)$$

where $J_1 = r = \left(\dfrac{n_2[m_2 - \phi - (1-\rho_2)\tau_2 e_2] - m_2(1-\rho_2)\tau_2 e_2}{m_2\theta_2}\right)$ and $J_2 = \dfrac{n_2(m_2 - 2[\phi + (1-\rho_3)\tau_3 e_3])}{m_2} - \dfrac{n_2[m_2 - \phi - (1-\rho_2)\tau_2 e_2] - m_2(1-\rho_2)\tau_2 e_2}{m_2} - (1-\rho_2)\tau_2 e_2$. The eigenvalues of (3.4.10) can be obtained from (3.4.11)

$$[n_1 - \theta_1 J_1 - (1-\rho_1)\tau_1 e_1 - \lambda][\lambda^2 - J_2\lambda + \theta_2 J_1(\phi + (1-\rho_3)\tau_3 e_3)] = 0 \qquad (3.4.11)$$

If $n_1 < 0$ and $J_1 > 0$, the characteristic equation (3.4.11) consists of at least one positive eigenvalue and the equilibrium point $\mathcal{E}_2$ is unstable. The equilibrium $\mathcal{E}_2$ can be stable if all the eigenvalues of (3.4.11) are negative. if there exists at least one imaginary root for (3.4.11) then the equilibrium $\mathcal{E}_2$ is unstable. The possibility of the existence of complex roots for the equilibrium point $\mathcal{E}_2$ instigates the concept of bifurcation for the equilibrium point $\mathcal{E}_2$ which shall be discussed later.

Case 3: *Only ophiocephalus fish is non-existence*

Suppose $\mathcal{E}_3$ exists then at the equilibrium

$$\mathcal{E}_3 = \left(\frac{\phi + (1-\rho_3)\tau_3 e_3}{m_1}, 0, \frac{n_1[m_1 - \phi - (1-\rho_3)\tau_3 e_3] - m_1(1-\rho_1)\tau_1 e_1}{m_1\theta_1}\right),$$

$$J(\mathcal{E}_3) = \begin{pmatrix} J_3 & -\alpha_1\left\{\dfrac{\phi+(1-\rho_3)\tau_3e_3}{m_1}\right\}^2 & -\theta_1\left\{\dfrac{\phi+(1-\rho_3)\tau_3e_3}{m_1}\right\} \\ 0 & n_2-\theta_2J_1-(1-\rho_2)\tau_2e_2 & 0 \\ m_1J_1 & m_2J_1 & 0 \end{pmatrix}, \quad (3.4.12)$$

where $J_1 = r = \left(\dfrac{n_2[m_2-\phi-(1-\rho_2)\tau_2e_2]-m_2(1-\rho_2)\tau_2e_2}{m_2\theta_2}\right)$ and $J_3 = \dfrac{n_1(m_1-2[\phi+(1-\rho_3)\tau_3e_3])}{m_1} - \dfrac{n_1[m_1-\phi-(1-\rho_1)\tau_1e_1]-m_1(1-\rho_1)\tau_1e_1}{m_1} - (1-\rho_1)\tau_1e_1$.

The eigenvalues of (3.4.12) can be obtained from (3.4.13)

$$[n_2-\theta_2J_1-(1-\rho_2)\tau_2e_2-\lambda][\lambda^2-J_3\lambda+\theta_1J_1(\phi+(1-\rho_3)\tau_3e_3)] = 0 \quad (3.4.13)$$

If $n_2 < 0$ and $J_1 > 0$, the characteristic equation (3.4.13) has at least one positive eigenvalue and the equilibrium point $\mathcal{E}_3$ is unstable. The equilibrium $\mathcal{E}_3$ is stable if all the eigenvalues of (3.4.13) are negative. if there exists at least one imaginary root for (3.4.13) then the equilibrium $\mathcal{E}_3$ is unstable and bifurcation occurs.

Case 4: *Only predator (African knife fish) is non-existence*

Suppose $\mathcal{E}_4$ exists then at the equilibrium

$$\left(\frac{n_2(1-q)-(1-\rho_2)\tau_2e_2}{\alpha_2q}, \frac{n_1(1-p)-(1-\rho_1)\tau_1e_1}{\alpha_1p}, 0\right),$$

$$J(\mathcal{E}_4) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (3.4.14)$$

where

$$
\begin{aligned}
a_{11} &= n_1(1-2p) - 2\alpha_1 pq - (1-\rho_1)\tau_1 e_1,\\
a_{12} &= -\alpha_1 \left\{ \frac{n_2(1-q) - (1-\rho_2)\tau_2 e_2}{\alpha_2 q} \right\}^2,\\
a_{13} &= -\theta_1 \left\{ \frac{n_2(1-q) - (1-\rho_2)\tau_2 e_2}{\alpha_2 q} \right\},\\
a_{21} &= -\alpha_2 \left\{ \frac{n_2 - (1-\rho_2)\tau_2 e_2}{p} \right\}^2,\\
a_{22} &= n_2(1-2q) - 2\alpha_2 pq - (1-\rho_2)\tau_2 e_2,\\
a_{23} &= -\theta_2 \left\{ \frac{n_1(1-p) - (1-\rho_1)\tau_1 e_1}{\alpha_1 p} \right\},\\
a_{31} &= 0,\\
a_{32} &= 0,\\
a_{33} &= -\phi + m_1 \left\{ \frac{n_2(1-q) - (1-\rho_2)\tau_2 e_2}{\alpha_2 q} \right\} + m_2 \left\{ \frac{n_1(1-p) - (1-\rho_1)\tau_1 e_1}{\alpha_1 p} \right\} - (1-\rho_3)\tau_3 e_3.
\end{aligned}
\tag{3.4.15}
$$

The characteristic equation of (3.4.14) is

$$
\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{3.4.16}
$$

Following Routh-Hurwitz criterion, $\mathcal{E}_4$ is stable if b_1, b_2, b_3 are nonnegative and $b_1 b_2 - b_3 > 0$.

Case 5: *Coexistence of the three fishes*

Suppose $\mathcal{E}_5$ exists then at the equilibrium

$$
,\left(\frac{n_2(1-q) - \theta_2 r - (1-\rho_2)\tau_2 e_2}{\alpha_2 q}, \frac{n_1(1-p) - \theta_1 r - (1-\rho_1)\tau_1 e_1}{\alpha_1 p}, \frac{A_1}{A_2} \right)
$$

$$
J(\mathcal{E}_5) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \tag{3.4.17}
$$

The equilibrium $\mathcal{E}_5$ is stable if all the eigenvalues of the characteristic equation of the matrix $J(\mathcal{E}_5)$ are negative otherwise $J(\mathcal{E}_5)$ is unstable.

3.4.4 Global Stability Analysis of the Equilibrium Points

We investigate the global stability of the equilibria $\mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$ and $\mathcal{E}_5$ for our model by relying on Bendixson-Dulac and Poincare-Bendixson theorems as well as Lyapunov functions approach. Bendixson-Dulac and Poincare-Bendixson theorems are to be used to examine global asymptotic stability in certain subspaces (e.g., the pq- or qr-planes). Lyapunov function method is used for constructive proof of global stability of equilibrium points in $\mathbb{R}^3$. Both theorems can be used to investigate global stability of 2-Dimensional systems.

Proof of global stability of $\mathcal{E}_2$ by Poincare-Bendixson-Dulac theorem

Theorem 3.5. *The equilibrium point $\mathcal{E}_2$ is globally asymptotically stable in the interior ${\mathbb{R}_+}^2$ of the $qr-$ plane.*

Proof

For any value in the interior ${\mathbb{R}_+}^2$ of the $qr-$ plane, the model reduces to the following subsystem.

$$\begin{aligned}\frac{dq}{dt} &= n_2q(1-q) - \alpha_2pq^2 - \theta_2qr - (1-\rho_2)\tau_2e_2q \\ \frac{dr}{dt} &= -\phi r + m_1pr + m_2qr - (1-\rho_3)\tau_3e_3r\end{aligned} \tag{3.4.18}$$

Following Bendixson-Dulac and Poincare-Bendixson theorem Dennis (2012) we have

$$H(q,r) = \frac{1}{qr} \text{ where } q > 0, r > 0$$

$$\begin{aligned}\nabla\cdot(HF) &= \frac{\partial}{\partial q}(H\cdot F_1) + \frac{\partial}{\partial r}(H\cdot F_2) \\ &= \frac{\partial}{\partial q}\left[\frac{1}{qr}(n_2q(1-q) - \alpha_2pq^2 - \theta_2qr - (1-\rho_2)\tau_2e_2q)\right] \\ &+ \frac{\partial}{\partial r}\left[\frac{1}{qr}(-\phi r + m_1pr + m_2qr - (1-\rho_3)\tau_3e_3r)\right], \\ &= \frac{\partial}{\partial q}\left[\left(\frac{n_2(1-q)}{r} - \frac{\alpha_2pq}{r} - \theta_2 - \frac{(1-\rho_2)\tau_2e_2)}{r}\right)\right] \\ &+ \frac{\partial}{\partial r}\left[\left(\frac{-\phi}{q} + \frac{m_1p}{q} + m_2 - \frac{(1-\rho_3)\tau_3e_3)}{q}\right)\right], \\ &= \frac{n_2}{r} - \frac{\alpha_2p}{r} \\ &< 0,\end{aligned} \tag{3.4.19}$$

if $\frac{n_2}{r} < \frac{\alpha_2p}{r}$. So, $\nabla\cdot(HF) < 0 \;\forall\; (q,r) \in \mathbb{R}^2_+$ provided $\frac{n_2}{r} < \frac{\alpha_2p}{r}$. According to

Bendixson-Dulac criterion, there is no periodic solution in the interior $\mathbb{R}^2_+$ of $qr-$ plane. Since all the solutions of the model are bounded and $\mathcal{E}_2$ is a unique positive equilibrium point in $\mathbb{R}^2_+$ of the $qr-$ plane, hence, by Poincare-Bendixson-Dulac theorem, $\mathcal{E}_2$ is globally asymptotically stable in $\mathbb{R}^2_+$ provided $\frac{n_2}{r} < \frac{\alpha_2 p}{r}$.

Proof of global stability of $\mathcal{E}_2$ by Lyapunov function

Theorem 3.6. *The boundary equilibrium $\mathcal{E}_2$ is globally asymptotically stable in the interior $\mathbb{R}^3_+$*

Proof

We want to investigate using Lyapunov functional approach if the time derivative of the Lyapunov functional could be negative. Suppose the Lyapunov function for the system at $\mathcal{E}_2$ is defined as

$$V_1(p,q,r) = l_1\left(q - q^* - q^* log\left(\frac{q}{q^*}\right)\right) + l_2\left(r - r^* - r^* log\left(\frac{r}{r^*}\right)\right)$$

$$\begin{aligned}\frac{dV_1}{dt} &= l_1\left(\frac{q-q^*}{q}\right)\frac{dq}{dt} + l_2\left(\frac{r-r^*}{r}\right)\frac{dr}{dt}\\ &= l_1\left(\frac{q-q^*}{q}\right)(n_2 q(1-q) - \alpha_2 p q^2 - \theta_2 q r - (1-\rho_2)\tau_2 e_2 q)\\ &+ l_2\left(\frac{r-r^*}{r}\right)(-\phi r + m_1 p r + m_2 q r - (1-\rho_3)\tau_3 e_3 r),\\ &= l_1(q-q^*)(n_2(1-q) - \alpha_2 p q - \theta_2 r - (1-\rho_2)\tau_2 e_2)\\ &+ l_2(r-r^*)(-\phi + m_1 p + m_2 q - (1-\rho_3)\tau_3 e_3).\end{aligned} \tag{3.4.20}$$

Suppose $n_2(1-q) = \alpha_2 p q^*$ then

$$\begin{aligned}\frac{dV_1}{dt} &= l_1(q-q^*)(\alpha_2 p q^* - \alpha_2 p q - (\theta_2 r + (1-\rho_2)\tau_2 e_2))\\ &+ l_2(r-r^*)(m_1 p + m_2 q - (\phi + (1-\rho_3)\tau_3 e_3))\\ &= l_1(q-q^*)(\alpha_2 p(q^* - q) - (\theta_2 r + (1-\rho_2)\tau_2 e_2))\\ & l_2(r-r^*)(m_1 p + m_2 q - (\phi + (1-\rho_3)\tau_3 e_3))\end{aligned} \tag{3.4.21}$$

l_1 and l_2 are nonnegative Lyapunov constants. Choosing $l_1 = \frac{1}{p}$, $l_2 = 1$ and suppose $m_1 p +$

$m_2 p < \phi + (1-\rho_3)\tau_3 e_3$

$$\frac{dV_1}{dt} = -(q-q^*)(\alpha_2(q-q^*) + \frac{1}{p}(\theta_2 r - (1-\rho_2)\tau_2 e_2)) \\ -(r-r^*)(m_1 p + m_2 q + (\phi + (1-\rho_3)\tau_3 e_3)). \tag{3.4.22}$$

The time derivative of the Lyapunov function V_1 is negative if

$m_1 p + m_2 p < \phi + (1-\rho_3)\tau_3 e_3$ then the system is globally asymptotically stable at the boundary equilibrium point $\mathcal{E}_2$.

proof of global stability of $\mathcal{E}_2$ by Poincare-Bendixson-Dulac theorem

Theorem 3.7. *The equilibrium point $\mathcal{E}_3$ is globally asymptotically stable in the interior $\mathbb{R}^2_+$ of the $pr-$ plane.*

Proof

For any value in the interior $\mathbb{R}^2_+$ of the $pr-$ plane, the model reduces to the following subsystem.

$$\begin{aligned} \frac{dp}{dt} &= n_1 p(1-p) - \alpha_1 p^2 q - \theta_1 pr - (1-\rho_1)\tau_1 e_1 p \\ \frac{dr}{dt} &= -\phi r + m_1 pr + m_2 qr - (1-\rho_3)\tau_3 e_3 r \end{aligned} \tag{3.4.23}$$

$$H(p,r) = \frac{1}{pr} \text{ where } p>0, r>0$$

$$\begin{aligned} \nabla\cdot(HF) &= \frac{\partial}{\partial p}(H\cdot F_1) + \frac{\partial}{\partial r}(H\cdot F_2) \\ &= \frac{\partial}{\partial p}\left[\frac{1}{pr}(n_1 p(1-p) - \alpha_1 p^2 q - \theta_1 pr - (1-\rho_1)\tau_1 e_1 p)\right] \\ &+ \frac{\partial}{\partial r}\left[\frac{1}{pr}(-\phi r + m_1 pr + m_2 qr - (1-\rho_3)\tau_3 e_3 r)\right], \\ &= \frac{\partial}{\partial p}\left[\left(\frac{n_1(1-p)}{r} - \frac{\alpha_1 pq}{r} - \theta_1 - \frac{(1-\rho_1)\tau_1 e_1}{r}\right)\right] \\ &+ \frac{\partial}{\partial r}\left[\left(\frac{-\phi}{p} + m_1 + \frac{m_2 q}{p} - \frac{(1-\rho_3)\tau_3 e_3}{p}\right)\right], \\ &= \frac{n_1}{r} - \frac{\alpha_1 q}{r} \\ &< 0, \end{aligned} \tag{3.4.24}$$

if $\frac{n_1}{r} < \frac{\alpha_1 q}{r}$. So, $\nabla\cdot(HF) < 0 \;\; \forall \;\; (p,r) \in \mathbb{R}^2_+$. Following Poincare-Bendixson-Dulac theorem, $\mathcal{E}_3$ is globally asymptotically stable in the interior $\mathbb{R}^2_+$ if $\frac{n_1}{r} < \frac{\alpha_1 q}{r}$.

Proof of global stability of $\mathcal{E}_3$ by Lyapunov function

Theorem 3.8. *The boundary equilibrium $\mathcal{E}_3$ is globally asymptotically stable in the interior $\mathbb{R}^3_+$*

Proof

For any value in the interior $\mathbb{R}^2_+$ of the $pr-$ plane, the model reduces to the following subsystem. Constructing Lyapunov function V_3 for the system at $\mathcal{E}_3$ as follows

$$V_3(p,q,r) = l_3\left(p - p^* - p^* log\left(\frac{p}{p^*}\right)\right) + l_4\left(r - r^* - r^* log\left(\frac{r}{r^*}\right)\right) \tag{3.4.25}$$

Differentiate equation (3.4.25) with respect to t gives

$$\begin{aligned}\frac{dV_3}{dt} &= l_3\left(\frac{p-p^*}{p}\right)\frac{dp}{dt} + l_4\left(\frac{r-r^*}{r}\right)\frac{dr}{dt}\\ &= l_3\left(\frac{p-p^*}{p}\right)(n_1p(1-p) - \alpha_1p^2q - \theta_1pr - (1-\rho_1)\tau_1e_1p)\\ &+ l_4\left(\frac{r-r^*}{r}\right)(-\phi r + m_1pr + m_2qr - (1-\rho_3)\tau_3e_3r),\\ &= l_3(p-p^*)(n_1(1-p) - \alpha_1pq - \theta_1r - (1-\rho_1)\tau_1e_1)\\ &+ l_4(r-r^*)(-\phi + m_1p + m_2q - (1-\rho_3)\tau_3e_3).\end{aligned} \tag{3.4.26}$$

Suppose $n_1(1-p) = \alpha_1p^*q$ then

$$\begin{aligned}\frac{dV_3}{dt} &= l_3(p-p^*)(\alpha_1p^*q - \alpha_1pq - (\theta_1r + (1-\rho_1)\tau_1e_1))\\ &+ l_4(r-r^*)(m_1p + m_2q - (\phi + (1-\rho_3)\tau_3e_3))\\ &= l_3(p-p^*)(\alpha_1q(p^*-p) - (\theta_1r + (1-\rho_1)\tau_1e_1))\\ & l_4(r-r^*)(m_1p + m_2q - (\phi + (1-\rho_3)\tau_3e_3))\end{aligned} \tag{3.4.27}$$

Choosing $l_3 = \dfrac{1}{q}$, $l_4 = 1$ (as always)

$$\begin{aligned}\frac{dV_3}{dt} &= -(p-p^*)(\alpha_1(p-p^*) + \frac{1}{q}(\theta_1r + (1-\rho_1)\tau_1e_1))\\ &- (r-r^*)(m_1p + m_2q + (\phi + (1-\rho_3)\tau_3e_3))\\ &< 0,\end{aligned} \tag{3.4.28}$$

provided that $m_1p + m_2p < \phi + (1-\rho_3)\tau_3e_3$. Hence, $\mathcal{E}_3$ is globally asymptotically stable if $m_1p + m_2p < \phi + (1-\rho_3)\tau_3e_3$.

Proof of global stability of $\mathcal{E}_4$ by Poincare-Bendixson-Dulac theorem

Theorem For any value in the interior $\mathbb{R}^2_+$ of the $pq-$ plane, the model reduces to the following subsystem.

$$\begin{aligned}\frac{dp}{dt} &= n_1p(1-p) - \alpha_1p^2q - \theta_1pr - (1-\rho_1)\tau_1e_1p\\ \frac{dq}{dt} &= n_2q(1-q) - \alpha_2pq^2 - \theta_2qr - (1-\rho_2)\tau_2e_2q\end{aligned} \tag{3.4.29}$$

$$H(p,r) = \frac{1}{pq} \quad \text{where} \;\; p>0, q>0$$

$$\begin{aligned}\nabla\cdot(HF) &= \frac{\partial}{\partial p}(H\cdot F_1) + \frac{\partial}{\partial r}(H\cdot F_2)\\ &= \frac{\partial}{\partial p}\left[\frac{1}{pq}(n_1p(1-p) - \alpha_1p^2q - \theta_1pr - (1-\rho_1)\tau_1e_1p)\right]\\ &+ \frac{\partial}{\partial q}\left[\frac{1}{pq}(n_2q(1-q) - \alpha_2pq^2 - \theta_2qr - (1-\rho_2)\tau_2e_2q)\right],\\ &= \frac{\partial}{\partial p}\left[\left(\frac{n_1(1-p)}{q} - \alpha_1p - \frac{\theta_1r}{q} - \frac{(1-\rho_1)\tau_1e_1)}{q}\right)\right]\\ &+ \frac{\partial}{\partial q}\left[\left(\frac{n_2(1-q)}{p} - \alpha_2q - \frac{\theta_2r}{p} - \frac{(1-\rho_2)\tau_2e_2)}{q}\right)\right],\\ &= \frac{n_1}{q} + \frac{n_2}{p} - (\alpha_1+\alpha_2)\\ &< 0,\end{aligned} \tag{3.4.30}$$

if $\frac{n_1}{q} + \frac{n_2}{p} < (\alpha_1+\alpha_2)$. Following Poincare-Bendixson-Dulac theorem, $\mathcal{E}_4$ is globally asymptotically stable in the interior $\mathbb{R}^2_+$ if $\frac{n_1}{q} + \frac{n_2}{p} < (\alpha_1+\alpha_2)$.

Proof of global stability of $\mathcal{E}_4$ by Lyapunov function

Theorem The boundary equilibrium $\mathcal{E}_4$ is globally asymptotically stable in the interior $\mathbb{R}^3_+$.

Proof

Constructing Lyapunov function V_4 for the system at $\mathcal{E}_4$ as follows

$$V_4(p,q,r) = l_5\left(p - p^* - p^*log\left(\frac{p}{p^*}\right)\right) + l_6\left(q - q^* - q^*log\left(\frac{q}{q^*}\right)\right)$$

$$
\begin{aligned}
\frac{dV_4}{dt} &= l_5\left(\frac{p-p^*}{p}\right)\frac{dp}{dt} + l_6\left(\frac{q-q^*}{q}\right)\frac{dq}{dt}\\
&= l_5\left(\frac{p-p^*}{p}\right)(n_1p(1-p) - \alpha_1p^2q - \theta_1pr - (1-\rho_1)\tau_1e_1p)\\
&+ l_6\left(\frac{q-q^*}{q}\right)(n_2q(1-q) - \alpha_2pq^2 - \theta_2qr - (1-\rho_2)\tau_2e_2q),\\
&= l_5(p-p^*)(n_1(1-p) - \alpha_1pq - \theta_1r - (1-\rho_1)\tau_1e_1)\\
&+ l_6(q-q^*)(n_2(1-q) - \alpha_2pq - \theta_2r - (1-\rho_2)\tau_2e_2).
\end{aligned}
\tag{3.4.31}
$$

Suppose $n_1(1-p) = \alpha_1p^*q$ and $n_2(1-q) = \alpha_2pq^*$ then

$$
\begin{aligned}
\frac{dV_4}{dt} &= l_5(p-p^*)(\alpha_1p^*q - \alpha_1pq - (\theta_1r + (1-\rho_1)\tau_1e_1))\\
&+ l_6(q-q^*)(\alpha_2pq^* - \alpha_2pq - (\theta_2r + (1-\rho_2)\tau_2e_2))\\
&= l_5(p-p^*)(\alpha_1q(p^*-p) - (\theta_1r + (1-\rho_1)\tau_1e_1))\\
&l_6(q-q^*)(\alpha_2p(q^*-q) - (\theta_2r + (1-\rho_2)\tau_2e_2))
\end{aligned}
\tag{3.4.32}
$$

Choosing $l_1 = \frac{1}{q}$ and $l_1 = \frac{1}{p}$ then

$$
\begin{aligned}
\frac{dV_4}{dt} &= -(p-p^*)(\alpha_1(p-p^*) + \frac{1}{q}(\theta_1r + (1-\rho_1)\tau_1e_1))\\
&- (q-q^*)(\alpha_2(q-q^*) + \frac{1}{p}(\theta_2r + (1-\rho_2)\tau_2e_2))\\
&< 0.
\end{aligned}
\tag{3.4.33}
$$

Hence, $\mathcal{E}_4$ is globally asymptotically stable .

Proof of global stability of $\mathcal{E}_5$ by Lyapunov function

The Poincare-Bendixson Theorem as well as Dulac's and Bendixon's criteria apply only in two dimensions (Gasull and Giacomini, 2012; Cherkas and Grin, 2010; Osuna and Villasenor-Aguilar, 2013). For this reason, the proof for global stability of $\mathcal{E}_5$ will be based on the Lyapunov function approach only.

Theorem The boundary equilibrium $\mathcal{E}_5$ is globally asymptotically stable in the interior $\mathbb{R}^3_+$

Proof

Constructing Lyapunov function V_5 for the system at $\mathcal{E}_5$ as follows:

$$\begin{aligned} V_5(p,q,r) &= l_7\left(p - p^* - p^* log\left(\frac{p}{p^*}\right)\right) + l_8\left(q - q^* - q^* log\left(\frac{q}{q^*}\right)\right) \\ &+ l_9\left(r - r^* - r^* log\left(\frac{r}{r^*}\right)\right) \end{aligned} \tag{3.4.34}$$

differentiating equation(3.4.34) with respect to t yields

$$\begin{aligned} \frac{dV_5}{dt} &= l_7\left(\frac{p-p^*}{p}\right)\frac{dp}{dt} + l_8\left(\frac{q-q^*}{q}\right)\frac{dq}{dt} + l_9\left(\frac{r-r^*}{r}\right)\frac{dr}{dt} \\ &= l_7\left(\frac{p-p^*}{p}\right)(n_1p(1-p) - \alpha_1p^2q - \theta_1pr - (1-\rho_1)\tau_1e_1p) \\ &+ l_8\left(\frac{q-q^*}{q}\right)(n_2q(1-q) - \alpha_2pq^2 - \theta_2qr - (1-\rho_2)\tau_2e_2q), \\ &+ l_9\left(\frac{r-r^*}{r}\right)(-\phi r + m_1pr + m_2qr - (1-\rho_3)\tau_3e_3r), \\ &= l_7(p-p^*)(n_1(1-p) - \alpha_1pq - \theta_1r - (1-\rho_1)\tau_1e_1), \\ &+ l_8(q-q^*)(n_2(1-q) - \alpha_2pq - \theta_2r - (1-\rho_2)\tau_2e_2), \\ &+ l_9(r-r^*)(-\phi + m_1p + m_2q - (1-\rho_3)\tau_3e_3). \end{aligned} \tag{3.4.35}$$

Suppose $n_1(1-p) = \alpha_1p^*q$ and $n_2(1-q) = \alpha_2pq^*$ then

$$\begin{aligned} \frac{dV_5}{dt} &= l_7(p-p^*)(\alpha_1p^*q - \alpha_1pq - (\theta_1r + (1-\rho_1)\tau_1e_1) \\ &+ l_8(q-q^*)(\alpha_2pq^* - \alpha_2pq - (\theta_2r + (1-\rho_2)\tau_2e_2) \\ &+ l_9(r-r^*)(m_1p + m_2q - (\phi + (1-\rho_3)\tau_3e_3) \\ &= l_7(p-p^*)(\alpha_1q(p^*-p) - (\theta_1r + (1-\rho_1)\tau_1e_1)) \\ &+ l_8(q-q^*)(\alpha_2p(q^*-q) - (\theta_2r + (1-\rho_2)\tau_2e_2)) \\ &+ l_9(r-r^*)(m_1p + m_2q - (\phi + (1-\rho_3)\tau_3e_3)) \end{aligned} \tag{3.4.36}$$

Choosing $l_7 = \dfrac{1}{q}$, $l_8 = \dfrac{1}{p}$ and $l_9 = 1$ then

$$\begin{aligned} \frac{dV_5}{dt} &= -(p-p^*)(\alpha_1(p-p^*) + \frac{1}{q}(\theta_1 r + (1-\rho_1)\tau_1 e_1)) \\ &\quad - (q-q^*)(\alpha_2(q-q^*) + \frac{1}{p}(\theta_2 r + (1-\rho_2)\tau_2 e_2)) \\ &\quad - l_9(r-r^*)(m_1 p + m_2 q - (\phi + (1-\rho_3)\tau_3 e_3)) \\ &\quad < 0, \end{aligned} \tag{3.4.37}$$

provided that $m_1 p + m_2 q < (\phi + (1-\rho_3)\tau_3 e_3)$. Hence, $\mathcal{E}_5$ is globally asymptotically stable if $m_1 p + m_2 q < (\phi + (1-\rho_3)\tau_3 e_3)$.

The complementary use of the Poincare-Bendixson-Dulac criterion and Lyapunov approach to study global stability of the system strengthens the overall global stability results.

Having studied the system stability, the dynamical behavior of the ecosystem under parameter variation needed to be studied. This concept is known as bifurcation in ecology.

3.4.5 Bifurcation Analysis

Bifurcation refers to a qualitative change in the system behavior as a parameter varies.

It is of various types with different biological interpretations: transcritical, tangent or ford, Hopf bifurcation, etc. (Majeed, 2021; Ayoade et al., 2022; Dong et al., 2024; Wu and Jiao, 2019).

The present analysis shall focus on Hopf bifurcation.

The existence of Hopf bifurcation is investigated for the model by relying on the conditions in Majeed (2021) which has been employed in Dong et al. (2024). If e_2, human harvesting effort rate on q, is chosen as the bifurcation parameter then from the equilibrium point $\mathcal{E}_5$ in (3.4.6),

$$e_2^* = \frac{\theta_1 + \theta_2 + n_1(1-p) + n_2(1-q) - pq(\alpha_1+\alpha_2) - (1-\rho_1)\tau_1 e_1}{(1-\rho_2)\tau_2}. \tag{3.4.38}$$

. The Jacobian matrix of the system evaluated with the bifurcation parameter is given as

$$J(\mathcal{E}_5)|_{e_2=e_2^*} = \begin{pmatrix} f_4' & -\alpha_1 p^2 & -\theta_1 p \\ -\alpha_2 q^2 & f_5' & -\theta_2 q \\ m_1 r & m_2 r & A_3 \end{pmatrix} \tag{3.4.39}$$

where $f_4' = n_1(1-2p) - 2\alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1$, $f_5' = n_2(1-2q) - 2\alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2^*$ and $A_3 = -\phi + m_1 p + m_2 q - (1-\rho_3)\tau_3 e_3$. The system undergoes a Hopf bifurcation when the eigenvalues are complex conjugate. Suppose the eigenvalues of the Jacobian $J(\mathcal{E}_5)$ are $-\omega,\ \ \mu_{1,2} = \omega(h) \pm i\beta(h)$, where according to Majeed (2021),

$$\begin{aligned} \omega(h) &= trace\ \ J(\mathcal{E}_5), \\ \beta(h) &= det\ \ J(\mathcal{E}_5). \end{aligned} \tag{3.4.40}$$

$$\begin{aligned} \omega(h) &= n_1(1-p) + n_2(1-q) - 2\alpha_1 pq - 2\alpha_2 pq - \theta_1 r - \theta_2 r - (1-\rho_1)\tau_1 e_1 - (1-\rho_2)\tau_2 e_2^* \\ &\quad - (\phi - m_1 p - m_2 q + (1-\rho_3)\tau_3 e_3), \end{aligned} \tag{3.4.41}$$

$$\begin{aligned} \beta(h) &= [n_1(1-p) - 2\alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1] \\ &\quad \times [(n_2(1-q) - 2\alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2^*)(-\phi + m_1 p + m_2 q + (1-\rho_3)\tau_3 e_3) + \theta_2 m_2 qr] \\ &\quad + \alpha_1 p^2[\alpha_2 q(\phi - m_1 p - m_2 q + (1-\rho_3)\tau_3 e_3) + \theta_2 m_1 qr] \\ &\quad - \theta_1 r[-\alpha_2 m_2 qr - m_1 r(n_2(1-q) - 2\alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2^*)]. \end{aligned} \tag{3.4.42}$$

Following Majeed (2021), as employed in Dong et al. (2024), the system undergoes Hopf bifurcation if the following conditions are met

$$\begin{aligned} \omega(e_2^*) &= 0, \\ \beta(e_2^*) &\neq 0, \\ \frac{\partial \omega(h)}{\partial e_2^*} &\neq 0. \end{aligned} \tag{3.4.43}$$

The condition shall be verified one after the other. Putting (3.4.38) into (3.4.41) and (3.4.42) then

$$
\begin{aligned}
\omega(e_2^*) &= -\phi - (\theta_1+\theta_2)(r+1) - pq(\alpha_1+\alpha_2) \\
&\quad - (1-\rho_3)\tau_3 e_3 + m_1 p + m_2 q.
\end{aligned}
\tag{3.4.44}
$$

$$
\begin{aligned}
\beta(e_2^*) = {} & [n_1(1-p) - 2\alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1] \\
& \times [(-n_1(1-p) - (\theta_1+\theta_2) - \theta_2 r - 2\alpha_2 pq + pq(\alpha_1+\alpha_2) + (1-\rho_1)\tau_1 e_1) \\
& (-\phi + m_1 p + m_2 q + (1-\rho_3)\tau_3 e_3) + \theta_2 m_2 qr] \\
& + \alpha_1 p^2[\alpha_2 q(\phi - m_1 p - m_2 q + (1-\rho_3)\tau_3 e_3) + \theta_2 m_1 qr] \\
& + m_1\theta_1 r^2(-n_1(1-p) - (\theta_1+\theta_2) - \theta_2 r - 2\alpha_2 pq + pq(\alpha_1+\alpha_2) \\
& + (1-\rho_1)\tau_1 e_1) + \alpha_2\theta_1 m_2 qr^2.
\end{aligned}
\tag{3.4.45}
$$

$$
\frac{\partial\omega(h)}{\partial e_2^*} = -(1-\rho_2)\tau_2. \tag{3.4.46}
$$

Going by the results in (3.4.44), (3.4.45) and (3.4.46) the model could undergo Hopf bifurcation. This is because the condition $\frac{\partial\omega(h)}{\partial e_2^*} \neq 0$ is satisfied. Besides, other two conditions, $\omega(e_2^*) = 0$ and $\beta(e_2^*) \neq 0$ could also be satisfied. Hence, there is the possibility of the existence of Hopf bifurcation for the model

3.4.6 Bioeconomic Equilibrium

The term bioeconomic equilibrium comes from the concepts of biological equilibrium and economic equilibrium. The concept was analyzed in Clark (1979), in Kar and Chaudhuri (2004), and was later studied by a number of researchers such as Raymond et al., (2019); Das et al., (2009); Bakht et al., (2017); Baba et al., (2021). Bioeconomic equilibrium for the proposed model in this study is as follows:

Suppose f_1, f_2 and f_3 are the cost per unit efforts for cat fishes, ophiocephalus fishes and knife fishes respectively. Suppose also, that g_1, g_2 and g_3 are the price per unit biomass of cat fishes, ophiocephalus fishes and knife fishes respectively while τ_1, τ_2 and τ_3 are catchability rates for cat fishes, ophiocephalus fishes and knife fishes respectively. Then, let

$$
\psi_1 = (g_1\tau_1 p - f_1)e_1, \psi_2 = (g_2\tau_2 q - f_2)e_2 \text{ and } \psi_3 = (g_3\tau_3 p - f_3)e_3.
$$

where ψ_1, ψ_2 and ψ_3 are the economic rent (net revenue) of cat fishes, ophiocephalus fishes and knife fishes respectively. Hence, the economic rent (net revenue) at any time is given by

$$\psi = \psi_1 + \psi_2 + \psi_3, ... \tag{3.4.47}$$

that is

$$\psi = (g_1\tau_1 p - f_1)e_1 + (g_2\tau_2 q - f_2)e_2 + (g_3\tau_3 p - f_3)e_3.$$

Therefore, $F_\infty(p_\infty, q_\infty, r_\infty, e_1\infty, e_2\infty, e_3\infty)$ is the bioeconomic equilibrium where $p_\infty, q_\infty, r_\infty, e_1\infty, e_2\infty$ and $e_3\infty$ are the bioeconomic values of cat fishes, ophiocephalus fishes, knife fishes, harvesting effort of cat fishes, harvesting effort of ophiocephalus fishes, and harvesting effort of knife fishes respectively, and it is given by setting each of $\frac{dp}{dt}, \frac{dq}{dt}$ and $\frac{dr}{dt}$ to zero so that the system (3.2.7)-(3.2.9) becomes

$$n_1 p(1-p) - \alpha_1 p^2 q - \theta_1 pr - (1-\rho_1)\tau_1 e_1 p = 0, \tag{3.4.48}$$

$$n_2 q(1-q) - \alpha_2 pq^2 - \theta_2 qr - (1-\rho_2)\tau_2 e_2 q = 0, \tag{3.4.49}$$

$$-\phi r + m_1 pr + m_2 qr - (1-\rho_3)\tau_3 e_3 r = 0. \tag{3.4.50}$$

The following cases are considered to determine bioeconomic equilibrium

case 1: If only ophiocephalus fish and knife fish fishery would remain in operation (i.e., $f_2 < g_2\tau_2 q$ and $f_3 < g_3\tau_3 r$.) Therefore,

$$q_\infty = \frac{f_2}{g_2\tau_2} \text{ and } r_\infty = \frac{f_3}{g_3\tau_3}. \text{ From (3.4.48) } e_1 = 0$$

$$\Longrightarrow n_1(1-p_\infty) - \alpha_1 p_\infty q_\infty - \theta_1 r_\infty = 0.$$

But, $q_\infty = \frac{f_2}{g_2\tau_2}$ and $r_\infty = \frac{f_3}{g_3\tau_3}$ and on substitution,

$$n_1(1-p_\infty) - \alpha_1 p_\infty\left(\frac{f_2}{g_2\tau_2}\right) - \theta_1\left(\frac{f_3}{g_3\tau_3}\right) = 0,$$

$$\Longrightarrow p_\infty = \frac{n_1 g_2\tau_2 g_3\tau_3 - \theta_1 f_3 g_2\tau_2}{n_1 g_2\tau_2 g_3\tau_3 + \alpha_1 g_3 f_2\tau_3}$$

case 2: If only cat fishes and knife fishes fishery would remain in operation (that is, $f_1 < g_1\tau_1 p$ and $f_3 < g_3\tau_3 r.$) so that we have

$$p_\infty = \frac{f_1}{g_1\tau_1} \text{ and } r_\infty = \frac{f_3}{g_3\tau_3}. \text{ From (3.4.49) } e_2 = 0 \Longrightarrow$$

$$n_2(1-q_\infty) - \alpha_2 p_\infty q_\infty - \theta_2 r_\infty = 0.$$

But, $p_\infty = \dfrac{f_1}{g_1\tau_1}$ and $r_\infty = \dfrac{f_3}{g_3\tau_3}$ and on substitution,

$$n_2(1-q_\infty) - \alpha_2 q_\infty \left(\frac{f_1}{g_1\tau_1}\right) - \theta_2\left(\frac{f_3}{g_3\tau_3}\right) = 0,$$

$$\Longrightarrow q_\infty = \frac{n_2 g_1\tau_1 g_3\tau_3 - \theta_2 g_1 f_3 \tau_1}{n_2 g_1\tau_1 g_3\tau_3 + \alpha_2 g_3 f_1 \tau_3}$$

case 3: If only cat fishes and ophiocephalus fishes fishery would remain in operation (i.e., $f_1 < g_1\tau_1 p$ and $f_2 < g_2\tau_2 q.$) so that we have

$$p_\infty = \frac{f_1}{g_1\tau_1} \text{ and } q_\infty = \frac{f_2}{g_2\tau_2}. \text{ From (3.4.50) } e_3 = 0 \Longrightarrow$$

$$-\phi + m_1 p_\infty + m_2 q_\infty = 0.$$

But, $p_\infty = \dfrac{f_1}{g_1\tau_1}$ and $q_\infty = \dfrac{f_2}{g_2\tau_2}$ and on substitution,

$$-\phi + m_1\left(\frac{f_1}{g_1\tau_1}\right) + m_2\left(\frac{f_2}{g_2\tau_2}\right) = 0,$$

$$\Longrightarrow \phi = -\frac{m_1 g_2\tau_2 + m_2 g_1\tau_1}{g_1 g_2\tau_1\tau_2}.$$

The result shows that human harvesting effort rates for the preys should be regulated to prevent the extinction of the predator.

case 4: if $f_1 > g_1\tau_1 p, f_2 > g_2\tau_2 q$ and $f_3 > g_3\tau_3 r,$ then the fishing cost is more than revenues for all three species and the whole fishery will be discontinued. This condition is more favorable to the organisms as they would be allowed to interact under normal condition without human interference.

case 5: if $f_1 < g_1\tau_1 p, f_2 < g_2\tau_2 q$ and $f_3 < g_3\tau_3 r,$ then the fishing cost is less than revenues for all three organisms, i.e., the fishery is more profitable and hence it would attract more fishermen and the whole fishery will be in full operation. In this case, government will have to come in with regulations when fishing operations become profitable for all the organisms to prevent overfishing. If the conditions $f_1 < g_1\tau_1 p$, $f_2 < g_2\tau_2 q$ and $f_3 < g_3\tau_3 r$ are fulfilled then

$$p_\infty = \frac{f_1}{g_1\tau_1}, q_\infty = \frac{f_2}{g_2\tau_2} \text{ and } r_\infty = \frac{f_3}{g_3\tau_3}.$$

So that from (3.4.48)-(3.4.50),

$$e_1\infty = \frac{n_1(1-p) - (\alpha_1 pq + \theta_1 r)}{(1-\rho_1)\tau_1}. \text{ Therefore, } e_1\infty > 0 \text{ iff}$$

$$n_1(1-p) > \alpha_1 pq + \theta_1 r \tag{3.4.51}$$

$$e_2\infty = \frac{n_2(1-q) - (\alpha_2 pq + \theta_2 r)}{(1-\rho_2)\tau_2}. \text{ Therefore, } e_2\infty > 0 \text{ iff}$$

$$n_2(1-q) > \alpha_2 pq + \theta_2 r \tag{3.4.52}$$

$$e_3\infty = \frac{m_1 p + m_2 q - \phi}{(1-\rho_3)\tau_3}. \text{ Therefore, } e_3\infty > 0 \text{ iff}$$

$$m_1 p + m_2 q > \phi \tag{3.4.53}$$

Hence, the bioeconomic equilibrium point $F_\infty(p_\infty, q_\infty, r_\infty, e_1\infty, e_2\infty, e_3\infty)$ exits if and only if the inequalities (3.4.51), (3.4.52) and (3.4.53) are true.

3.4.7 Optimal conservation policy

The conservation parameters ρ_1, ρ_2 and ρ_3 shall remain intact to regulate human effort rates on p, q and r respectively. In addition, time dependent variables $u_1(t)$ and $u_2(t)$ shall be introduced to increase the reproductions of p and q respectively. In addition to the available natural foods for p and q in Itebunkunmi river, artificial foods are added to boost the populations of the organisms in terms of a time-dependent variable $u(t)$ so that the proportion of $u(t)$ converted by p and q are $u_1(t)$ and $u_2(t)$ respectively. Since only p and q are to be boost to maximize the coexistence of all the organisms, we consider only the equations for p and q and incorporate the time-dependent boost variables which is the optimal control model given below.

$$\frac{dp}{dt} = [n_1(1-p) - \alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1]u_1(t)p, \tag{3.4.54}$$

$$\frac{dq}{dt} = [n_2(1-q) - \alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2]u_2(t)q. \tag{3.4.55}$$

The intervention set is given as

$$\Omega_1 = (u_1(t), u_2(t))|0 < u_1(t) < u_{1max}, 0 < u_2(t) < u_{2max}).$$

u_{1max} and u_{2max} represent the maximum limits for the impact of the boost on p and q respectively. Then, the boost on p and q is maximized and the costs of implementing the boost in a given period of time is minimized subject to the objective functional

$$\Gamma(u_1, u_2) = \min_{0 \leq u_1, u_2 \leq 1} \int_0^T [p(t)+q(t)+w_{21}u_1(t)p(t)+w_{22}u_1^2(t)+w_{31}u_2(t)q(t)+w_{32}u_2^2(t)]dt \tag{3.4.56}$$

The coefficients $w_{i,j}(i = 1, 2, 3; j = 1, 2)$ in monetary terms give the cost implications of the intervention strategies while the quadratic terms involved in the objective functional indicate the nonlinear nature of the costs particularly at the high level of intervention which is in agreement with the nonlinear nature of the model. The minimization procedure is based on the model (3.4.54)-(3.4.55) whose equations are regarded as the *state equations* in the optimal control context and the variables p and q are regarded as the *state variables*. Our main concern is to establish the optimal solutions, $u_1^*(t)$ and $u_2^*(t)$ that minimize the objective functional (3.4.56)

Following the approach in Ayoade et al. (2018) and by invoking the Pontryaging's maximum principle in Pontryagin et al. (1962), the necessary conditions for the optimality of the intervention are derived, using the Hamilton Adjoint Equations, Optimality Equation and Optimality conditions as follows

3.4.8 The Hamiltonian Adjoint Equations

Suppose the adjoint functions introduced are λ_1 and λ_2. Since there are two state variables, p and q then λ_1 and λ_2 corresponds to p and q respectively. The Hamiltonian, H can then be derived by finding the product of each adjoint function with its corresponding state equation and adding each of these products to the integrand of the objective functional. Hence,

$$\begin{aligned}H &= p(t) + q(t) + w_{21}u_1(t)p(t) + w_{22}u_1^2(t) + w_{31}u_2(t)q(t) + w_{32}u_2^2(t),\\ &+ \lambda_1[n_1(1-p) - \alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1]u_1(t)p,\\ &+ \lambda_2[n_2(1-q) - \alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2]u_2(t)q.\end{aligned} \tag{3.4.57}$$

The adjoint function (3.4.57) is characterized as:

$$\begin{aligned}\frac{\partial\lambda_1}{\partial t} &= -\frac{\partial H}{\partial p} = -(1 + w_{21}u_1(t)) + \lambda_2\alpha_2 q^2 u_2(t) - \lambda_1[n_1(1-2p) - 2\alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1]u_1(t),\\ \frac{\partial\lambda_2}{\partial t} &= -\frac{\partial H}{\partial q} = -(1 + w_{31}u_2(t)) + \lambda_1\alpha_1 p^2 u_1(t) - \lambda_2[n_2(1-2q) - 2\alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2]u_2(t)\end{aligned} \tag{3.4.58}$$

3.4.9 The Optimality Equation/ Optimality Condition

We may wish to minimize or maximize in optimal control theory depending on our intention. In the present study, we want to maximize the boost of p and q to enhance the conservation of the organisms in Itebunkunmi river. The optimal control is minimal if the second derivative of the Hamiltonian equation with respect to a control variable is positive otherwise it is maxima. that is

$$\frac{\partial^2}{\partial u^2} < 0 - \text{ maximum } and \ \frac{\partial^2}{\partial u^2} > 0 - \text{ minimum.}$$

In respect to the present study, the optimality equations are derived by determining the differential coefficients of the Hamiltonian equation (3.4.57) with respect to the control variables. The resulting equations are equated to zero and the control variables are solved for from which the optimal solutions, $u_1^*(t)$ and $u_2^*(t)$ are obtained subject to the lower and upper constraints $0 < u_1(t) < u_{1max}$ and $0 < u_2(t) < u_{2max}$. So

$$\frac{\partial H}{\partial u_1} = \frac{\partial H}{\partial u_2} = 0 \Longrightarrow$$

$$
\begin{aligned}
\frac{\partial H}{\partial u_1} &= w_{21}p(t) + 2w_{22}u_1(t) + \lambda_1[n_1(1-p) - \alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1]p = 0,\\
\frac{\partial H}{\partial u_2} &= w_{31}q(t) + 2w_{32}u_2(t) + \lambda_2[n_2(1-q) - \alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2]q = 0.
\end{aligned}
\tag{3.4.59}
$$

From which the optimal values for $u_1^*(t)$ and $u_2^*(t)$ are obtained as

$$
\begin{aligned}
u_1^*(t) &= -\left(\frac{w_{21} + \lambda_1[n_1(1-p) - \alpha_1 pq - \theta_1 r - (1-\rho_1)\tau_1 e_1]}{2w_{22}}\right)p(t),\\
u_2^*(t) &= -\left(\frac{w_{31} + \lambda_2[n_2(1-q) - \alpha_2 pq - \theta_2 r - (1-\rho_2)\tau_2 e_2]}{2w_{32}}\right)q(t)
\end{aligned}
\tag{3.4.60}
$$

To distinguish between the *minima* and *maxima*, we find the second derivative of (3.4.59) when $\frac{\partial H}{\partial u_1} \neq 0$ and $\frac{\partial H}{\partial u_2} \neq 0$, that is,

$$
\begin{aligned}
\frac{\partial^2 H}{\partial u_1^2} &= 2w_{22} > 0,\\
\frac{\partial^2 H}{\partial u_2^2} &= 2w_{32} > 0.
\end{aligned}
\tag{3.4.61}
$$

Since $\frac{\partial H}{\partial u_1} > 0$ and $\frac{\partial H}{\partial u_2} > 0$, the optimality is minimum. That is our objective is achieved under minimum cost. We are able to conserve the ecosystem for the benefit of the present and future generations with the least cost.

CHAPTER FOUR

Numerical Simulation

4.1 Simulation and Discussion of Results

Mathematical tools are employed to numerically simulate the formulated prey-predator system of equation. Numerical simulations and associated graphs offer numerical and visual perspectives on the long term behavior of the three fish species in the Itebukunmi river.

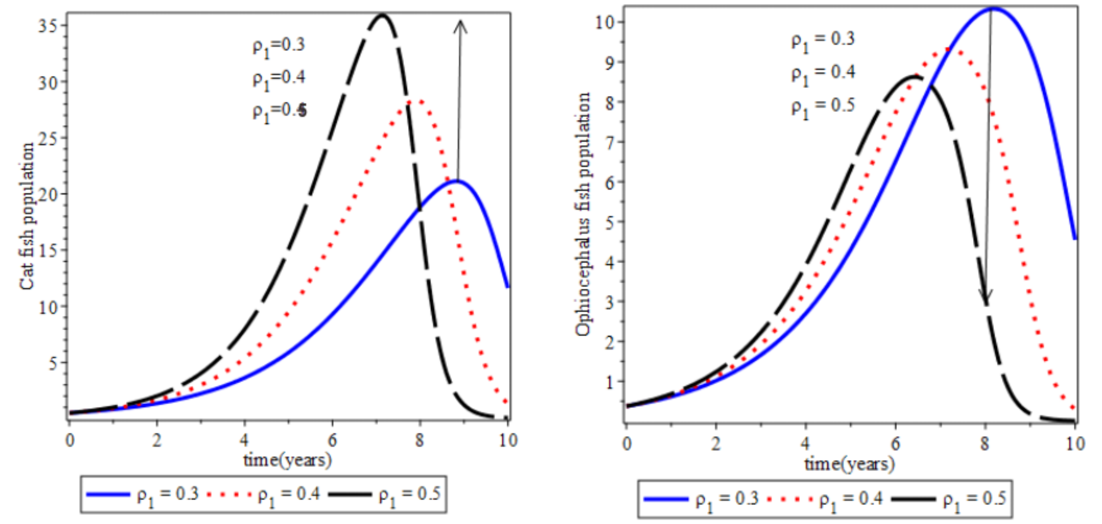
A computer-based mathematical package called MAPLE 17 is used to simulate the model, and the parameter values indicated in Table 4.1 are used to plot the graphs. The purpose of the simulation is to validate analytical results and analyze the long term behavior of the systems. It is difficult to come about real data for the variables and parameters of the model.

Therefore, we rely on the prevailing realities surrounding the dynamics of the three fish species in the Itebukunmi river as well as interaction with the fisher men and women from Itebukunmi community, to provide a reasonable approximation for the model parameters and initial conditions as given in Table 4.1.

Table 4.1: Nomenclatures, values and sources for the model parameters

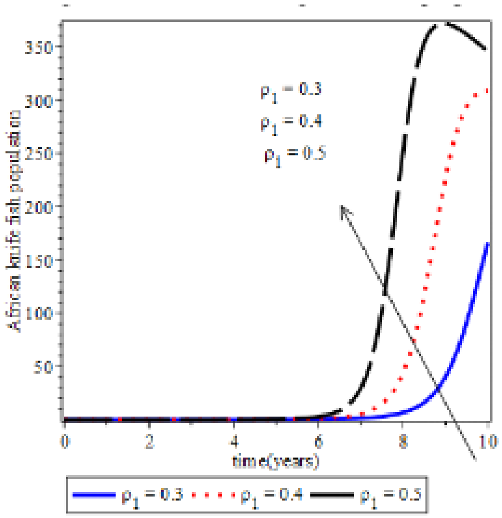
parameters	Nomenclatures	Values	Source
n_1	Growth rate of p	0.5	EDIC
n_2	Growth rate of q	0.5	EDIC
α_1	Reduction rate for p due to intra and inter competition	0.001	EDIC
α_2	Reduction rate for q due to intra and inter competition	0.002	EDIC
θ_1	Decrease rate effects of r on p	0.01	EDIC
θ_2	Decrease rate effects of r on q	0.01	EDIC
ϕ	Decrease rate effects on r due to scarceness of p and q	0.5	EDIC
τ_1, τ_2, τ_3	Catchability rates for p, q and r respectively	0.003, 0.02, 0.001	EDIC
m_1, m_2	Increase rate effects of p on r and of q on r respectively	0.1, 0.2	EDIC
e_1, e_2, e_3	Human harvesting effort rates on p, q and r respectively	0.01, 0.02, 0.03	EDIC EDIC
ρ_1, ρ_2, ρ_3	Regulatory rates to conserve p, q and r respectively	0.3 0.4, 0.5	EDIC
k_1, k_2	environmental carrying capacity for p and q respectively	3.5, 2.5 (in millions)	EDIC

The obtained in Table 4.1 are estimated from the interaction with the fisher men and women from Itebukunmi community, in Ondo state, Nigeria. We denote EDIC to be Estimated Data from Itebukunmi community. With the parameter values in Table 4.1 and the initial conditions $(p, q, r) = (200, 150, 50)$respectively, for the ecological variables, plots are generated to visualize the effects of parameter perturbations on the long-term behavior of the systems.



(a) Effect of regulating harvesting rate of p on the population of p.

(b) Effect of regulating harvesting rate of p on the population of q.



(c) Effect of regulating harvesting rate of p on the population of r.

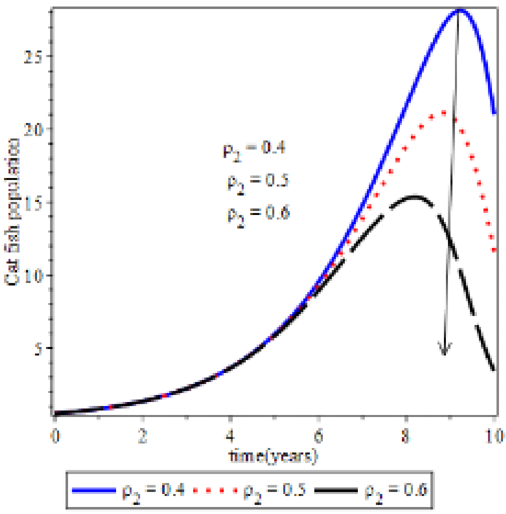
Figure 4.1: Simulation showing the effects of conservation of only cat fish on the population of the three fish species.

The plots in Figure 4.1 illustrate a situation where conservation efforts are placed on only cat fish.

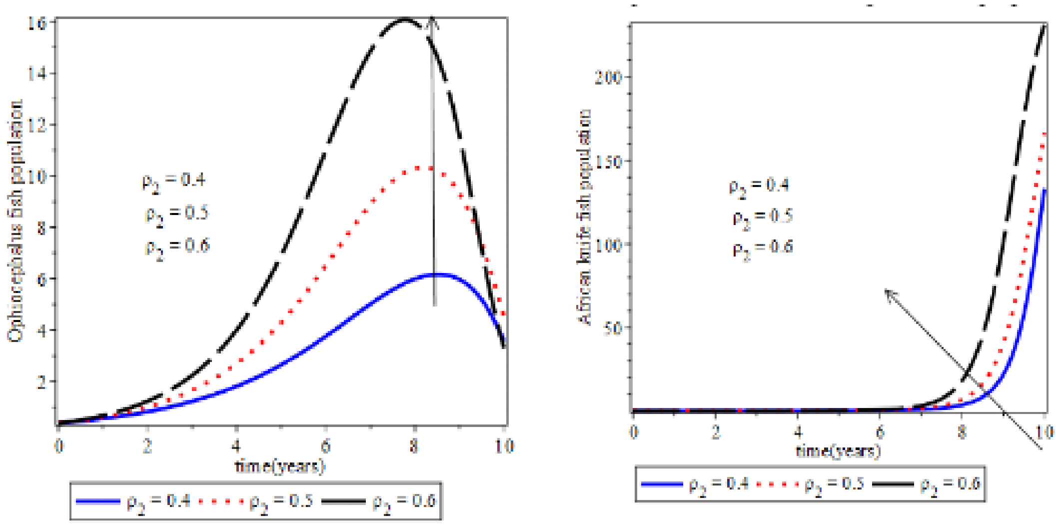
Figure 4.1(a): Shows the percentage of cat fish population in Itebukunmi river when the control of harvesting Cat fish is considered . It is observed that as ρ_1 increases from 0.3 to 0.5 the percentage of cat fish population in Itebukunmi river will increase for the period of 0 to 10 years. This can be seen with the direction of the arrow in the plot. Hence at 10 years when $p_1 = 0.3$ the percentage of cat fish in Itebukunmi river will be about 1100%, and when $\rho_1 = 0.4$ the percentage will be about 100% and when $\rho_1 = 0.5$ the percentage will be about 0% It is also observed that when $\rho_1 = 0.3$ cat fish attained its highest percentage at about 9 years period of time which is about 2000%. And when $\rho_1 = 0.4$, the cat fish population attained its highest percentage at about 2800% Also when $\rho_1 = 0.5$, the cat fish population attained its highest percentage at about 7 years 8 months period of time which is about 3600%

Figure 4.1(b) Show the percentage of Ophiocephalus fish population in the Itebukunmi river when the control of harvesting Cat fish is considered. It is observed that as ρ_1 increases from 0.3 to 0.5 the percentage of ophiocephalus fish population in Itebukunmi river will decrease for a period of 0 to 10 years .This can be seen with the direction of the arrow in the plot. Hence at 10 years when $\rho_1 = 0.3$ the percentage will be about 500%, when $\rho_1 = 0.4$ the percentage will be about 20% And when $\rho_1 = 0.5$ the percentage will be about 0%. It is also observed that when $\rho_1 = 0.3$ it attained the highest percentage at about 7 years 8 months period of time which about 400% Also when $\rho_1 = 0.4$, the fish attained its highest percentage at about 7 years period of time which is about 900% Also when $\rho_1 = 0.5$, the cat fish attained its highest percentage at about 7 years 3 months period of time which is about 850%

Figure 4.1(c) Shows the percentage of Africa knife fish in Itebukunmi river, when the control of harvesting Cat fish is considered. It is observed that as ρ_1 increases from 0.3 to 0.5 the percentage of African knife fish will increase for the period of 0 to 10 years. This can be seen with the direction of the arrow in the plot. Hence at 10 years when $\rho_1 = 0.3$ the percentage of the fish will be about 15000%, when $\rho_1 = 0.4$, the percentage will be about 31000%, and when $\rho_1 = 0.5$ the percentage will be about 37000%



(a) Effect of regulating harvesting rate of q on the population of p.



(b) Effect of regulating harvesting rate of q on the population of q.

(c) Effect of regulating harvesting rate of q on the population of r.

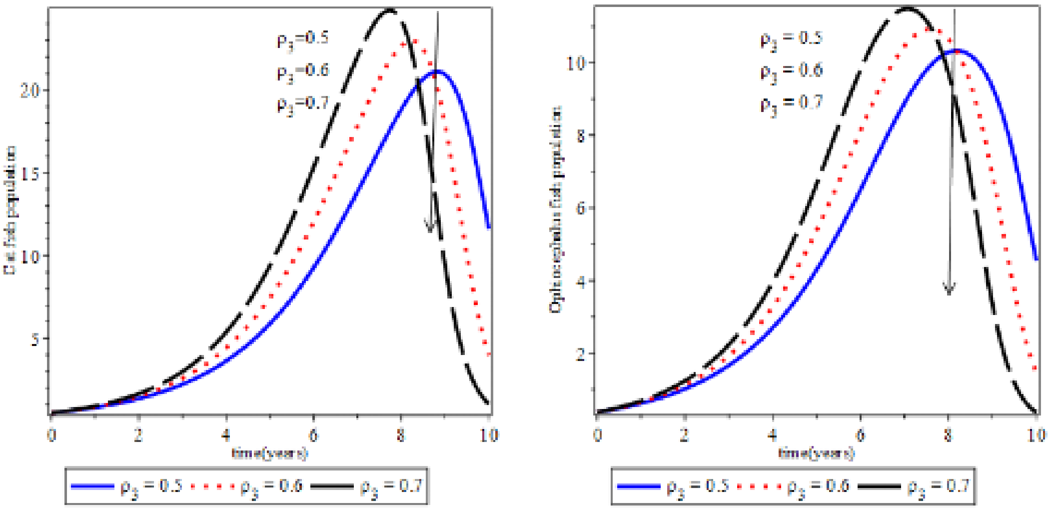
Figure 4.2: Simulation showing the effects of conservation of only ophiocephalus fish on the population of the three fish species.

Figure 4.2(a) shows cat fish population in Itebukunmi river when the control of harvesting Ophiocephalus fish is considered. It is observed that as ρ_2 increases from 0.4 to 0.6, the percentage of cat fish population in Itebukunmi river will decrease for the period of 0 to 10 years. This can be seen with the direction of the arrow in the plot. Hence, at 10 years, when $\rho_2 = 0.4$ the percentage of cat fish in Itebukunmi river will be about 2100%, when $\rho_2 = 0.5$, the percentage of cat fish in Itebukunmi river will be about 1200%, while when $\rho_2 = 0.6$, the percentage will be about 300%. It is also observed that when $\rho_2 = 0.4$, cat fish attained its highest percentage at about 9 years, period of time, which is about 2800%. And when $\rho_2 = 0.5$

the fish attained its highest percentage at about 8 years 10 months period of time, which is about 2100%. Also when $\rho_2 = 0.6$ the fish attained its highest percentage at about 8 years period of time which is about 1500%

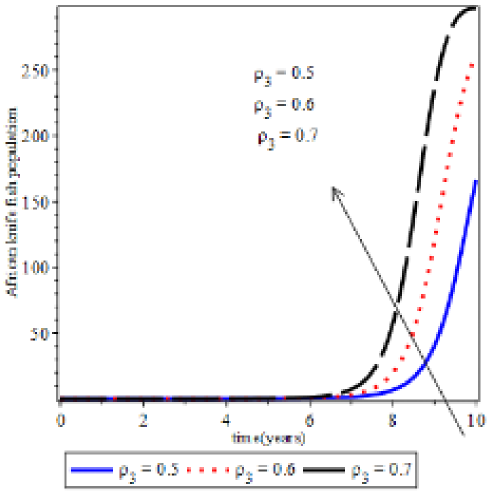
Figure 4.2(b): shows the percentage of Ophiocephalus fish in Itebukunmi river when the control of harvesting Ophicephalus fish is considered. It is observed that as ρ_2 increases from 0.4 to 0.6 the percentage of ophiocephalus fish increases from period of 0 years to 10 years. This can be seen with the direction of the arrow in the plot. Hence at 10 years when $\rho_2 = 0.4$, the percentage of ophiocephalus fish will be about 300%, when $\rho_2 = 0.5$, the percentage of ophiocephalus fish will be about 250%, and when $\rho_2 = 0.6$, the percentage of ophiocephalus fish will be about 240%. It is also oberved that when $\rho_2 = 0.4$ the Ophiocephalus fish attained its highest percentage at about 8 years period of time which is about 500%. And when $\rho_2 = 0.5$ the fish attained its highest percentage at about 8 years 10 months period of time, which is about 1000%. Also And when $\rho_2 = 0.6$ the ophiocephalus fish attained its highest percentage at about 7 years 10 months period of time, which is about 1600%.

Figure 4.2(c): shows the percentage of gymnachus niloticus fish in Itebukunmi river when the control of harvesting Ophiocephalus fish is considered. It is observed that as ρ_2 increases from 0.4 to 0.6 the percentage of gymnachus niloticus fish increases from period of 0 years to 10 years. This can be seen with the direction of the arrow in the plot. Hence at 10 years when $\rho_2 = 0.4$, the percentage of gymnachus niloticus fish will be about 14000%, when $\rho_2 = 0.5$, the percentage of gymnachus niloticus fish will be about 150%, and when $\rho_2 = 0.6$, the percentage of gymnachus niloticus fish will be about 25000%..



(a) Effect of regulating harvesting rate of r on the population of p.

(b) Effect of regulating harvesting rate of r on the population of q.



(c) Effect of regulating harvesting rate of r on the population of r.

Figure 4.3: Simulation showing the effects of conservation of only the African knife fish on the population of the three fish species.

Figure 4.3(a) Shows the percentage of Cat fish in Itebukunmi river when the control of harvesting African knife fish is considered. It is observed that as ρ_3 increases from 0.5 to 0.7, the percentage of the fish population in Itebukunmi river will decrease for the period of 0 to 10 years. This can be seen with the direction of the arrow in the plot. Hence, at 10 years, when $\rho_3 = 0.5$ the percentage of the fish in Itebukunmi river will be about 1200%, when $\rho_3 = 0.6$, the percentage of the fish in Itebukunmi river will be about 1800%, while when $\rho_3 = 0.7$, the percentage will be about 1700%. It is also observed that when $\rho_3 = 0.5$ Cat fish attained its highest percentage at about 9 years period of time, which is about 2100% also that when $\rho_3 = 0.6$ it attain its highest percentage at about 9 years 6 months period of time, which is about 2200%, And when $\rho_3 = 0.7$ the fish attain its highest percentage at about 6 years 6 months period of time, which is about 3000% .

Figure 4.3(b): show the percentage of Ophiocephalus fish in Itebukunmi rive rwhen the control of harvesting of African knife fish is considered. It is observed that as ρ_3 increases from 0.5 to 0.7, the percentage of the fish population in Itebukunmi river will decrease for the period of 0 to 10 years. This can be seen with the direction of the arrow in the plot. Hence at 10 years when $\rho_3 = 0.5$, the percentage of the fish in Itebukunmi river will about 500% when $\rho_3 = 0.6$ the percentage of the fish would be about 100% and when $\rho_3 = 0.7$, the percentage of the fish in Itebukunmi river will be about 0% It is also observed that when $\rho_3 = 0.5$ Ophiocephalus fish attained its highest percentage at about 8 years period of time which is about 1000%, also when $\rho_3 = 0.6$, it attain its highest percentage at about 7years 8months period of time which is about 1300%. And when $\rho_3 = 0.7$, the fish attained its highest percentage at about 7years period of time, which is about 1500%.

Figure 4.3(c): Show the percentage of African knife fish in Itebukunmi river when the control of harvesting African knife fish is considered; It is observed that as ρ_3 increases from 0.5 to 0.7, the percentage of African knife fish will increase for the period of 0 to 10 years. This can be seen with the direction of the arrow in the plot. Hence at 10 years when $\rho_3 = 0.5$, the percentage of the fish increases to about 16000% when $\rho_3 = 0.6$ the percentage will be about 25000% and when $\rho_3 = 0.7$, the percentage will be about 30000%

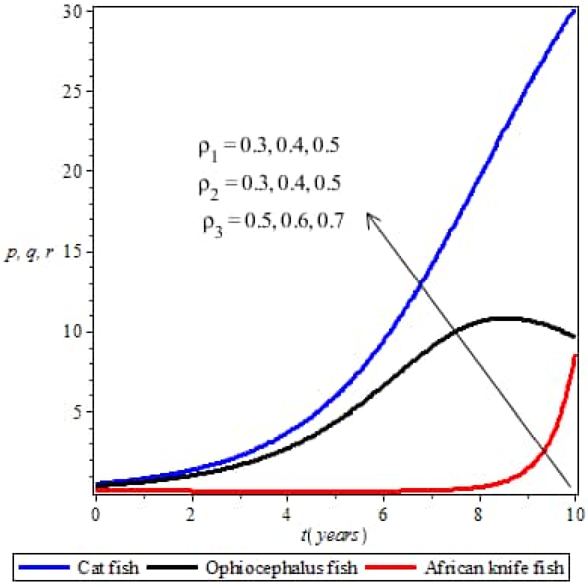


Figure 4.4: Simulation showing the effects of conservation of the three fishes on their populations.

Figure 4.4: Shows the percentage of the three species each varying p_1, p_2,and p_3. For the cat fish when p_1 varied form 0.3 to 0.5, the percentage increases from about 0% to 1000% within 10 years. Also for the ophiocephalus fish the percentage of population increases from about 0% to about 700%.within 10 years but reaching it highest percentage at about 9years, when p_2 is varied from 0.3 to 0.5, while the gymnachus niloticus fish increases from about 0% to about 800% within 10 years when p_3 is varied from 0.3 to 0.7.

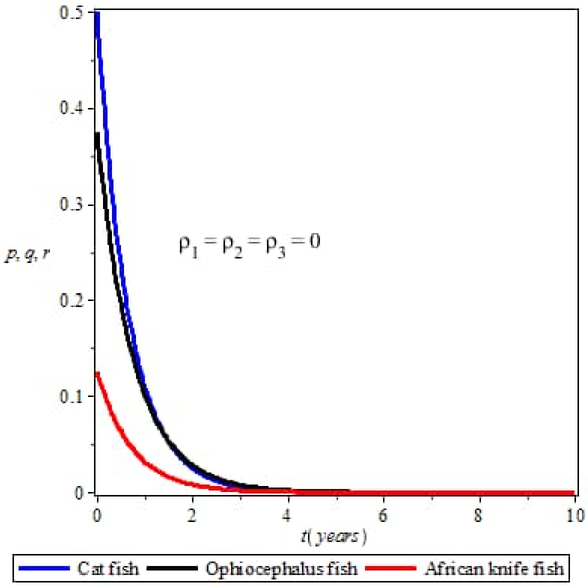
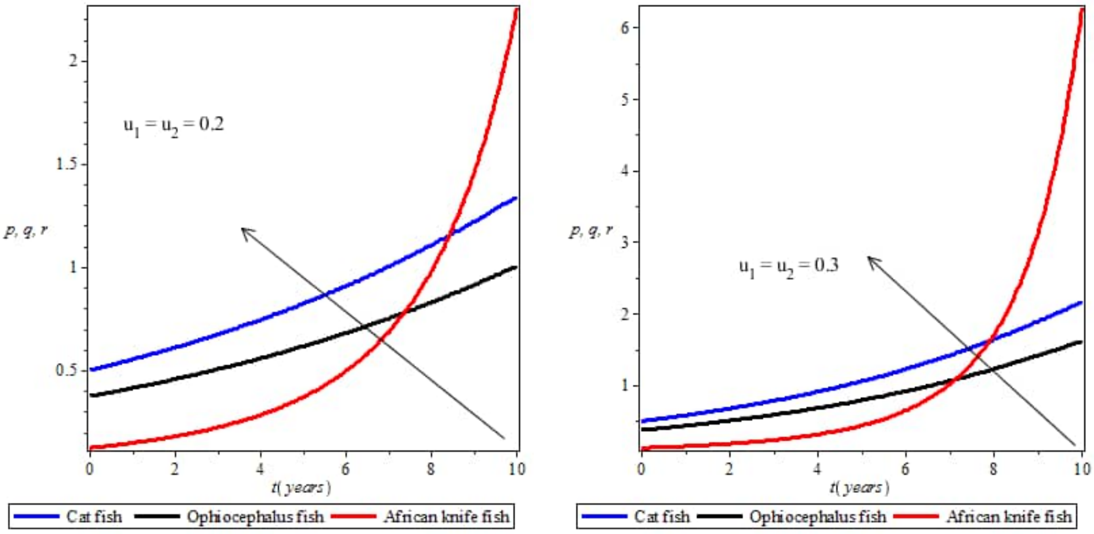


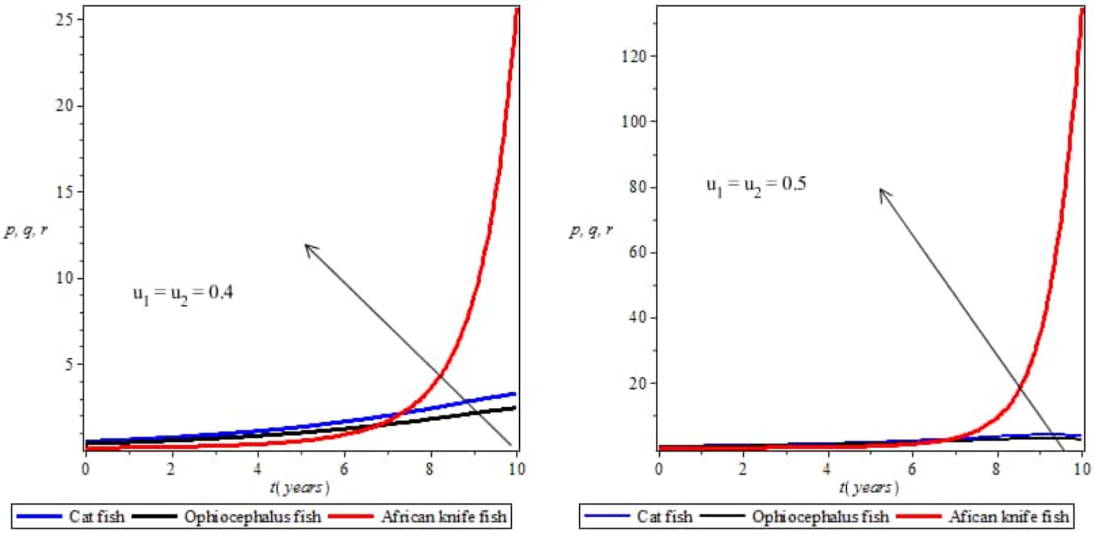
Figure 4.5: Simulation showing the effects of no regulation on the harvesting rates of the three fishes.

In Figure 4.5 shows the percentage of the three species in Itebukunmi river. where there is no regulation. The three fishes populated will be decreasing for the period of 0 to 10 years in the absence of regulation, Cat fish decreases from about 50% to about $1.5 \times 10^{-7}\%$ at the end of 10 years. While Ophiocephalus decreases from about 37% to $8.4 \times 10^{-7}\%$. Also gymnachus niloticus fish decreases from the percentage of 12% to $1.1 \times 10^{-7}\%$ at the end of 10 years.



(a) Effect of 20% boost on the populations of the three fish species.

(b) Effect of 30% boost on the populations of the three fish species.



(c) Effect of 40% boost on the populations of the three fish species.

(d) Effect of 50% boost on the populations of the three fish species.

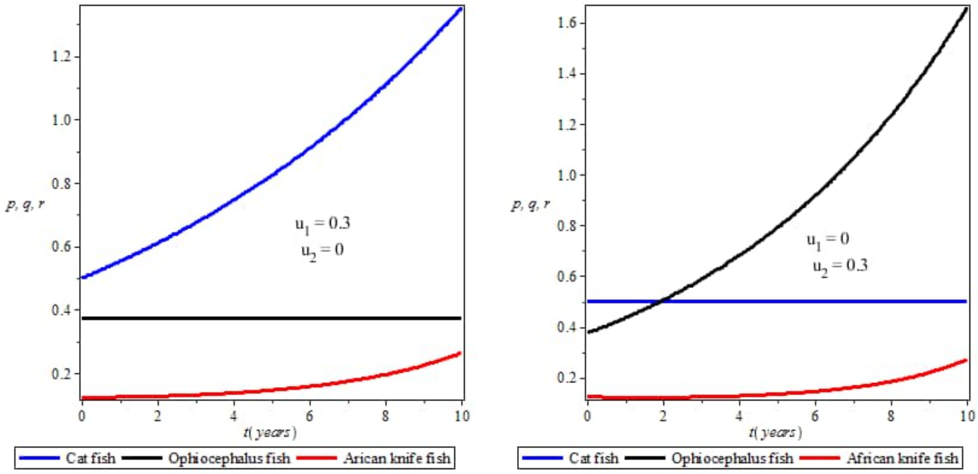
Figure 4.6: Simulation showing the effects of artificial approaches to boost the reproductions of p and q on the populations of the three fish species.

Figure 4.6(a): Shows the percentage of the three species when the cat fish and the Ophiocephalus, the preys were boosted. As the cat fish and the ophiocephalus fish were boosted with $u_1 = u_2 = 0.2$, the population of the Gymnachus niloticus (African knife fish) increases sharply with the 10 years, as shown in the plot.

Figure 4.6(b): Shows the plot of when cat fish and the ophiocephalus fish were boosted with $u_1 = u_2 = 0.3$ the population of the Gymnachus niloticus fish, the predator increases also sharply with the 0 to 10 years, but the population of the prey increases less

Figure 4.6(c): Shows the plot of when cat fish and the ophiocephalus fish were boosted with $u_1 = u_2 = 0.4$ the population of the predator, Gymnachus niloticus fish also increases from 10% to 2500% within 0 to 10 years.

But at Figure 4.6(d) Show the plot of when cat fish and the ophiocephalus fish were boosted with $u_1 = u_2 = 0.5$ the population of the preys were down,very low to about 1000% increase with 0 t0 10 years, but the population of the Gymnachus niloticus increases up to 13500% within 0 to the 10 years.



(a) Effect of boosting only p on the populations of p, q, r.

(b) Effect of boosting only q on the populations of p, q, r.

Figure 4.7: Simulation showing the effects of boosting only one prey on the populations of the three fish species.

Figure 4.7(a): Shows when only cat fish is boosted.The African knife fish increases at a very low rate within the 0 to the 10 years. While the Ophiocephalus fish was stationary at about 35% And the cat fish was increasing sharply at 50% in population to 130% within 0 to 10 years. Figure 4.7(b) Shows the percentage of the three fishes when only ophiocephalus fish is boosted with $U_2 = 0.3$. the Ophiocephalus fish increases sharply from 40% to 170% within 0 to 10 years. but the cat fish was stationary at 50% in population. Also the African knife fish increases at a very low rate within the 0 to the 10 years.

4.2 Summary of Discussion

Mathematical models have been revealed as powerful tools to analyze the dynamics of ecological variables, to understand their behaviors, to predict their social and economic impacts and to find out how external factors influence the impacts (Raymaond et al., 2019). In this thesis, a mathematical model has been proposed and analyzed to study the dynamics of a two-prey-one predator system with harvesting and conservation aspects. The model is used to study the ecological dynamics of the African knife-cat-ophiocephalus fishes prey-predator system of the Itebukunmi river fishery in Ondo State of Nigeria. The regulatory rates to conserve the fish resources are found to play an important role in stabilizing the system. Based on the simulation results, the regulation of the harvesting rates is so important that the populations of the three fishes could be conserved if the harvesting rate of just one of the fishes is seriously regulated.

Generally, if the harvesting rates of the cat and ophiocephalus fishes exceed their intrinsic growth rates, the populations of the three fish reources would become extinct with time. Therefore it is necessary that government include the boost of the reproductive tendencies of the cat and ophiocephalus fishes in its conservation policies. Again, it is found from both theoretical and numerical results that absence of regulation of harvesting rates of the three fish species would lead to collapse of the system and push the organisms to extinction whereas judicious regulation of their harvesting rates would ensure continuous co-existence and conservation of the fish resources. Thus, in order to use fish as a resource and produce maximum economic benefit while maintaining sustainable fishery species in the Itebukunmi river, the harvesting rates of species should be judiciously regulated.

CHAPTER FIVE

Summary and Conclusion

5.1 Summary

Despite the problem of fishing methods that endangered the aquatic life of Itebukunmi river, fishing is still one of the major occupation in Itebukunmi town. This thesis contribute to mathematical modeling in a two-prey, one predator model with harvesting and conservation management strategies, in which the gynnarchus niloticus acts as the predator while the cat fish and the ophilocephalus fish act as the preys, to boost the knowledge base of Itebukunmi aquatic activities .

Hence, activities surrounding prey-predator Mathematical model have been discussed as related to fishing in Itebukunmi river. Also relevant Literature focusing on the Mathematical modeling of prey-predator activities have be presented and open research questions with statement of problems which formed the basis of the research study has been stated therein. A proper formulated Mathematical model, consisting of system of non linear ordinary differential equarions and incorporating control mechanisms of harvesting strategies, has been used to investigate the dynamics of prey-predator in fisheries with respect to the Itebukunmi fishing ground, Ondo state, Nigeria. Standard mathematical analysis (both local and global), bifurcation analysis, bio-economic equilibrium and optimal conservation policy had been properly investigated. Also a computer-based Mathematics package called MAPLE 17 is used to simulate the model.

5.2 Findings

The findings from this study are as follows:

(i) as the control of the harvesting rate of cat fish increases, the population of cat fish and

African knife fish, in Itebukunmi river increases, while the Ophiocephalus fish population decreases.

(ii) as the control of the harvesting rate of Ophiocephalus fish increases, the population of Ophiocephalus fish and African knife fish, in Itebukunmi river increases, while the cat fish population decreases.

(iii) as the control of the harvesting rate of African knife fish increases, the population of African knife fish , in Itebukunmi river increases, while the cat fish and Ophiocephalus fish population decreases

5.3 Contributions to Knowledge

The study has contributed to knowledge in the following ways:

1. mathematical model for the study of aquatic life in Itebukunmi fishing ground was developed
2. a two prey- predator model that incorporated the activities of fishermen in Itebukunmi river was formulated.
3. model that would effectively guide policy makers in regulating fishing activities in Nigerian waterways, especially for Itebukunmi fishing ground was developed.

5.4 Conclusion

The model is rich in dynamical behaviour and establishes various conditions under which the preys can exist with predation. Both quantitative and numerical analysis results indicates that the harvesting rate could be controled to increase the population of the three fish species via the regulatory parameters and by the boosting of the cat and *Ophiocephalus* fishes at a conservation policies of fish resources in the Itebukunmi river.

5.5 Recommendation for Further Studies

This study can be extended in a number of ways such as;

(i) regarding how to estimate parameter values used in the model from field data.

(ii) The model can be improved by incorporating more prey/predator for more equations to capture more species.

(iii) Qualitative analysis on how to find the co-existence equilibrium point and conditions for local and global stability can be done following a different approach.

(iv) Other bifurcation beside Hopf bifurcation can be carried out on the model.

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Appendix

Figure 1: Pictures of some fish species, local canoe used for fishing, some of the river terminals, and traps used for catching fishes about Itebukunmi river are shown(See Appendix Figure A1-A10 for details)

Figure 1: Pictures of some fish species, local canoe used for fishing, some of the river terminals, and traps used for catching fishes about Itebukunmi river are shown(continued)

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