ESTIMATING THE PARAMETERS OF AUTOREGRESSISVE MODELS USING YULE-WALKER EQUATIONS

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A PROJECT REPORT SUBMITTED TO THE DEPARTMENT OF STATISTICS, FACULTY OF PHYSICAL SCIENCES, UNIVERSITY OF BENIN, BENIN CITY, EDO STATE, NIGERIA, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF BACHELOR OF SCIENCE (B.Sc.) DEGREE IN STATISTICS.

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CERTIFICATION

This is to certify that this project work was carried out by **ANU FAVOUR AKUNNA** with Matriculation Number **PSC2008396** under my supervision and it is adequate in scope and content for the award of Bachelor of Science **(B.Sc.)** Degree in Statistics of the University of Benin.

PROF. F. EWERE **DATE**

PROJECT SUPERVISOR

PROF. A. IDUSERI **DATE**

HEAD OF DEPARTMENT

DEDICATION

I dedicate this work to God, for giving me the wisdom and guidance to properly carry out the work and also for his protection throughout my stay in the University of Benin.

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Abstract:

This research will undertake a comprehensive statistical analysis of Nigeria's Exchange rate spanning a decade, with a focus on estimating Autoregressive (AR) models using a prominent statistical methods: the Yule-Walker method. The study aims to provide statistical insights into the underlying dynamics of Nigeria's economic performance during this period.

The research will commence by delineating the statistical framework of AR models, which offer a statistical representation of a time series based on its past values. Subsequently, the Yule-Walker method will be introduced, a statistical technique leveraging autocorrelation functions to estimate AR model parameters. The statistical properties of Yule-Walker estimators will be elucidated in the context of Nigeria's Exchange rate data.

In contrast, the Least Squares method will be presented as an alternative statistical approach, characterized by its objective to minimize the sum of squared prediction errors. A statistical framework for the least squares estimators will be outlined, providing insights into the statistical properties of parameter estimates and their significance in explaining variations in Nigeria's Exchange rate.

The core of the research involves the statistical analysis of Nigeria's Exchange rate time series data over the forty-three year period. The Yule-Walker method will be applied to estimate AR models tailored to the Exchange rate data. The statistical comparison will be based on goodness-of-fit statistics, such as the Akaike Information Criterion (AIC), to evaluate the models' adequacy in capturing the statistical patterns within the Exchange rate dataset.

CHAPTER ONE

INTRODUCTION

Time series is a set of observations $\{x_t\}$ which is recorded at a specific time t sequentially, over equal time increments or continuous time. If the set is of single observations, the series is called a univariate time series (Tingyan, 2010).

Autoregressive models, commonly referred to as AR models, are a form of statistical time series models used to forecast future values by analyzing past observations of a variable. In these models, the current value of a variable is presumed to have a linear relationship with its previous values, along with some added random noise. The term "auto" in autoregressive indicates that the model uses its own historical values for making predictions. Recently, there has been significant interest in working with time series data. There are two primary approaches for analyzing time series: frequency-domain methods and time-domain methods. Frequency-domain methods include techniques like spectral analysis and wavelet analysis, whereas time-domain methods involve autocorrelation and cross-correlation analysis. These approaches are widely applied in areas such as astronomy, weather forecasting, financial market analysis, economic activity monitoring, and more.

To better understand and represent time series data, various models have been developed. These include basic autoregressive (AR) models, simple moving-average (MA) models, combined autoregressive moving-average (ARMA) models, seasonal models, unit-root non-stationarity models, and fractionally differenced models for long-term dependencies. Of these, the autoregressive moving average (ARMA) model is one of the most essential and extensively utilized methods in time series analysis (Tingyan, 2010).

In this project, we will explore different methods for estimating the parameters of AR models and apply these estimation techniques to real-world time series data.

1.1 BACKGROUND OF THE STUDY

The autoregressive (AR) model, developed by Box and Jenkins in 1970, provides a framework for representing how the current value of a time series is influenced by its previous values through a linear regression approach. Introduced in the mid-1970s, autoregressive modeling first gained traction in nuclear engineering and soon became popular in various other fields (Beran, 1994). Today, AR modeling is extensively used for system identification, performance monitoring, malfunction detection, and diagnostic purposes across different industries.

Constructing an autoregressive model involves estimating a select number of parameters from time series data. Several techniques exist for calculating the AR coefficients, with the most prominent being the Least Squares method and Burg's method (Brockwell & Davis, 2016). The Least Squares method focuses on minimizing the sum of squared differences between the predicted values and the actual observations in the time series. Burg's method, on the other hand, utilizes a recursive algorithm to estimate the coefficients by leveraging the Yule-Walker equations.

The Yule-Walker equations are a set of linear equations that play a crucial role in autoregressive modeling. These equations relate the autocovariance of a time series at different lags to the AR model parameters. The Yule-Walker method applies least squares regression to solve these equations, providing estimates of the autoregressive coefficients. This approach is particularly effective because it ensures that the estimated coefficients satisfy the stationarity conditions required for AR models.

The Yule-Walker method is widely regarded as one of the most reliable techniques for parameter estimation in AR models due to its simplicity and efficiency. It is especially useful when the time series is stationary, as it directly relates the autocovariances to the model parameters, thereby simplifying the estimation process. By focusing on the Yule-Walker method, this project aims to enhance the understanding of its application and performance in estimating autoregressive models, contributing valuable insights to the field of time series analysis.

1.2 AIM AND OBJECTIVES OF THE STUDY

The primary objective of this project is to assess performance of the Yule-Walker method in estimating the parameters of an Autoregressive (AR) model. The specific goals of this project are outlined as follows:

1. To review the foundational concepts and theories of Autoregressive modeling and its importance in time series analysis. This involves exploring the basic principles behind AR models, understanding their development, and examining their role in various applications, particularly in forecasting and analyzing time-dependent data.
2. To investigate the estimation methods for autoregressive models, with an emphasis on the Yule-Walker method. This objective focuses on a detailed examination of different techniques used to estimate AR model parameters, with a special focus on the Yule-Walker method. The study will consider how this method utilizes the Yule-Walker equations to estimate AR coefficients, and how it compares to other methods in terms of efficiency, accuracy, and applicability.
3. To empirically compare the performance of these estimation methods on selected datasets, analyzing and interpreting the results to uncover their respective strengths and weaknesses. This objective aims to apply both the Least Squares and Yule-Walker

methods to real-world datasets, performing a comparative analysis to evaluate their performance. The results will be analyzed to identify the conditions under which each method excels or falls short, thereby providing insights into their practical use in time series analysis.

This project seeks to enhance the understanding of these estimation techniques by providing a thorough comparison based on empirical evidence, ultimately contributing to more informed decisions in the application of AR models.

1.3 SIGNIFICANCE OF THE STUDY

The study of Autoregressive (AR) models holds immense significance in the field of time series analysis and various application domains. Understanding and accurately estimating AR models offer valuable insights and practical benefits that contribute to advancements in research, decision-making, and problem-solving. Thus, this project work offers valuable insights and benefits in several key areas, making it a critical subject of investigation for statisticians and researchers. Some of the significance of estimating AR models are outlined below:

1. **Forecasting and Prediction:** AR models provide a powerful framework for forecasting future values based on past observations. Accurate predictions are crucial in decision-making processes for industries such as finance, economics, and meteorology. AR models enable stakeholders to anticipate

trends, identify potential risks, and make informed strategic choices, leading to improved planning and resource allocation. (Box et al., 2015) (Taylor, 2017)

2. **Process Control and Quality Assurance:** In industries where continuous monitoring is important, such as manufacturing and healthcare, AR models facilitate real-time process control and quality assurance. By detecting deviations from expected patterns, these models help identify anomalies and malfunctions, allowing for timely interventions and optimization of processes. (Montgomery, 2019) (Box et al., 2020)
3. **Financial Market Analysis:** AR models play a fundamental role in financial time series analysis. They assist in modeling stock prices, exchange rates, and other financial indicators, aiding investors, traders, and financial analysts in making well-informed decisions. Moreover, AR models contribute to risk assessment and volatility prediction, supporting portfolio diversification and risk management strategies. (Tsay, 2018) (Brooks, 2019)
4. **Economic Policy and Planning:** In the realm of economics, AR models are utilized to analyze economic indicators, such as GDP, Exchange rates, inflation rates, and unemployment. These models help policymakers understand the dynamics of the economy, identify trends, and evaluate the effectiveness of various policy interventions. The insights derived from AR

models guide economic planning and contribute to achieving stable and sustainable economic growth. (Enders, 2015) (Hamilton, 2020).

1.4 SCOPE OF THE STUDY

In this project work, the main focus will be on the method for parameter estimation for Autoregressive (AR) model parameters. Particularly, we shall focus on the Yule-Walker estimation method.

Furthermore, this study will be limited to application of AR models on secondary time series data on Nigeria Exchange rate, sourced from World-bank official site.

1.5 ORGANIZATION OF THE STUDY

The organization of the study is as follows:

The first chapter gives concise information on the background of the study. The second chapter will present a brief history and literature review on Autoregressive (AR) models by many scholars. The third chapter will examine the methodologies and statistical assumptions of AR models followed by the theories and concepts behind AR models parameter estimation techniques considered in this project work. The fourth chapter shows the application of AR models on real life time series data. The fifth chapter gives a brief discussion and conclusion drawn based on the results obtained in fourth of this work.

1.6 DEFINITION OF BASIC TERMS

The study of the Methods of Estimating AR models will not be comprehensive enough without some Statistical terms. For the purpose of research, some terms have been chosen and the meaning given to enable us carry out analysis. The following are the definitions:

1. **Autoregressive (AR) Model:** A time series model in which the current value of a variable is regressed on one or more of its past values. The AR model captures the temporal dependencies and patterns in the data, making it a valuable tool for forecasting and time series analysis.
2. **Time Series:** A sequence of data points observed at successive time intervals. Time series data is collected over time and is characterized by its temporal ordering, where each observation is dependent on previous observations.
3. **Lag:** Refers to the time interval between consecutive data points in a time series. It represents the time delay or gap between an observation and its preceding observation. The concept of lag is essential in understanding the temporal dependencies and patterns within the time series data
4. **Linear Regression:** A statistical technique that models the relationship between a dependent variable and one or more independent variables by fitting a linear equation to the observed data. In the context of AR models,

linear regression is used to represent the relationship between the current value of the time series and its lagged values.

5. **Parameter Estimation:** The process of determining the values of model parameters that best fit the data. In AR models, parameter estimation involves finding the coefficients that minimize the difference between the actual values and the predicted values.
6. **Model Order:** The number of lagged observations (past values) considered in the AR model. It determines the memory or dependence of the time series on its past values. Selecting the appropriate model order is crucial in AR modeling.
7. **Least Squares Method:** A common approach to estimate the parameters of an AR model by minimizing the sum of squared differences between the actual data points and the predicted values obtained from the model.
8. **Yule-Walker Equations:** A set of linear equations that relate the auto-covariance function of the time series at different lags to the unknown AR coefficients. By solving these equations, one can obtain the estimates of the autoregressive coefficients, which define the AR model.
9. **Forecasting:** The process of predicting future values of a time series based on historical data and the fitted AR model. Forecasting is essential for planning, decision-making, and risk assessment.

10. **Time Series Analysis:** The examination of time series data to identify patterns, trends, and underlying relationships. Time series analysis involves various techniques, including AR modeling, to gain insights into the behavior of the data over time.

SUMMARY

This chapter examines the definition of Autoregressive models and gives an introduction to the project work. In the next chapter we shall examine the historical background of AR models and attempt a review of some academic scholars work.

CHAPTER TWO

LITERATURE REVIEW

2.0. INTRODUCTION

Autoregressive (AR) and Autoregressive Moving Average (ARMA) models are foundational to time series analysis. This literature review seeks to examine the definitions, theoretical concepts, applications, and distinctions between AR and ARMA models by critically evaluating and comparing the contributions of various scholars. Additionally, accurate parameter estimation in AR models is crucial for reliable forecasting and for capturing the underlying temporal relationships within the data. Two widely used methods for estimating parameters in AR models are the Least Squares method and the Yule-Walker equations. This review will assess and contrast different studies addressing these estimation techniques. Moreover, the review will evaluate scholarly work on the application of Autoregressive Integrated Moving Average (ARIMA) models, a related class of AR models commonly employed in time series analysis.

2.1. LITERATURE REVIEW OF ARMA MODELS

According to Tingyan (2010), A Time series model for the observed time series $\{x_t\}$ is a specification of the joint distributions of the sequence of random variables $\{x_t\}$. Time series models takes up different forms and explains different stochastic processes, but are broad categorized into three categories namely: Autoregressive (AR) models, Moving Average (MA) models and Integrated (I) models. In the coming sections, we shall consider the definitions and concepts behind the above mentioned Time series models as opined by several authors and scholars in the past.

2.1.1 DEFINITIONS AND CONCEPTS OF AUTOREGRESSIVE (AR) AND AUTOREGRESSIVE MOVING AVERAGE MODELS (ARMA)

AR models represent a time series as a linear combination of its past values, where the current value depends on a fixed number of lagged observations. Brockwell and Davis (2019) provide a comprehensive definition of AR models and explain their mathematical formulation. They highlight that the order of an AR model, denoted by "p," determines the number of past observations used in the model. Thus an AR model of order p can be denoted as AR(p). Mathematically, an AR(p) model can be formulated as shown below:

$$x_t = \propto_0 + \propto_1 x_{t-1} + \propto_2 x_{t-2} + \dots + \propto_p x_{t-p} + \varepsilon_t$$

If we assume $\propto_0 = 0$, $we\ have\ x_t = \propto_1 x_{t-1} + \propto_2 x_{t-2} + \dots + \propto_p x_{t-p} + \varepsilon_t$

Which can be written as in linear form as: $\propto (\boldsymbol{\beta})\boldsymbol{X_t} = \boldsymbol{\varepsilon}_t$

Where $\propto (\boldsymbol{\beta}) = 1 - \propto_1 \beta - \propto_1 \beta^2 - \dots - \propto_1 \beta^p$ is a polynomial of degree p. The roots of $\propto (\boldsymbol{\beta}) = \mathbf{0}$ must lie outside the unit circle to ensure stationarity.

Where, x_t represents the current value of the time series at time t.

$\propto_0$ is the intercept or constant term in the model, representing the baseline level of the time series when all lagged terms are zero.

$\propto_1, \propto_2, \ldots, \propto_p$, are the autoregressive coefficients representing the influence of past values on the current value.

$x_{t-1}, x_{t-2}, \ldots, x_{t-p}$, are the lagged values of the time series up to order p, which are used to predict the current value.

ε_t is the error term at time t, representing the unpredictable or residual part of the time series not explained by the lagged terms.

AR(p) models are widely used in various fields, including finance, economics, and signal processing, for their ability to capture temporal dependencies and forecast future values. Box and Jenkins (1970), in their seminar work introduced the autoregressive (AR) model as a linear regression of a time series against one or more of its past values. They defined an AR(p) model of order p as follows:

"*The AR(p) model expresses the current value of a time series as a weighted sum of its p previous values, plus an error term. The model captures the temporal dependencies within the data, where the weights (coefficients) represent the influence of past observations on the current one*." Box and Jenkins (1970)

The definition provided by Box and Jenkins offers a clear and succinct introduction to autoregressive (AR) models, describing them as a linear regression of a time series on its previous values. The reference to the model's order "p" and the influence of past observations on current values underscores the key principles of AR models. This definition serves as a foundational reference for other definitions in time series analysis.

Hamilton (1994) argued that in an AR(p) model, the current value of a time series is determined by a linear combination of its past p values, along with an error term. This model captures the persistence of the series over time, as previous values influence future outcomes. Hamilton's statement highlights the importance of selecting an appropriate value for p, as the model’s predictive accuracy is affected by this choice. Consequently, careful selection of p is necessary to avoid biased estimates in the AR(p) model.

According to Granger (1980), an autoregressive model of order p, denoted AR(p), represents a time series as a linear combination of its previous p observations, each weighted by a coefficient. The model reflects the persistence and memory of the series, relying on its past values. Granger's definition aligns with Hamilton’s (1994) view and emphasizes the role of autoregressive models in studying causality and relationships within time series data. This perspective is particularly relevant for researchers investigating causality and interdependencies in time series.

The Moving Average (MA) model, as described by Tingyan (2010), involves a linear regression of the current value of the series on the error terms of one or more previous values. Similar to the AR model, the MA model has an order, denoted as "q," which specifies the number of past error terms used in the model. The selection of q is crucial for capturing short-term dependencies and fluctuations in the time series. Thus, the mathematical formulation of the MA(q) model is shown below:

$$x_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q}$$

$$if\ we\ assume\ \mu = 0, therefore\ \boldsymbol{X_t} = \boldsymbol{\theta}(\boldsymbol{\beta})\boldsymbol{\varepsilon}_t$$

$$where\ \boldsymbol{\theta}(\boldsymbol{\beta}) = 1 + \theta_1\, \beta^1 + \theta_2 \beta^2 + \dots + \boldsymbol{\theta}_q \beta^q$$

Where, μ the mean or constant term in the model, representing the expected or average value of the time series, ε_{t-1}, ε_{t-2}, …, ε_{t-q}, are lagged values of the error terms and θ_1, $\theta_2, \dots, \theta_q$ are the parameters of the model, known as the moving average coefficients. They represent the weights assigned to each lagged error term (ε) in the model.

It is worthy to note that AR(p) models are easy to estimate compared to the MA(q) models.

The combination of Autoregressive (AR) and Moving Average (MA) models results in the Autoregressive Moving Average (ARMA) model. ARMA models extend the basic AR framework by incorporating the MA component, representing a time series as a combination of its previous values and a linear combination of past white noise error terms. Shumway and Stoffer (2017) provide a comprehensive explanation of ARMA models, emphasizing the significance of both the autoregressive and moving average elements. They discuss the process of selecting the model orders, "p" for the number of autoregressive terms and "q" for the number of moving average terms, leading to the ARMA(p,q) model formulation. ARMA models are particularly advantageous for capturing both autocorrelation—where past error terms are correlated with current values—and the moving average effects present in the data. A mathematical representation of an ARMA(p,q) model is shown below:

$$x_t = \mu + \propto_1 x_{t-1} + \propto_2 x_{t-2} + \ldots + \propto_p x_{t-p} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

$$if\ we\ assume\ \ \mu = 0,\ we\ have \propto (\boldsymbol{\beta})\boldsymbol{X_t} = \ \boldsymbol{\theta}(\boldsymbol{\beta})\boldsymbol{\varepsilon}_t$$

Where all of the above have their usual meaning from the AR and MA models.

AR and ARMA models have their application in various fields, for example, in climate science AR and ARMA models are applied to forecast various meteorological variables. Bermejo et al. (2019) employed ARMA models to forecast monthly rainfall patterns in a specific region, highlighting the significance of accounting for lagged autocorrelations and white noise errors to accurately capture seasonal variations and short-term fluctuations. The authors also emphasized the challenges associated with model selection and the necessity of employing robust diagnostic measures to ensure model adequacy. In the realm of finance, AR and ARMA models are frequently utilized for forecasting financial time series. Bollerslev (2018) explored the use of ARMA models in modeling and predicting volatility in financial asset returns. The study demonstrated that ARMA models effectively capture time-varying volatility patterns, making them valuable tools in risk management and option pricing. However, Bollerslev also acknowledged the limitations of ARMA models in capturing long-term volatility persistence, suggesting that more advanced models, such as GARCH (Generalized Autoregressive Conditional Heteroskedasticity), may be necessary.

Overall, AR and ARMA models serve as essential tools in time series analysis, offering valuable insights into temporal dependencies and enabling accurate forecasting. While AR models focus on past values to predict current outcomes, ARMA models extend this approach by incorporating moving average components. Both models are widely applied across diverse fields, including finance, climate science, and economics.

2.1.2. APPLICATION OF AR MODELS AND AUTOREGRESSIVE INTEGRATED MOVING AVERAGE MODELS (ARIMA)

ARIMA models (AutoRegressive Integrated Moving Average) are an extension of AR and MA models, integrating their characteristics while adding a differencing component. The "I" in ARIMA stands for the integrated aspect, referring to the number of differencing steps applied to the time series to achieve stationarity. Time differencing is a widely used technique in time series analysis to eliminate trends and seasonality, transforming a non-stationary series into a stationary one by calculating the differences between consecutive observations. Aghabozorgi and Talei (2021) provide an in-depth examination of ARIMA models, highlighting their versatility in addressing non-stationary data through differencing. They emphasize the critical importance of selecting an appropriate differencing order "d" to ensure model stability and accurate forecasts. While the mathematical structure of ARMA(p,q) and ARIMA(p,d,q) models is similar, the key distinction lies in the fact that ARIMA is applied to differenced data, whereas ARMA is used on non-differenced data.

Several studies have demonstrated the applicability of AR and ARIMA models. For example, Chen and Zheng (2018) illustrated the use of AR models for forecasting stock price movements, showing that AR models can effectively capture short-term trends and provide valuable insights for investors. However, they noted that AR models may struggle during periods of high volatility or when dealing with non-stationary behavior. In a different context, Ahmed and Rahman (2020) employed ARIMA models to analyze the impact of foreign direct investment on economic growth, finding a significant positive correlation between the two variables. Similarly, Hatemi-J (2018) explored the relationship between exchange rates and interest rates using AR models, demonstrating their effectiveness in uncovering dynamic interactions in economic time series. Additionally, Sung et al. (2017) applied ARIMA models to forecast tourist arrivals, concluding that ARIMA models can provide accurate short-term predictions, aiding in tourism management and resource planning. However, they cautioned that ARIMA's long-term forecasting ability may be limited due to the influence of external events and evolving patterns.

Both AR(p) and ARIMA(p,d,q) models are essential tools in time series analysis and forecasting. AR models are well-suited for capturing short-term trends, while ARIMA models are effective for handling non-stationary data. Their applications span diverse fields, including finance, economics, tourism, and energy forecasting. Nonetheless, researchers must carefully consider the underlying assumptions and ensure the appropriate selection of model orders (p, d, q) to produce accurate and reliable forecasts.

SUMMARY

In this chapter the literatures of various scholars and authors about AR, MA, ARMA and ARIMA models were presented and reviewed and some examples on the applications of these models in various fields was considered. In the next chapter we shall study some methodologies of AR models and the methods of parameter estimation of AR models.

CHAPTER THREE

METHODOLOGY

3.0. INTRODUCTION

This research concentrates on a particular category of time series models: the Autoregressive (AR) model. The AR(p) model, first introduced by Box and Jenkins in 1970 (Box, 1994), is a linear regression model that relates the present value of a series to its previous values. The parameter p represents the order of the AR(p) model, indicating that the current value depends on the p most recent observations in the series. In this chapter, we examine methods for estimating the parameters of the AR(p) model, specifically the Yule-Walker and Least Squares estimation methods, analyzing their underlying assumptions, theoretical foundations, and relevant concepts.

3.1. STATISTICAL ASSUMPTIONS OF AUTOREGRESSIVE MODELS

In order to model a time series $\{X_t\}$ using the AR(p) model, there are certain assumptions that must hold regarding the nature of $\{X_t\}$. These assumptions are given below:

3.1.1. STATIONARITY

The bedrock of time series analysis is stationarity. A time series is considered stationary if its statistical properties, such as mean, variance, and autocorrelation, remain constant over time. In simpler terms, the characteristics of the data do not change with respect to time. There are basically two types of stationarity namely:

1. **Strict Stationarity:** A time series is strictly stationary if the joint distribution of any subset of its time points is the same for all time points. This implies that the statistical properties of the data do not change even when considering different time intervals. Mathematically; $\{X_t\}$ to be strictly stationary if the joint distribution of (X_{t1} , ..., X_{tk}) is identical to that of (X_{t1} , ..., X_{tk+1}) for all t, where k is an arbitrary positive integer and (t_1, , ..., t_k) is a collection of k positive integers represent the recorded time (You, 2010).
2. **Weak Stationarity:** A time series is weakly stationary if its mean and variance are constant over time and the autocovariance function (or autocorrelation function) between any two time points depends only on the time lag between them. In other words, the overall pattern of the data remains consistent, and it exhibits constant mean, variance, and covariance structure. That is to say, for a time series $\{X_t\}$ to meet the requirement of weakly stationary, it should satisfy two conditions:

(a) Constant mean: $E(X_t) = \mu$; and (b) $Cov(X_t, X_{t-i}) = \gamma_i$ only depends on i.

In order to model a time series using AR(p) model, the time series must satisfy the condition of being Stationary; Either strictly or weak stationary. There are various methods of detecting stationarity in time series. These methods are broadly categorized into: Visualizations and Statistical testing methods.

3.1.1.1. Visualization Method of Detecting Stationarity

One widely used visualization method is by taking a time plot. A time plot is a plot of the observations in a time series against time. A time plot helps illustrate the existence of stationarity in a time series. The existence of a trend in a time series plot implies that the time series is not stationary, while the existence of no trend in time series plot indicate that the series in stationary. An illustration is shown below.

3.1.1.2. Statistical testing methods

Testing for the presence of stationarity in time series using plots is not efficient enough. Thus more robust statistical techniques are required to capture the presence of stationarity. One major statistical testing tool employed in testing for stationarity otherwise known as unit root in time series is the Augmented Dickey Fuller test. Where the null hypothesis of the test is there exist non-stationarity and the Alternative hypothesis is the existence of stationarity. i.e.

Augmented Dickey Fuller Test:

$$H_0: Time\ series\ is\ stationary$$

$$H_1: Time\ series\ is\ non-stationary$$

Thus, in the advent of rejection of the null hypothesis, we may conclude that the time series is stationary and thus good for the AR(p) model.

3.1.2. AUTOCORRELATION

Autocorrelation in an AutoRegressive (AR) model refers to the correlation between a time series observation and its past values at different lags. In other words, it measures how a value in the time series is related to its own previous values.

In an AR(p) model, where "p" represents the order of autoregression, the autocorrelation is a key factor in understanding the dependence structure of the time series. The autocorrelation function (ACF) and Partial autocorrelation function (PACF) are commonly used to visualize the autocorrelation at various lags.

1. **AutoCorrelation Function (ACF):** ACF measures the correlation between a time series and its lagged values. It helps identify the nature of the

correlation between the current value and past values at different lags. In an AR(p) model, ACF can help determine the order of the autoregressive component (p) by observing which lag values have significant correlations.

1. If the ACF shows a significant correlation at lag 1 and a gradual decline for subsequent lags, it suggests an AR(1) model.
2. If the ACF shows significant correlations at lags 1, 2, and a gradual decline afterward, it suggests an AR(2) model.
3. Similarly, significant correlations at specific lags can suggest the appropriate order for the AR model.

2. **Partial AutoCorrelation Function (PACF):** PACF measures the direct correlation between a time series value and its lagged values, while controlling for the influence of other intermediate lag values. It helps identify the direct effect of each past value on the current value, which is particularly important for determining the AR order in AR(p) models.

    1. If the PACF shows a significant correlation only at lag 1 and no significant correlations at other lags, it suggests an AR(1) model.
    2. If the PACF shows significant correlations at lags 1 and 2 and no significant correlations afterward, it suggests an AR(2) model.

3. The PACF helps differentiate the direct influence of each lag on the current value, allowing for a more accurate determination of the AR order.

3.2. PARAMETER ESTIMATION IN AUTOREGRESSIVE MODELS

The AR model is widely used in science, engineering, econometrics, biometrics, geophysics, etc. When a series is to be modeled by the AR model, the appropriate order p should be determined using the ACF and PACF plot and the parameters of the model must be estimated after testing for stationarity.

There are a number of methods available for estimating its parameters for this model and of these the following two maybe the most commonly used.

3.2.1. Least Squares Estimation Method

Regression analysis is perhaps the most commonly employed statistical technique in data analysis. Among the assortment of regression methods, Least Squares is extensively applied in linear regression models and is frequently utilized for estimation purposes. The core principle of the Least Squares method involves minimizing the sum of squared error terms (ε_t). An autoregressive (AR) model, a basic linear regression model, operates by making use of the principle of the least squares method to fit a model by minimizing the sum of square errors for estimating parameters. Consider the following $AR(p)$ model:

$$x_t = \propto_1 x_{t-1} + \propto_2 x_{t-2} + \dots + \propto_p x_{t-p} + \varepsilon_t$$

$$here\ we\ assume\ that\ x_t has\ mean\ zero\ or\ it\ has\ been\ demean$$

$$i.e\ x_t = x_t^* - \bar{x}$$

Here, x_t represents the current value of the time series at time t.

$\propto_1, \propto_2, ..., \propto_p$, are the autoregressive coefficients representing the influence of past values on the current value.

$x_{t-1}, x_{t-2}, ..., x_{t-p}$, are the lagged values of the time series up to order p, which are used to predict the current value.

ε_t is the error term at time t, representing the unpredictable or residual part of the time series not explained by the lagged terms.

The white noise ε_t is under the following assumptions:

1. E(ε_t) = 0

2. E($\varepsilon_t{}^2$) = σ^2

3. E($\varepsilon_t \varepsilon_k$) = 0 for t ≠ k

4. E($x_{t-k}\varepsilon_t$) = 0, k=1, 2, …..

5. ε_t *are independently and identically Normally distributed(iid)* .

Our objective is to find the parameter estimates $\propto_0$, $\propto_1, \propto_2, ..., \propto_p$, that minimize the sum of squared errors , $SSE = \sum_{t=1}^{n} \varepsilon_t{}^2$ where *n* is the total number of observations.

To minimize the sum of squared errors, we differentiate the sum of squared errors with respect to each parameter $\propto_{i,}$ set the derivatives to zero, and solve for the parameter estimates. Here's the general procedure:

1. Formulate the sum of squared errors *SSE* as:

$$SSE = \sum_{t=1}^{n} \varepsilon_t{}^2 = \sum_{t=1}^{n} (x_t - \propto_0 - \propto_1 x_{t-1} - \propto_2 x_{t-2} - \dots - \propto_p x_{t-p})^2$$

2. Differentiate *SSE* with respect to each parameter $\propto_i$:
   - Differentiate with respect to $\propto_0$: Set $\frac{\partial SSE}{\partial \propto_0} = 0$
   - Differentiate with respect to $\propto_1, \propto_2, \dots, \propto_p$: Set $\frac{\partial SSE}{\partial \propto_i}$ =0 for each *i,* and write down the normal equations where $\propto_0, \propto_1, \propto_2, \dots, \propto_p$ are parameters in the system of equations
3. Solve the resulting equations for $\propto_0, \propto_1, \propto_2, \dots, \propto_p$.

3.2.2. Yule-Walker Estimation Method

Yule-Walk Method, also called the autocorrelation method, is a numerically simple approach to estimate the AR parameters of the ARMA model. In this method, an autoregressive (AR) model is also fitted by minimizing the forward prediction error in a sense of least-squares regression. The difference is that Yule-Walker method is to solves the Yule-Walker equations, which is formed from sample covariances. (You, 2010)

A derivation of Yule-walker estimates for an AR(p) model as outlined by (Young and Jakeman 1979) is given below:

AR(p) Model: Consider an AR(p) model of order "p":

$$x_t = \propto_1 x_{t-1} + \propto_2 x_{t-2} + \dots + \propto_p x_{t-p} + \varepsilon_t$$

Covariance Structure: The Yule-Walker method is based on the covariance structure of the time series. For an AR(p) model, the covariance between x_t and x_{t-k} can be expressed as:

Cov(x_t, x_{t-k}) $= \gamma_k = \alpha_1 \gamma_{k-1} + \alpha_2 \gamma_{k-2} + \dots + \alpha_p \gamma_{k-p}$ Where γ_k is the covariance between x_t and x_{t-k}, and γ_{k-1} to γ_{k-p} are covariances at previous lags.

Yule-Walker Equations: The Yule-Walker method aims to find estimates for the coefficients α_1 to α_p using the sample covariances γ_k, $using\ \rho_k = \frac{\gamma_k}{\gamma_0}$. This leads to a set of p equations, known as the Yule-Walker equations:

$$\rho_1 = \alpha_1 \rho_0 + \alpha_2 \rho_{-1} + \dots + \alpha_p \rho_{-p+1}$$

$$\rho_2 = \alpha_1 \rho_1 + \alpha_2 \rho_0 + \dots + \alpha_p \rho_{-p+2}$$

$$\vdots \qquad \dots \qquad \vdots$$

$$\rho_p = \alpha_1 \rho_{p-1} + \alpha_2 \rho_{p-2} + \dots + \alpha_p \rho_0$$

i.e.

$$\rho_k = \sum_{j=1}^{p} \alpha_j \rho_{k-j} \ , k = 1,2,3,4, \dots, p$$

using the fact that $\rho_0 = 1, \rho_{-k} = \rho_k, \ \ k = 1,2,3, \dots$

In Matrix form we have:

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_3 \end{pmatrix} = \begin{pmatrix} \rho_0 & \rho_1 & \cdots & \rho_{p-1} \\ \rho_1 & \rho_2 & \cdots & \rho_{p-2} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \cdots & \rho_p \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{pmatrix}$$

thus when p = 1, we have an AR(1) model. So the yule-walker equations becomes:

$$\rho_1 = \sum_{j=1}^{1} \alpha_j \rho_{k-j} = \alpha_1 \rho_0$$

With $\rho_0 = 1$

$$we\ have\ that\ \ \rho_1 = \widehat{\alpha_1}$$

When p =2, we have an AR (2) model, so that the walker equations

$$\rho_k = \sum_{j=1}^{2} \alpha_j \rho_{k-j} \, , k = 1,2$$

$$\rho_1 = \alpha_1\rho_0 + \alpha_2\rho_{-1}$$

$$\rho_2 = \alpha_1\rho_1 + \alpha_2\rho_0$$

i.e.

$$\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = \begin{pmatrix}1 & \rho_1\\ \rho_1 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$$

When p =3, we have an AR (3) model, so that the yule-walker equations becomes

$$\rho_k = \sum_{j=1}^{3} \alpha_j\rho_{k-j}\,, k = 1,2,3$$

$$\rho_1 = \alpha_1 + \alpha_2\,\rho_1 + \alpha_3\,\rho_2$$

$$\gamma_2 = \alpha_1\,\rho_1 + \alpha_2 + \alpha_3\,\rho_1$$

$$\rho_2 = \alpha_1\,\rho_2 + \alpha_2\rho_1 + \alpha_3$$

i.e.

$$\begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\end{pmatrix} = \begin{pmatrix}1 & \rho_1 & \rho_2\\ \rho_1 & 1 & \rho_1\\ \rho_2 & \rho_1 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$$

These equations gotten by By Multiplying both side of the AR(p) model by x_{t-k}, $k = 0, 1, \cdots, p$, then taking expectation, forming a system of linear equations in terms of the unknown coefficients $\alpha_1, .., \alpha_p$.

By solving this system of equations for α, the estimates for the AR(p) model coefficients are obtained.

3.3. COMPARISON BETWEEN LEAST-SQUARE ESTIMATION METHOD AND YULE-WALKER ESTIMATIOH METHOD

Time series analysis is a critical domain in statistics and econometrics, with various estimation methods being employed to model temporal dependencies accurately. Two commonly used methods for estimating the parameters of AutoRegressive (AR) models are the Least-Squares Estimation Method and the Yule-Walker Estimation Method. This section attempt to compare and contrast these two approaches based on their underlying principles, assumptions, applicability, efficiency, and advantages.

The Least-Squares Estimation Method focuses on minimizing the sum of squared residuals between observed data and model predictions, emphasizing the overall fit of the model to the entire dataset. The Yule-Walker Estimation Method, on the other hand, leverages the autocovariance structure of the time series. It estimates parameters based on the autocovariance and autocorrelation functions, offering insights into the temporal relationships among data points. (Yule, 1927). Least-Squares method is applicable to a wide range of regression problems, but it assumes that errors are independent and identically distributed (i.i.d.). Yule-Walker method is specifically designed for stationary AR models, assuming constant mean, variance, and autocorrelation over time. Least-Squares estimation aims to provide parameter estimates that minimize the overall error, ensuring a balanced fit across the entire dataset. Yule-Walker estimation provides parameter estimates that closely match the autocovariance structure, capturing the temporal dependencies within the time series. Least-Squares method is versatile and widely applicable, but its sensitivity to outliers can affect its robustness. Yule-Walker method is particularly useful for stationary time series with short-term dependencies and offers insights into the time lag structure of the data. (Kutner, *et al.,* 2004)

In conclusion, both the Least-Squares Estimation Method and the Yule-Walker Estimation Method have their strengths and limitations. The choice between these methods depends on the specific characteristics of the dataset, the assumptions that can be reasonably made, and the research objectives. Researchers and practitioners need to carefully consider the nature of the time series, computational efficiency, and the interpretability of results when selecting the appropriate estimation method.

SUMMARY

In this chapter we have considered the methodologies behind Autoregressive (AR) models, the method and parameter estimation and a comparison between two of these methods. In the chapter four of this project work we shall apply this method in finding parameter estimates to an AR model using data on Nigeria Gross Exchange rate gotten from World Bank.

CHAPTER FOUR

APPLICATION OF AUTOREGRESSIVE MODELS TO REAL LIFE TIME SERIES DATA

4.0. INTRODUCTION

In this chapter we will be applying the Autoregressive (AR) model on real life time series data. We will be using a secondary data on the Exchange Rate (EXR) of Nigeria, collected from the World Bank data site. Furthermore, we will be adopting the Yule-walker and Least Squares Method in estimating the model parameters and attempt a comparison between these models, leveraging on R programming Language for all of the analysis in the study.

4.1. DATASET DESCRIPTION

The dataset adopted in this study comprises of the amount in Naira per USD of the value of Nigeria Exchange Rate (EXR) over the span of forty-three years (1980 - 2023). This dataset was extracted from the World Bank data site and analysis conducted.

Year	Exchange rate (NGN_per_USD)
1980	0.55
1981	0.62

1982	0.67
1983	0.72
1984	0.76
1985	0.89
1986	2.02
1987	4.02
1988	4.54
1989	7.39
1990	8.04
1991	9.91
1992	17.3
1993	22.05
1994	21.89
1995	21.89
1996	21.89
1997	21.89
1998	21.89
1999	92.34
2000	102.11
2001	111.94
2002	120.97
2003	129.36

2004	133.5
2005	131.27
2006	128.27
2007	125.83
2008	118.57
2009	148.88
2010	150.3
2011	153.74
2012	157.5
2013	159.45
2014	164.04
2015	197
2016	304.5
2017	305.84
2018	306.08
2019	306.5
2020	380
2021	411.5
2022	445
2023	770

4.2. CHECK FOR STATIONARITY

In order to check for stationarity in our dataset an Augmented Dickey Fuller (ADF) test was conducted and the result shown below:

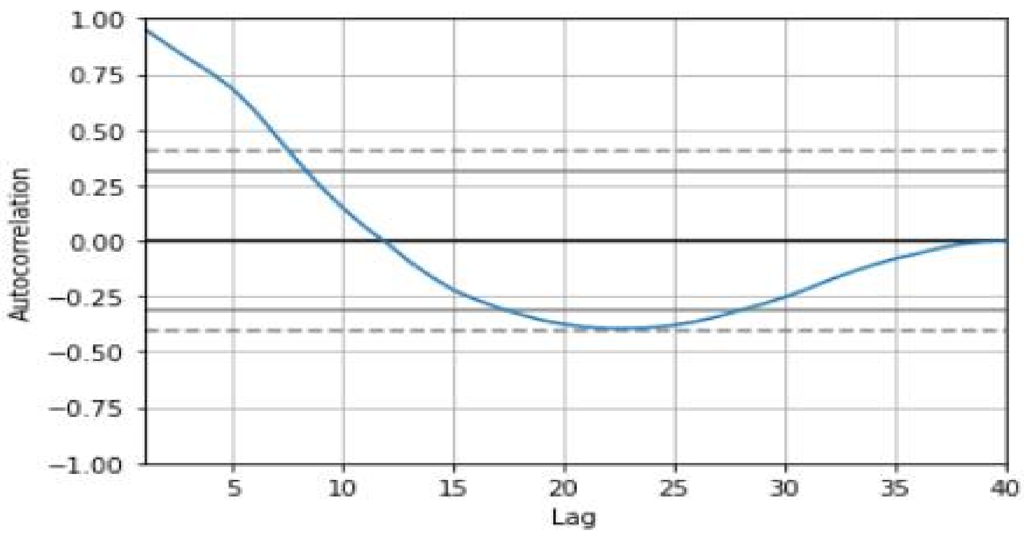
Augmented Dickey-Fuller Test
data: data$`EXR(Billion US $)`
Dickey-Fuller = -2.347, Lag order = 4, p-value = 0.5230
alternative hypothesis: stationary

Source: Author's compilation using R programming

The result indicates that the time series is non-stationary as the p-value is greater than 0.05, which indicates that the ADF test statistic is insignificant at 5% level of significance. Thus, the time series is non-stationary

4.3. FINDING THE ORDER OF THE AR MODEL

In order to find the order (p) of the AR (p) model to be used, we will be making an Autocorrelation Function (ACF) plot as shown below:



Based on this ACF—which remains significant through about lag 7 before converging back toward zero—one interpretation is that an **AR(7)** model could be appropriate. In other words, the correlation structure suggests you may need up to seven lags to adequately capture the dependence in the data. While typically we'd also review the **Partial Autocorrelation Function (PACF)** and formal model selection criteria (AIC/BIC), the visible "cutoff" around lag 7 in this plot provides a strong hint that an **AR(7)** specification is worth considering. Thus, we will be estimating an AR (7) model. Mathematically we will be estimating the model:

$$EXR_t = \alpha_0 + \alpha_1\, EXR_{t-1} + \alpha_2\, EXR_{t-2} + \ldots + \alpha_7\, EXR_{t-7} + \varepsilon_t$$

4.4. ESTIMATION USING YULE-WALKER METHOD

Yule walker method was applied in estimating the model parameters of the AR (7) model and the result presented below:

Call:
ar(x = data$`EXR(Billion US $)`, order.max = 7, method = "yule-walker")
Coefficients:
1 2 3 4 5 6 7
1.264 -0.2422 -0.4941 0.5042 -0.0146 0.0239 -0.1290
Order selected 7 MSE: 123.21

Source: Author's compilation using R programming

Interpretation:

The results presented above shows that the estimated parameters of the AR (7) model are 1.264, -0.2422, -0.4941, 0.5042, -0.0146, 0.0239 and -0.1290 and the model Mean Squared Error (MSE) is 123.21. The MSE indicates the accuracy of the model in using past values of EXR in predicting future values of EXR.

4.5. ESTIMATION USING LEAST SQUARES METHOD

Least squares was applied in estimating the model parameters of the AR (7) model and the result presented below:

Call:
ar(x = data$`EXR(Billion US $)`, order.max = 7, method = "ols")
Coefficients:
1 2 3 4 5 6 7
1.4102 -0.3301 -0.6158 0.6125 0.2119 0.1901 -0.5131
Intercept: -3.432
Order selected 7 MSE: 110.11

Source: Author’s compilation using R programming

Interpretation:

The results presented above shows that the estimated parameters of the AR (7) model are 1.4102, -0.3301, -0.6158, 0.6125, 0.2119, 0.1901, and -0.5131 with an intercept of -3.432 and the model Mean Squared Error (MSE) is 110.11.

The MSE indicates the accuracy of the model in using past values of EXR in predicting future values of EXR.

4.6. COMPARISON BETWEEN YULE-WALKER METHOD AND LEAST SQUARES ESTIMATION METHOD

The Yule-Walker and Least-Squares estimation methods, both yields similar AR(7) coefficients. While the Yule Walker estimation technique yielded an MSE of 123.21, the Least Squares estimation technique yielded an MSE of 110.11. Thus, we may conclude that the accuracy in predicting future values of EXR using the Least squares estimation technique is higher than the accuracy of the yule-walker method. These results reflect in the importance of the requirements of stationarity when applying the Yule-Walker method, as the Yule-Walker method of estimations solemnly relies heavily on the stationarity of the time series data for accurate predictions compared to the least squares method.

CHAPTER FIVE

SUMMARY OF FINDINGS, RECOMMENDATIONS AND CONCLUSION

5.0. SUMMARY OF FINDINGS

In the chapter four of this project work, an Autoregressive (AR) model was applied to real-life time series data representing the Exchange rate (EXR) of Nigeria over a period of forty-three years (1980 - 2022). Two estimation methods, the Yule-Walker method and the Least Squares method, were employed to estimate the model parameters. The following key findings were observed:

1. **Stationarity Check:** An Augmented Dickey-Fuller (ADF) test was conducted to check for stationarity in the EXR time series data. The test result indicated that the time series is non-stationary, as the p-value exceeded the 0.05 significance level.

2. **Order of AR Model:** To determine the order (p) of the AR model, an Autocorrelation Function (ACF) plot was created. The ACF plot suggested an appropriate order of seven (AR(7)) based on the lags that remained inside the cut-off lines.

3. **Estimation Using Yule-Walker Method:** The Yule-Walker method was applied to estimate the parameters of the AR(7) model. The estimated coefficients for the model were presented, along with a Mean Squared Error

(MSE) of 123.21, which reflects the model's accuracy in predicting future EXR values.

4. **Estimation Using Least Squares Method:** The Least Squares method was also employed to estimate the parameters of the AR(7) model. The estimated coefficients for the model, along with an intercept, were provided, and the model's MSE was found to be 110.11

Comparison Between Estimation Methods:

A comparison between the Yule-Walker and Least Squares estimation methods was conducted. Both methods yielded similar AR(7) coefficients, indicating the consistency of the model. However, it was observed that the Least Squares method had a lower MSE (110.11) compared to the Yule-Walker method (123.21). This suggests that the Least Squares method exhibited higher accuracy in predicting future EXR values.

5.1. Recommendations:

Based on the findings of this analysis, the following recommendations can be made:

1. **Model Selection:** When applying AR models to time series data, it is advisable to consider the stationarity of the data. If the data is non-stationary, appropriate differencing techniques should be applied to make it stationary before modeling.

2. **Estimation Method:** The choice of estimation method can significantly impact the accuracy of predictions. In cases where stationarity assumptions are met, the Yule-Walker method can be effective. However, if stationarity is a concern, the Least Squares method may be more suitable, as demonstrated by its lower MSE in this study.

5.2. Suggestions for Further Study

Future research could explore additional factors that influence EXR fluctuations in Nigeria, such as economic policies, external factors, and sectoral contributions. This could lead to more robust models for EXR forecasting.

5.3. Conclusion

This project work has demonstrated the application of AR models to real-life time series data and highlighted the importance of choosing an appropriate estimation method based on stationarity considerations. The findings contribute to the understanding of EXR fluctuations in Nigeria and provide insights into accurate modeling techniques.

This study has contributed valuable insights into the application of AR models in a real-world context. It underscores the significance of data stationarity and the choice of estimation methods, ultimately highlighting the role of rigorous statistical analysis in understanding and predicting economic phenomena. As the world of time series analysis continues to evolve, the lessons from this study remain relevant for economists, policymakers, and analysts seeking to make informed decisions based on time series data.

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