APPLICATION OF LAPLACE TRANSFORMS TO DIFFERENTIAL EQUATIONS USING BOUNDARY CONDITIONS

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CERTIFICATION

I certify that this work was carried out by OSIKI MARK OSAGIE in the department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, under my Supervision.

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(Head of Department)

Date______________________

DEDICATION

This work is dedicated to God Almighty for His infinite love, care and mercy upon my life. I also dedicated this work to my dearest mum.

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I wish to thank all, but first and foremost, I want to thank God Almighty for his sufficient mercy, love and care upon my life, because the road to success and greatness is never without help from God, who always uses men to pave ways. My special thanks goes to my supervisor Mr. Daniel Okuonghae for his patience and assistance to the completion of this work.

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ABSTRACT

Laplace transforms involves the mathematical study of Laplaces, or using it to obtain solution of differential equations. The formation of differential equations is, of course, a common phenomenon that occurs whenever the demand to solve systems of differential equations. Principles regarding the various approaches to it, which may be made frequent in technology like the transfer system used. In this project work, we look at some different system of different equations and consider such procedure as how long it takes on contribution to technology to pass through the system of introduction to application. The kinds of method with which we are all familiar in mathematical field, mechanics. However, the result of Laplace theory have equal if not more important applications in computer such as Data transfer system, Data and signal impulse response and needing to join a Laplace system to be considered. Hence the need in technology is of another usefulness.

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CHAPTER ONE

INTRODUCTION

The laplace transform which is a part of the growing topic known as operational calculus is easily and effectively applicable to the initial value problems of differential equations arising in physics, mathematics and engineering. The subject was mainly originated in the work of Heaviside who found it useful to solve the equation of electromagnetic theory in the end of the nineteenth century.

The laplace transform is used to solve linear constant coefficient of differential equation, this is achieved by transforming them to algebraic equation. The algebraic equation are solved, then the inverse laplace transform is useful to obtain a solution in terms of the original variables. This techniques can be applied to both single and system of differential equations.

The topic "Laplace transforms" has its origin or originated in attempts to justify rigorously certain operational rules used by Heaviside.

The laplace transform play another dominant roles and also provide powerful tools in numerous field of technology such as control theory where in knowledge of the system transfer comes into is use. The laplace transform can be used to solve different methods of differential equations, but first we need to consider certain properties of laplace transform studied by some scientist like Anthony Crofts, Robert Davison, Martin Hargreaves (Engineering Maths).

Some of these properties that are exploited are examined as follows:

a) Linearity Property

A laplace transform $L[f(t)]$ is said to be linear if for every pair of function $f_1(t)$ and $f_2(t)$ and for every pairs of Constant C_1 and C_2, we have

$$L\{C_1 f_1(t) \ + C_1 f_2(t)\} = c_1 L[f_1(t)] + c_2 L[f_2(t)] = c_1 f_1(s) + c_2 f_2(s)$$

where $f_1(s)$ and $f_1(s)$ is the linear transformation of $f_1(t)$ and $f_2(t)$ respectively. We have that

$$L[f_1(t)] = f_2(s) = \int_0^\infty \ell^{-st}\ f_1(t)dt \ and\ \ L\{f_2(t)\} = f_2(s) = \int_0^\infty \ell^{-st} f_2(t)dt$$

So that $L[c_1 f_1(t)] = c_1 f_1(s) = \int_0^\infty \ell^{-st} c_1\, f_1\ \ (t)dt = c_1\, L[f_1(t)]$

$\therefore L\{c_1\, f_1(t) + c_2\, f_2(t)\} = \int_0^\infty \ell^{-st}\ \ c_1 f_1(t)dt + c_2 f_2(t)dt$

By definition $= \int_0^\infty \ell^{-st} c_1 f_1(t)dt + \int_0^\infty \ell^{-st} c_2 f_2(t)dt$

$= c_1\{L\, f_1(t)\} + c_2\{f_2(t)\} c_1\, f_1(s) + c_2\, f_2(s)$

The result may be generalized for any number of functions and for the same number of arbitrary constants i.e

$$L\left[\sum_{r=1}^{n} Crf_r(t)\right] = \sum_{r=1}^{n} CrL[f_r(t)]$$

Problem: find the laplace transform of $4\ell^{5t} + 6t^3 - 4\ cos3t + 3sin4t$

Hence, $4L\{\ell^{5t}\} + 6L\{t^3\} - 4L[cos3t] + 3L[sin4t]$

$$4 \cdot \left(\frac{1}{s-5}\right) + 6\left(\frac{3}{s^{3+1}}\right) - 4\left(\frac{3}{s^2+3^2}\right) + 3\left(\frac{4}{s^2+4^2}\right)$$

$$4\left(\frac{1}{s-5}\right) + 6\left(\frac{3!}{s^4}\right) - 4\left(\frac{3}{s^2+9}\right) + 3\left(\frac{4}{s^2+16}\right)$$

<B> first translation (or shifting) property

If f(s) be the laplace transform of f(t), then the laplace transform of $\ell^{at}\ f(t)\ is\ f(s-a)$, where a is any real or complex number ie if

$L[f(t)] = f(s), then\ L[\ell^{at} f(t)] = F(s-a)$

Given, $L[f(t)] = \int_0^\infty \ell^{-st} F(t)dt = f(s)$

$\therefore L[\ell^{at} f(t)] = \int_0^\infty \ell^{st} f(t)dt \cdot \ell^{at}$

$= \int_0^\infty \ell^{-(s-a)t} f(t)dt$

$= \int_0^\infty \ell^{-ut} f(t)dt\ by\ putting\ u = s-a$

$= f(u) = 0\ f(s-a)\ since\ u = (s-a)$

(c) second translation (or shifting) property

If $L[f(t)] = f(s) and\ G(t) = \{f(t-a), t > a$

{0, t<a}

Then $L[G(t) = \ell^{-as} f(s)$

We have that $\ell[G(t)] = \int_0^\infty \ell^{-st} G(t)dt$

$= \int_0^\infty \ell^{-st} F(t-a)dt$

$\int_0^\infty \ell^{-s(u+a)} f(u)du \quad taking\ u = t-a \quad i.e\ \ du = dt$

$when\ t = a, u = 0 \quad when\ t = \infty, u = \infty$

$= \ell^{-as} \int_0^\infty \ell^{-su} F(u)du = \ell^{-sa} f(s)$

(d) The change of scalar property

If $\{f(t)\} = f(s)\ then\ L(\ f(at) = \frac{1}{a} f\left(\frac{s}{a}\right)$

We have $L\{f(t)\} = \int_0^\infty \ell^{-st} F(t)dt = f(s)$

$$\therefore L[f(at)] = \int_0^\infty \ell^{-st}\ f\ (at)dt\ \ (on\ replacing\ t\ by\ a)$$

$$= \int_0^\infty \ell^{-su/a} F(u)\frac{du}{a}\ by\ taking\ at = u$$

$$= \frac{1}{a}\int_0^\infty \ell^{-pu} f(u)du \quad where\ p = \frac{s}{a}$$

$$= \frac{1}{a}\int_0^\infty \ell^{pu} F(u)dt \quad (replacing\ u\ by\ t)$$

$$= \frac{1}{a} f(p)$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right) \therefore P = \frac{s}{a}$$

(e) Derivatives of laplace transforms

If the function F(t) is sectionaly continuous for $t \geq 0$ and

If $L[F(t)] = f(s), then\ f(s) = L[(-t)f(t)]$

Where $F^n(s) = \frac{d^n}{ds^n} f(s)$ for all integral values of n

We may state as

$$L[t^n\ F(t)] = (\ -1)^n \frac{d^n}{ds^n}\ f(s)$$

1.1 **LIMITATIONS OF LAPLACE TRANSFORM**

As we have seen later in the study, there are many methods on laplace transforms that are developed by mathematicians/analysts. Methods are available for more complex situations, for example, the problem in which the general solutions to odes is finite, the laplace principles, methods and also the procedure guiding such problems and other principles. In general, complex solutions to problems involving variables are less exploited in the course of the study, but other method gives a better support to the limitation of the background of the study.

In some application, the laplace transform system never reaches a final stage, so the method of solution is of less value.

1.2 **IMPORTANCE OF LAPLACE TRANSFORM**

The content of this topic has gained a vast recognition in our modern world. Its attempt to solve or determine rigorously solutions to either ordinary differential equation, partial differential equation, laplace transforms with respect to some ivp (initial value problems) solve simultaneous equations. It can also be used or applied to the field of physics where it is used to solve problems in mechanics arising from wave equations, basis criment appliances.

It can also be useful in solving beam equations in options, laplace transform can also be used intensively in solving integral equations, integral equations of covolution types, the tauto chrome problems, difference differential equation. Laplace transforms can be of great importance in evaluating certain special functions like gamma and beta function error function, Bassel functions and also in evaluating exponential integrals.

1.3 **DEFINITION OF TERMS/THEOREMS**

(a) the gamma function

$$\Gamma(n) = \int_0^\infty u^{n-1} \ell^{-n} d_n, \quad if\ n > 0$$

(b) Bassel function

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)} \left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4(2n+2)(2n+4)} + \dots\right]$$

The error function is defined as

$$Erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t \ell^{-u^2}\, du$$

Complementary error function is defined as

$$erfc(t) = 1 - \text{erf}\,(t)$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^t \ell^{-u^2}\, du = \frac{2}{\sqrt{\pi}} \int_0^t \ell^{-u^2}\, du$$

The sine and cosine integrals are defined by

$$\sin(t) = \int_0^t \frac{\sin u}{u} du$$

$$C_i(t) = \int_t^\infty \frac{\cos u}{u} du$$

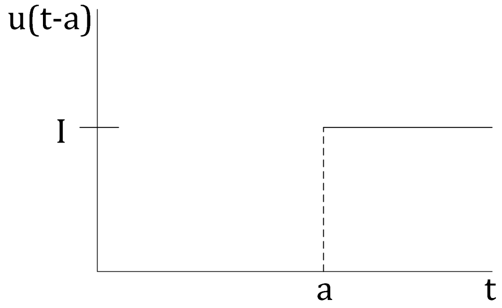
The exponential integral is defined by

$$E_i(t) = \int_t^\infty \frac{\ell^{-u}}{u} du$$

The unit step function, also called Heaviside unit function, is defined as

$$U(t-a) = \begin{Bmatrix} 0, t < a \\ 1, t > a \end{Bmatrix}$$

Diagram below



Initial value theorem

$$If\ L\{f(t)\} = f(s) then \quad \lim_{t\to 0} F(t) = \lim_{s\to\infty} sf(s)$$

We have that $L\left\{f^{'}(t)\right\} = s\,F(s) - f(0)$

$$i.e \int_0^\infty \ell^{-st} f^{'}(t)dt = sf\ (s) - f(0)$$

Taking the limit as $t \to \infty, or$

$$\lim_{s\to\infty} sf(s) = f(0) + \int_0^\infty \left(\lim_{s\to\infty} \ell^{-st}\right) f^{'}(t)dt$$

$$= f(0) + 0 \qquad\qquad \therefore \lim_{s\to\infty} \ell^{-st} = 0$$

$$= \lim_{t\to 0} f(t)$$

Verify the initial value theorem for the function

$$F(t) = \ell^{-3t}$$

$: f(s) = L\{f(t)\} = L[\ell^{-3t}] = \frac{1}{s+3}$

$Now, \ \lim_{t \to 0} f(t) = \lim_{t \to 0} \ell^{-3t} = 1$

$and \ \lim_{t \to 0} sf(s) = \lim_{s \to \infty} \frac{s}{s+3} = 1$

$Hence, \ \lim_{t \to 0} f(t) = \lim_{s \to \infty} sf(s)$

Final value theorem

If $L\{f(t)\} = f(s), then$

$\lim_{s \to 0} sf(s) = \lim_{t \to \infty} f(t)$

We have $L\{f'(t)\} = sf(s) - f(0)$

$i.e \int_0^\infty \ell^{-st} f^I(t)dt = sf(s) - f(0)$

$Taking\ limit\ as\ s \to 0,$

$\lim_{s \to 0} \int_0^\infty \ell^{-st} f^I(t)dt = \lim_{s \to 0} sf(s) - f(0)$

$or \ \lim_{s \to 0} sf(s) = f(0) + \lim_{s \to 0} \int_0^\infty \ell^{-st} f^I(t)dt$

$= F(0) + \int_0^\infty \left(\lim_{s \to 0} \ell^{-st}\right) f^I(t)dt$

$= f(0) + \int_0^\infty 1 \cdot f^I(t)dt$

$= f(0) + \int_0^\infty \frac{d}{dt} f(t)dt$

$= F(0) + \{F(t)\}_0^\infty$

$F(0) + \lim_{t=\infty} f(t) - f(0)$

$\lim_{t=\infty} f(t)$

The first shift theorem

The first shift theorem states or defined as

$If\ L\{f(t)\} = f(s), \quad L\{\ell^{-at} f(t)\} = f(s+a)$

Because $L\{\ell^{-at} f(t)\} = \int_0^\infty \ell^{-at} f(t)\, \ell^{-st} dt$

$$\Rightarrow \int_0^\infty f(t)\, \ell^{-(s+a)t} dt = f(s+a)$$

That is $L\{\ell^{-at}\, f(t)\} = f(s+a)$

The transform $L\{\ell^{-at}\, f(t)\}$ is the same as $L\{f(t)\}$ with s everywhere in the result replace by (s + a)

Dirac-delta function or unit impulse function

A function $\delta(t)$ such that

$$\lim_{\varepsilon \to 0} f\delta(t) = \delta(t) \quad where\ f\delta(t) = \begin{Bmatrix} 1/\varepsilon, 0 \le t \le \varepsilon \\ 0, \quad t > \varepsilon \end{Bmatrix}$$

and $\int_0^\infty f_\delta(t) dt = 1$

With the following properties,

i) $(a) \int_0^\infty \delta(t) dt = 1$

ii) $\int_0^\infty \delta(t) G(t)\, dt = G(a)$ for any continuous function of G(t)

iii) $\int_0^\infty \delta(t-a)G(t)dt = G(a)$ for continuous function G(t) is called unit impulse function or Dirac-Delta function.

We have $L\{s(t)\} = \int_0^\infty \ell^{-st} \lim_{\varepsilon \to 0} f_\delta(t)dt$ which is not in existence and hence it is useful to consider $\delta(t) = \lim_{\varepsilon \to 0} f_\delta(t)$ to be such $L\{\delta(t)\} = 1$

Also $L\{f\delta(t)\} = \int_0^\infty \ell^{-st} f\delta(t)\, dt = \int_0^\varepsilon \ell^{-st} \frac{1}{\varepsilon} dt + \int_\varepsilon^\infty .0.dt$

$$= \frac{1}{\varepsilon}\int_0^\varepsilon \ell^{-st}\, dt = \frac{1-\ell^{-st}}{\varepsilon_s}$$

Null function: A null function N(t) for all t > 0 is defined as $\int_0^t N(x)dx = 0$

$$e.g\ if\ F(t) = \begin{cases} 1, & t = 1 \\ 0, & otherwise \end{cases}$$

Then it is null function, since $\int_0^t f(x)dx = 0\ for\ all\ t > 0$. As such the laplace transform of a null function is zero i.e $L\{N(t)\} = 0$

Inverse laplace transforms. Here we have reverse process to find the value of t to which it belong. This is denoted by $L^{-1}\{f(s)\} = f(t)$.

CHAPTER TWO

SYSTEM OF DIFFERENTIAL EQUATION AND ITS APLLICATIONS

(ODES with Constant Coefficients)

2.0 INTRODUCTION

In earlier studies on functions and differentiation we have expressions such as $y = f(x)$ where x is independent variable and y is the dependent variable. We also encountered $\frac{dy}{dx}$ as derivative of y.

With respect to x (differential coefficient) and where order is one. Other derivatives of higher order are as follows

Derivatives	Order of Derivatives
$\frac{d^2y}{dx^2}$	2
$\frac{d^3y}{dx^3}$	3
⋮	⋮
$\frac{d^ny}{dx^n}$	n

2.1 WHAT IS AN ORDINARY DIFFERENTIAL EQUATION (ODES)?

This is an equation which involves one independent variable x, the dependent variable y and at least one of the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}$

If the dependent variable y depends on two or more independent variables let say x and t (i.e $y = f(x, t)$) then the differential equation is called a partial differential equation.

2.2 SOLUTION AND FORMATION OF ORDINARY DIFFERENTIAL EQUATION

Let a function $y = f(x)$ which expresses a functional relationship between x and y satisfies a given differential equation can be defined and differentiable throughout some interval, $a < x < b$. Then $y = f(x)$ is a solution of differential equation in the given interval.

The following example clarifies two basic concepts.

a) The solution of a differential equation always satisfies the differential equation.

b) By eliminating arbitrary constants from a given solution (where they exist), we can obtain the differential equation which has the given solution.

2.3 METHOD OF FINDING LAPLACE TRANSFORMS

Various means are available for determining laplace transform as indicated in the following list.

1. Direct method

2. Series method

3. Method of differential equation (i.e unit impulses)

4. Differentiation with respect to a parameter

Example using method of differentiation (unit impulse)

a) Evaluate $x + 6x + 8 = g(t)$

where $g(t)$ is an impulse of 4 unit applied at $t = 5$.

...$g(t) = 4 \cdot \delta(t-5)$

$\therefore x + 6x + 8x = 4 \cdot \delta(t-5) \quad at\, t = 0, \quad x = 0, \quad x = 3$

$(s^2x - sx_0 - x_1) + 6(sx - x_0) + 8x = 4e^{-5s}$

Now $x_0 = 0, \;\; x_1 = 3$

$\therefore s^2x - 3 + 6sx + 8x = 4e^{-5s}$

$\therefore (s^2 + 6s + 8)x = 3 + 4e^{-5s}$

$$\therefore = \left(3 + 4e^{-5s}\right)\frac{1}{(s+2)(s+4)}$$

Writing $\frac{1}{(s+2)(s+4)}$ in partial fractions, we get

$$X = \left(3 + 4e^{-5s}\right)\left\{\frac{1}{2}\frac{1}{(s+2)} - \frac{1}{2}\left(\frac{1}{s+4}\right)\right\}$$

$$\therefore x = \frac{3}{2}\left[\frac{1}{s+2} - \frac{1}{s+4}\right] + 2\left\{\frac{e^{-5s}}{s+2} - \frac{\ell^{-5s}}{s+4}\right\}$$

Taking inverse transform

$$x = \frac{3}{2}[\ell^{-2t} - \ell^{-4t}] + 2\{\ell^{-2(t-5)}u(t-5) - \ell^{-4t(t-5)}u(t-5)\}$$

$$= \frac{3}{2}\{\ell^{-2t} - \ell^{-4t}\} + 2\{\ell^{10} \cdot u(t-5) - \ell^{-4t}\ell^{20}\, u(t-5)\}$$

$$x = \ell^{-2t}\left\{\frac{3}{2} + 2\ell^{10} \cdot u(t-5)\right\} - \ell^{-4t}\{\frac{3}{2} + 2\ell^{20} \cdot u(t-5)\}$$

2) The series method: This involves if $f(t)$ has a power series or is being expressed in power series standard given by $f(t) = a_0 + a_1 + a_2t^2 + \dots \sum_{n=0} a_nt^n$

Taking the laplace transform of either sides, the equation becomes

$$L\{f(t)\} = L\{a_0\} + L\{a_1t\} + L\{a_2t^2\} + \dots + L\left[\sum_{n=0}^{\infty} a_nt^n\right]$$

where $L\{f(t)\}$ is the laplace transform of $f(t)$ in terms of other parameters, $L(a_0) = \frac{1}{s}a_0$ because L{1} =, hence

$$L\{f(t)\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2!\,a_2}{s^3} + \dots + \sum_{n=0}^{\infty}\frac{a_nn!}{s^{n+1}}$$

Hence,

$$L\{(t)\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \sum_{n=0}^{\infty}\frac{a_nn!}{s^{n+1}}$$

3. Direct method: This method is a straight forward approach to solve or evaluate simple linear problem (i.e a constant variable) by applying laplace method directly.

4. Differentiation with respect to a parameter (i.e the unit step function). Now consider function $f(t)$, and $f'(t)$ denotes the first derivatives of $f(t)$ with respect to t. $f''(t)$ denotes the second derivatives of $f(t)$ with respect to t, etc.

Then $L\{F'(t)\} = \int_0^t \ell^{-st}\, F'(t)dt$ by definition

Integrating by part

$L\{f^{'}(t)\} = [\ell^{-st}f(t)]_0^\infty - \int_0^\infty f(t)\{-se^{st}\}dt$

When $\ell \to \infty,\ \ell^{-st}f(t) \to 0$

because s is positive and large enough to ensure that decays faster than any positive growth of $f(t)$.

$L\{f(t)\} = f(0) + sL\{f(t)\},$

Replacing $f(t)$ by $f'(t)$ gives

$L\{f(s) - sf(t) - f(0)$

because $L\{f^{'}(t)\} =- f(0) + sL\{f(t)\}$

$= f^{'}(0) + s\{-f(0) + sL\{f(t)\}$

writing $L\{f(t)\} = f(s)$ as usual, we have

$L\{f(t)\} = f(s)$

$L\{f^{'}(t)\} = sf(t) - f(0)$

$L\{f^{''}(t) = sf(s) - s^2f(0) - f'(t)$

We can see a pattern emerging

$L\{f^{'''}(t)\} = s^3f(s) - sf(0) - s^2f(0) - f^{'}(0) - f''(0)$

The usual notation of the derivatives symbols,

$L\{f^{'''}(t)\} = s^3f(s) - sf(0) - f'(0)$

$L\{f^{''}(t)\} = s^2f(s) - sf(0) - f'(0)$

And finally

$L\{f^{'}(t)\} = sL\{f(t)\} - f(0)$

2.4 PROBLEMS AND APPLICATION OF ORDINARY DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORM

Solve $y^{''} + 2y^{''} + 5y = \ell^{-t}sut, \;\; y(0) = 0, y^{'}(0) = 1$

We have that $L\{y^{''}\} + 2L\{y^{''}\} + 5L\{y\} = \ell\{\ell^{t}sut\}$

From our earlier definition

$$\{s^2y - sy(0) - y^{'}(0)\} + 2\{sy - 0\} + 5y = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$

$$\{s^2y - s(0) - 1\} + 2\{sy - 0\} + 5y = \frac{1}{s^2 + 2_s + 2}$$

$$(s^2 + 2s + 5)y - 1 = \frac{1}{s^2 + 2s + 2}$$

$$y = \frac{1}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2, + 2)(s^2 + 2s + 5)}$$

$$y = L^{-1}\left\{\frac{s^2 + 2s + 2}{(s^2 + 2s + 2)(s^2 + 2s + 5)}\right\}\frac{1}{3}\ell^{-t}(\sin t + \sin 2t)$$

2.5 THE USE OF INVERSE LAPLACE TRANSFORMS

Example 1: Evaluate $L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

Method: $\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2,+5)} = \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2+25+5)}$

Multiply by $(s^2 + 2s + 2)(s^2 + 2s + 5)$

$s^2 + 2s + 3 = (As + B)(s^2 + 2s + 5) + (Cs + D)(s^2 + 2s + 2)$

$= (A + C)s^3 + (2A + B + 2c + D)s^2 + (5a + 2b + 2c + 2D)s + 5B + 2D$

The $A + C = 0, 2A + BY\ 2X + D = 1, 5A + 2B + 2C + 2D = 2$

$5A + 2D = 3$, solving by the company coefficients to obtain $A = 0, B = \frac{1}{3}, C = 0, D = \frac{2}{3}$ thus

$$L^{-1}\left\{\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}\right\} = L^{-1}\left\{\frac{1/3}{(s^2 + 2s + 2)} + \frac{2/3}{(s^2 + 2s + 5)}\right\}$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{s^2 + 2s + 2} + \frac{2}{3}L^{-1}\left\{\frac{1}{(s^2 + 2s + 5)}\right.\right.$$

$$\frac{1}{3}\ell^{-t}\sin t\frac{2}{3}\frac{1}{2}e^{t}sin2t$$

$$\frac{1}{3}\ell^{-t}(sint + sin2t)$$

2.6 PARTIAL DIFFERENTIAL EQUATIONS

The laplace transform is also used in solving various partial differential equations subject to boundary condition. Such problems are often referred to as boundary – value problems. We consider a few such simple problems in this chapter.

A more complete discussion of boundary – value problem is seen in complex inversion formula where it is of great importance or has an advantage.

2.7 APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Given the function $u(x, t)$ defined for $a \leq x \leq b, t > 0$

Find

a) $\left\{\frac{\partial u}{\partial t}\right\} = \int_0^\infty \ell^{-st}\ \frac{\partial u}{\partial t}\partial t;$

b) $L[\frac{\partial u}{\partial t}] \int_0^\infty \ell^{-st} \frac{\partial u}{\partial t} \partial t$

using suitable restriction on $u = u(x, t)$

a) integrating by part, we have

$$L\left[\frac{\partial u}{\partial t}\right] = \int_0^\infty \ell^{-st} \frac{\partial u}{\partial t} \partial t = \lim_{p \to \infty} \int_0^p \ell^{-st} \frac{\partial u}{\partial t} \partial t$$

$$= \lim_{p \to \infty} \left[\ell^{-st} u(x, t)|_0^p + s \int_0^p \ell^{-st} \, u(u, t) dt \right]$$

$$= s \int_0^\infty \ell^{-st} \, u(x, t) dt - U(x, 0)$$

$$su(x, s) - u(x, 0) = su - u(x, 0)$$

where $u = u(x, s) = L\{U(x, t)\}$

We have assumed that $u(x, t)$ satisfied the restriction of theorem 1-1 which states that if $f(t)$ is sectionally continuous in every finite interval $0 \leq t \leq N$ and of exponential order y for $t > N$, then its laplace transform $f(s)$ exist for all $s > y$.

b) We have, using Leibnitz's rule for differentiating under the integral sign

$$L\left\{\frac{\partial u}{\partial x}\right\} \int_0^\infty \frac{\partial u}{\partial x} \partial t = \frac{d}{dx} \int_0^\infty \ell^{-st} U dt = \frac{du}{dx}$$

Referring to problem (a) we show that

a) $L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 u(x, s) - su(x, 0) - Ut(x, 0)$

b) $L\left\{\frac{\partial^2 u}{2t^2}\right\} = L\left\{\frac{\partial v}{\partial t}\right\} = SL\{v\} - v(x, 0)$

$= \{sL(U) - U(x,0)\} - U_t(x,0)$

$= s^2x - su(x,0) - U_t(x,0)$

Find the solution of $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, U(x,0) = 6e^{-3t}$

Taking the laplace transform of the given partial equation with respect to t and using problem (a), we find $\frac{du}{dx} = 2\{su - u(x,0)\} + u \quad or \quad \frac{du}{dx} - (2s+1)u =- 12\ell^{-3s}$

From the given boundary conditions, note that the laplace transformation has transformed the partial differential equation into an ordinary equation (1).

To solve; (1) we multiply both side by integrating factor $\ell^{-(2s+1)} = \ell^{-}(2s+1)x$. Then (1) becomes or can be written as

$$\frac{d}{dx}\{ue(2s+1)x\} = 12\ell^{-(2s+1)x}$$

Integration yields

$$u \cdot e(2s+1)_n = \frac{6}{s+2}\ell^{-(2s+4)x} + C \; or \; u = \frac{6}{s+2}\ell^{3u} +$$

Now since $u(x,t)$ must be bounded as $n \to \infty$, as $n \to \infty$ and it follows that we must choose $c = 0$. Then

$$u = \frac{6}{s+2}\ell^{-3x}$$

And so on taking the inverse, we find

$U(u,t) = 6e^{-2t-3x}$

This is easily checked as the required solution

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial u^2}, 9(x,0) = 3\sin 2\pi x \; U(0,t) = 0$

$U(1, t) = 0 \; where \; 0 < x < 1, t > 0$

Taking the transform the laplace transform of the partial equation using problem (a) and (b)

We find $su - u(x, 0) = \frac{d^2u}{dx^2} \; or \frac{d^2u}{du^2} - su =- 3 \sin 2\pi x$

where $u = u(u, s) = L\{U(x, t)\} =$ The general solution of (1) is

$$U = c_1 \ell^{\sqrt{su}} + c_2 \ell^{-\sqrt{su}} + \frac{3}{s + 4\pi^2} sin 2\pi x$$

Taking the laplace transform of these boundary conditions

Which involves t, we have $L\{U(0, t)\} = U(0, s) = 0 \; and \; L\{U(1, t)\} = U(1, s) = 0$ …… (4)

Using the first condition $\left(u(0, s)\right) = 0 \; of \; 3 \; in$ (2), we have $C_1 + C_2 = 0$

Using the second condition $\{U(1, s) = 0 \; of \; (3) in$ (2), we have $C_1 \ell^{\sqrt{s}} + C_2 \ell^{-\sqrt{s}} = 0$……(5)

From (4) and (5), we find $C_1 = 0, C_2 and \; so$ (2)

Becomes $U = \frac{3}{s+4\pi^2} \sin 2\pi x$

From which we obtain on inversion

$U(u, t) = 3\ell^{-4\pi x} \sin 2\pi x$

This problem has an interesting physical interpretation. If we consider a solid bounded by the infinite plane faces $x = 0$ and $u = 1$, the equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial t^2}$$

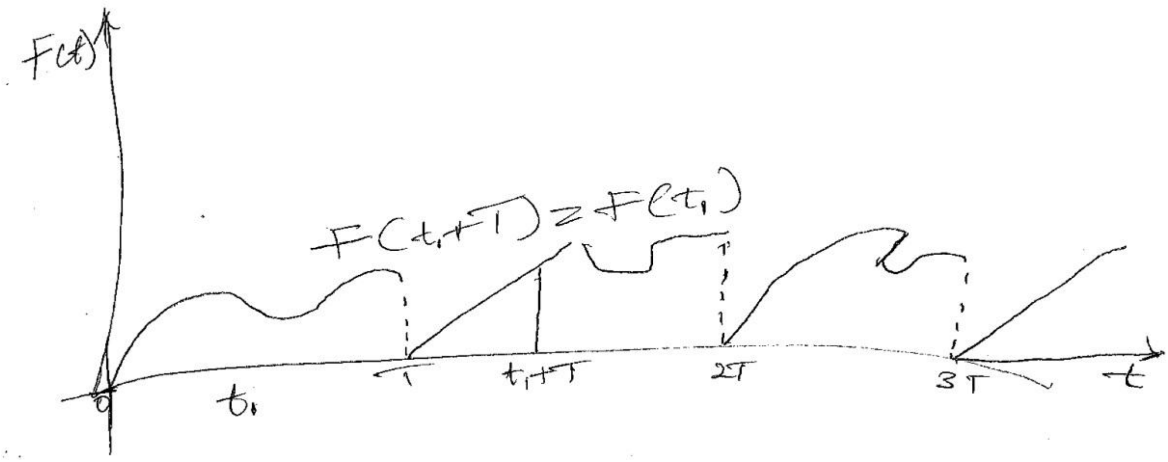
is the equation for heat conduction in this solid where $U = u(u, t)$ is the temperature at any plane force x at any time t and k is called constant and is referred to as diffusivity.

Some second linear order differential can also be solved by partial differential method provided the boundary value conditions are stated or given.

2.8 LAPLACE TRANSFORMS OF PERIODIC FUNCTION

2.8.1 PERIODIC FUNCTIONS (INTRODUCTION)

Let $f(t)$ represent a periodic function with period T so that $F(t + nT) = f(t)$ with a graph of the following form



If we describe the first cycle of $f(t)$, then,

$$f(t) = \begin{Bmatrix} f(t) for = 0 \leq t < t \\ 0 \quad\quad otherwise \end{Bmatrix}$$

The second cycle is identical to the first cycle except that is shifted by T units of time along the t −axis. Therefore the second cycle can be described in terms of the Heaviside unit step functions as $f(t - T)u(t - T)$. that is

$$f(t - T)u(t - T) = \begin{Bmatrix} f(t)\ for\ T \leq t \leq 2T \\ 0, \quad\quad otherwise \end{Bmatrix}$$

By reasoning, the periodic function $f(t)$ is represented by $f(t) = f(t)u(t) + ...$

$f(t) = f(t)u(t) + f(t-T)u(t-T) + f(u-2T)\,u(t-2T)...$

Because $u(t)$ switches on $f(t)$ at $t = 0, u(t-T)$ switches on $f(t-T)$ at time $t = T$ on $u\,(t-2T)$ switches on $f(t-2T)$ at $t = 2T$ etc.

Now consider the laplace transform of $f(t)$. By definition

$$L\{f(t\} = \int_0^\infty \ell^{-st}\,F(t)dt = \int_0^T \ell^{-st}\,f(t)ds = F(s)$$

Because for $t > T, f(t) = 0$ and so the semi-infinite integral becomes an integral just over the period of $f(t)$. Using the second shift theorem, the laplace transform of $f(t)$ is

$$L\{f(t)\} = L\{f(t)\,u(t)\} + L\{f(t-T)\,u(t-T)\} + L\{f(t-2T)\,u(t-2T)\} + ...$$

That is $L\{f(t)\} = f(s) + e^{-st} + e^{-2st}\,f(s) + ...$

Because

$L\{f(t)\,u(t-c)\} = e^{-sc}L\{f(t)\}$ by the second shift theorem, we can factor out $f(s)$ and write

$L\{f(t)\}$ as $L\{f(t)\} = (1 + e\ell^{st} + \ell^{-2st} + ...)f(s)$

Now, do you remember the series $1 + x + x^2 + x^3 + ...$ which can be written as

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

Because $\frac{1}{1-x}(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$

Either by the binomial theorem or by the long division

So if we let $x = \ell^{-st}\ then, 1 + \ell^{-st} + \ell^{-2st} + ... = \frac{1}{1-\ell^{-st}}$

And so the laplace transform of $f(t)$ is given as

$L\{f(t)\} = (1 + e^{-st} + e^{-2st} + \ldots)\, f(s)$ where f(s) is equal to $L\{f(t)\} = \frac{1}{(1-\ell^{-st})} f(s)$ where

$$f(s) = \int_0^T \ell^{-st}\, f(t)\, dt$$

Note that we integrate $\ell^{st}\, f(t)$ over one cycle, that is from $t = 0\ to\ t = T$ and from $t =$

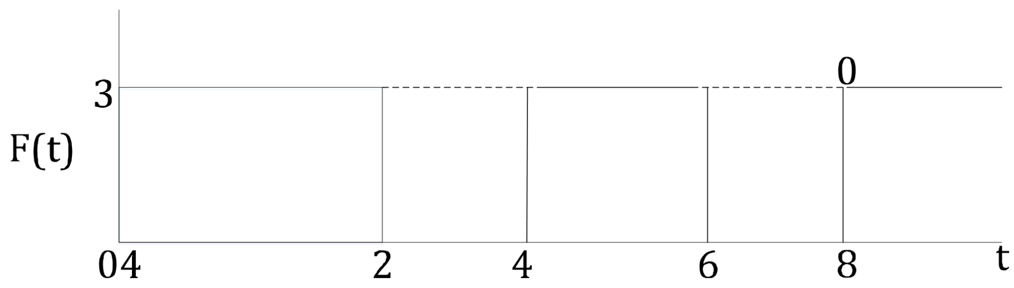
$0\ \ to\ t = \infty$ as we did previously.

Example 1

Find the laplace transform of the function $f(t)$ is defined by

$$f(t) = \begin{cases} 3 & 0 < t < 2 \\ 0 & 2 < t < 4 \end{cases}\ f(t+4) = f(t)$$

DIAGRAM



The expression for $L\{f(t)\}$ is

$$L\{f(t)\} = \frac{1}{1 - \ell^{-4s}} \int_0^4 \ell^{-st}\, f(t)\, dt$$

Because the period $= 4$ i.e $T = 4$

The function $f(t) = 3\ for\ 0 < t < 2\ and\ f(t) = 0$

For $2 < t < 4$

$$\therefore L\{f(t)\} = \frac{1}{1-\ell^{4s}}\int_0^2 \ell^{-st}\,3dt$$

$$L\{f(t)\} = \frac{3}{s(1+\ell^{-2s})}$$

Because

$$L\{f(t)\} = \frac{3}{1-\ell^{-4s}}\left[\frac{\ell^{-st}}{-s}\right]_0^2 = \frac{3}{1-\ell^{-4s}}\left\{\left(\frac{\ell^{-2s}}{-s}\right)-\left(\frac{1}{-s}\right)\right\}$$

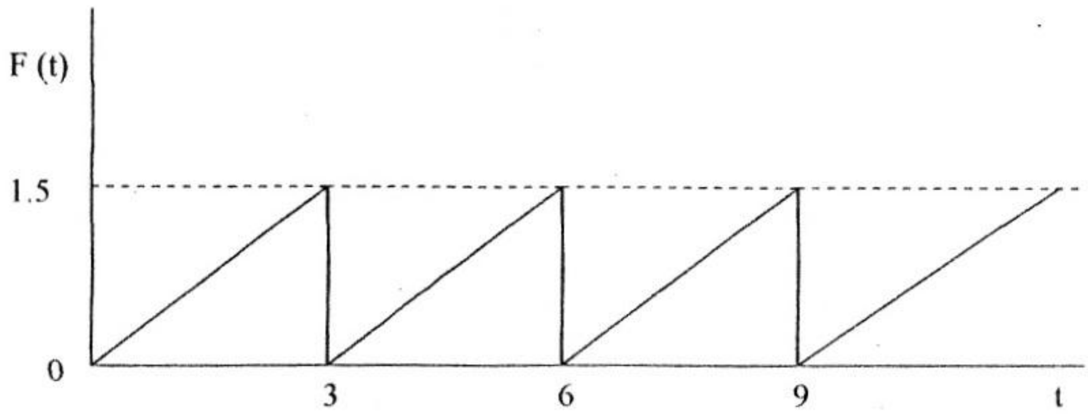
$$= \frac{3}{1-\ell^{-4s}}\left[\frac{1-\ell^{-2s}}{s}\right] = \frac{3}{s(1+\ell^{-2s})}$$

Example 2

Find the laplace transform of the periodic function defined by $f(t)$

$= {}^{t}/_{2} \quad 0 < t < 3$

$f(t+3) = f(t)$



Because in this case period = 3, i.e. T = 3

$$\therefore \text{L}\{\text{f(t)}\} = \frac{1}{1-\ell^{-3s}}\int_0^T \ell^{-st}\ .f(t)\,dt$$

$$= \frac{1}{1-\ell^{-3s}}\int_0^T \ell^{-st}({t}/{2})dt$$

$$\therefore 2(1-\ell^{3s-})\text{L}\{\text{f(t)}\} = \int_0^3 t.\ell^{-st}dt$$

integrating by parts and simplifying the result to give $\text{L}\{\text{f(t)}\} = \frac{1}{2s^2}\left\{1-\frac{3s}{\ell^{3s}-1}\right\}$

because

$$2(1-\ell^{3s})\text{L}\{\text{t}\} = \int_0^3 t\ \ell^{-st}dt$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^3 + \frac{1}{s}\int_0^3 \ell^{-st}\,dt = -\frac{3\ell^{-3s}}{s} + \frac{1}{s}\left[\frac{\ell^{-st}}{-s}\right]_0^3$$

$$= -\frac{3\ell^{-3s}}{s} - \frac{\ell^{-3s}}{s^2} + \frac{1}{s^2}$$

$$\therefore L\{f(t)\} = \frac{1}{2s^2}\left[1-\frac{3s\ \ell^{-s}}{1-\ell^{-3s}}\right]$$

$$= \frac{1}{2s^2}\left[1-\frac{3s}{\ell^{-3s}-1}\right]$$

2.8.2 LAPLACE TRANSFORMATION IN THE FORM OF $\boldsymbol{\delta(t-a)}$

We already shown that,

$\int_p^q f(t) - \delta(t-a)dt = f(a)$, provided $p < a < q$ still holds

$\therefore$ If $p = 0$ and $q = \infty$

$$\int_0^\infty f(t).\,\delta(t-a)dt = f(a)$$

Hence, if $f(t) = \ell^{-st}$, this becomes

$$\int_0^\infty \ell^{-st}(t-a)\,dt = \mathrm{L}\{\delta(\mathrm{t}-\mathrm{a}\} = \ell^{-\mathrm{as}}$$

i.e. the value of $f(t)$, i.e. $\ell^{-st}, at\ \ t = a$

$\mathrm{L}\{\delta(\mathrm{t}-\mathrm{a})\} = \ell^{-\mathrm{as}}$

It follows from the definition that the laplace transform of the impulse function at the origin is 1.

Because, for $a = 0$, $\mathrm{L}\{\delta(\mathrm{t}-\mathrm{a})\} = \mathrm{L}\{\delta(\mathrm{t})\} = \ell^0 = 1\ \therefore\ L\{\delta(t)\} = 1$

Finally, Let us deal with the more general case of $\mathrm{L}\{\mathrm{f}(\mathrm{t}).\,\delta(\mathrm{t}-\mathrm{a})\}$

We have L: $f(t).\,\delta(t-a)$

$= \int_0^\infty \ell^{-st} f(t)\,.\,\delta(t-a)dt$. Now the integrand

$\ell^{-st}\ .\,f(t).\,\delta(t-a) = 0$ for all values of t except at $t = a$ at which point $e^{-st} = e^{-st}$ and $f(t) = f(a)$

$\therefore\ \{Lf(t)\,.\ \delta(t-a)\} = f(a)\,e^{-as}(1)$

$\therefore \{Lf(t)\,.\ \delta(t-a)\} = f(a)\,e^{-as}(1)$

We have that

$L\,\{f(t)\,.\,\delta(t-a)\} = f(a)\,.\,e^{-as}$

Therefore (a) $\mathrm{L}\{6\ .\,\delta(\mathrm{t}-4)\}$ at $a = 4$

$\therefore \mathrm{L}(6\ .\,\delta(\mathrm{t}-4)\} = 6\mathrm{e}^{-4\mathrm{s}}$

and also (b) $L\{t^3 . \delta(t-2)\} = t \;\; a = 2$

$\therefore \; L\{t^3 \;\; \delta(t-2)\} = 8 \;\; e^{-2s}$

2.8:3 THE DIRAC – DELTA FUNCTION WITH ITS FORMAL PROPERTIES

Consider a function $\delta(x)$ which is zero every where except at $x = 0$ and tend to ∞ in such a manner that $\int_{-\infty}^{\infty} \delta(u)\, dx = 1$…………………………………………………….. (1)

$$with \;\; \delta(t) \begin{matrix} = 0 \text{ if } t \neq 0 \\ = \infty \text{ if } t = 0 \end{matrix}\Bigg\} \qquad \text{………………………………………………………….. (2)}$$

This is known as Dirac–delta function and it is used in mathematical physics whenever functions exist with non–zero values in very short intervals, e.g. an impulsive force acting for a short while is defined as $\delta(x-\xi)$ by

$$\lim_{a \to 0} Ca^{-(x-\xi)^2/a}$$

where the constant $C(a)$ is chosen such that

$\int_{-\infty}^{\infty} \delta(x-\xi)\, dx = 1$ and hence using the mean value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} f(x)\delta(x-\xi)\, dx = f(\xi)$$

Let us again consider

$$\delta_a(x) = \frac{1}{2a}, \begin{matrix} -a < x < a \\ = 0, |x| > a \end{matrix}\Bigg\} \qquad \text{…………………………………………………………… (3)}$$

$$\text{Then } \int_{-\infty}^{\infty} \delta(x)dx = \int_{-\infty}^{\infty} \delta_a(x)dx + \int_{-\infty}^{\infty} \delta_a(x)dx + \int_{a}^{\infty} \delta_a(x)dx$$

$$= 0 + \int_{-a}^{a} \frac{1}{2a} dx + 0 = \frac{1}{2a} [a - (-a)]$$

$= \frac{1}{2a}\ 2a = 1$………………………………………………………………………..…….. (4)

Incase $f(x)$ is integrable in the interval $(-a, a)$, then from the mean value theorem,

$$\int_{-\infty}^{\infty} f(x)\delta_a(x)dx = \frac{1}{2a}\int_{-a}^{a} f(x)dx = f(\theta a); |\theta| \leq 1$$

Let us now define $\delta(x) = \lim_{a\to 0} \delta_a(x)$

and such (3) and (4) becomes

$\delta(x) = 0,$ when $x \neq 0$

$and\ \int_{-\infty}^{\infty} d(x)dx\ \ 1$, which define Dirac-delta function.

Further, since we have

$$\int_{-\infty}^{\infty} f(x)\delta_a(x)dx = \frac{1}{2a}\int_{-a}^{a} f(x)dx = f(\theta\ a), |\theta| \leq 1$$

$\therefore\ \int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$……………………………………………………………. (7)

which by change of variable, reduce to

$\int_{-\infty}^{\infty} f(x)\ \delta(x-a)dx = f(a)$…………………………………………………………….. (8)

A symbolically, $f(x)\delta(x-a) = f(a)\delta(x-a)$………………………………………….... (9)

in case $f(x) = x,$ (9) field, $xdx = 0$ ……………………………………….…..... (10)

In a similar manner we can show

$\delta(-x) = \delta(x)$……………………………………………………………………………….(11)

$\delta(-ax) = \frac{1}{a}\delta(x), a > 0$……………………………………………………………….(12)

$\delta(a^2 - x^2) = \frac{1}{2a}\{\delta(x - a) + \delta(x + a)\}, a > 0$………………………………………(13)

Now assuming that $\delta^1(x)$ i.e. differential of $\delta(x)$ exists and regarding $\delta(x)$ and $\delta'(x)$ both as ordinary functions in the rule for integrating by parts, we have

$$\int_{-\infty}^{\infty} f(x)\delta'(x)dx = [f(x)\delta(x)]_{\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x)dx$$

$= 0 - f'(0)$ by (7)

If $\delta^{(n)}$ be the nth derivative of $\delta(x)$, then similarly we find on repeating this process n times,

$$\int_{-\infty}^{\infty} f(x)\delta^n(x)dx = (-1)^n f^n(0)$$

Problem that the function $\delta(x) = \lim_{\varepsilon \to 0} \frac{\sin(2\pi\varepsilon\, x)}{\pi x}$ is a Dirac–Delta function

We have that $\delta(x) = \lim_{\varepsilon \to 0} \frac{\sin(2\pi\varepsilon x)}{\pi x}$

$\therefore\ \delta(x) = 0$ when $x \neq 0$ ……………………………………………………………….(1)

and $\int_{-\infty}^{\infty} \frac{\sin(2\pi\varepsilon)}{\pi x} dx = 2\int_{\infty}^{\infty} \frac{\sin(2\pi\varepsilon\, x)}{\pi x} dx$, the function being even

$= \frac{2}{\pi}\ .\ \frac{\pi}{2} = 1$ …………………………………………………………………………(2)

It follows from (1) and (2) that the given function is Dirac–Delta function.

2.8.4 SOME SPECIAL SOLUTION OF DIRAC-DELTA FUNCTION USING LAPLACE TRANSFORMS

The use of laplace transform to evaluate problems on dirac – delta function using $p < a < q$.

This is usually denoted by the integral given by $\int_q^p f(t)\delta(t - a)dt = f(a)$, provided on each cases equation (1), holds

Note: If $f(t)$ represents a function, the dirac – delta $\delta(t)$ is often referred to the above integral even though it is not a function in the conventional sense being completely defined in terms of its output for corresponding inputs

$$\delta(t) = \begin{cases} 0, t \neq 0 \\ udf, t = 0 \end{cases}$$

From the definition, if $f(t) = 1$, then

$$\int_{-\infty}^{\infty} \delta(t-a)dt = 1$$

Because $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$ and $f(t) = 1$, so $f(a) = 1$

2.8.5 LAPLACE TRANSFORM METHODS

If $f(t)$ is a function defined for some values of a in the given formula

$f(t)\delta(t-a) = f(a) \,.\, e^{-as}$ where a is the value of the impulse from the factor $\delta(t-a)$.

Some examples

(a) $L\{6 \,.\, \delta\,(t-4)\} = a = 4$, i.e. appoint where impulse of the above formula occur hence, $L\{6 \;.\, \delta(t-a)\} = 6e^{-4s}$

(b) Evaluate L{ $\sin 3t \;.\, \delta\left(t - {}^{\pi}/_{2}\right)$ $i.e.$ $a = \frac{\pi}{2}$

$\therefore$ L($\sin 3t \,.\, \delta\left\{t - {}^{\pi}/_{2}\right\} = [\,\sin 3t\,]_{{}^{\pi}/_{2}} \cdot e^{-\pi s/_{2}}$

Hence L($\sin 3t \,.\, \delta\left\{t - {}^{\pi}/_{2}\right\} = [\,\sin 3t\,]_{{}^{\pi}/_{2}} \cdot e^{-\pi s/_{2}}$

CHAPTER THREE

MORE ON PDES AND THE WAVE EQUATION

The formulation of ordinary linear differential equations and their solution by various methods already discussed in chapter.

The main results obtained are listed here for convenience and easy reference

1. Equation of the from $a\frac{\partial^2 y}{\partial x^2} + b\frac{\partial y}{\partial x} + Cy = 0$

Auxiliary equation $am^2 + bm + C = 0$. Solution depend on the roots of this equation

(a) Real different roots: $m = m_1$ and $m = m_2$

Solution $y = Ae^{mix} + Be^{m_2}$ ……………………………………………………………(1)

(b) Real and equal roots: m = m_1 (twice)

Solution $y = e^{mix}(A + Bx)$ ……………………………………………………………(2)

(c) Complex roots:$m = a \pm j\,\beta$

Solution $= e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ …………………………………………………(3)

Equation of the form

2. (a) $\frac{d^2y}{dx^2} \pm n^2 y = 0$ $\quad m^2 + n^2 = 0 \quad \therefore m^2 =- n^2 \quad \therefore m =\pm jn$

Solution $y = A\cos nx + B\sin nx$

(b) $\frac{d^2y}{dx^2} - n^2 y = 0 \quad \therefore m^2 - n^2 = 0 \quad \therefore\ m^2 - n^2 \qquad \therefore\ M = \Sigma n$

Solution $y = A\cosh nx + B\sinh nx$

or $y = Ae^{nx} + Be^{-nx}$

or $y = A \sinh n(x + \phi)$

In each case, A and B are arbitrary constants depending on the initial conditions, and in the last from ϕ is an arbitrary constant.

A partial differential equation is the relationship between a dependent variables x and two or more independent variables $(x, y, t, ...)$ and partial derivatives of x with respect to these independent variables. The solution is therefore of the rom $U = f(x, y, t...)$.

3.1 SOLUTION BY DIRECT INTEGRATION

The simplest form of partial equation is such that a solution can be determined by direct partial integration.

Example 1

Solve the equation $\frac{\partial^2 u}{\partial x^2} = 12x^2(t + 1)$ given that at $x = 0, u = \cos 2t$ and $\frac{du}{dx} = \sin t$. Notice that the boundary conditions are functions of t and not just constants $\frac{\delta^2 u}{\delta x^2} = 12x^2(t + 1)$

Integrating partially,

with respect to x, we have $\frac{\partial u}{\partial x} = 4x^3(t + 1) + \phi(t)$

where $\phi(t)$ is the arbitrary constants takes the place of the normal constant of ordinary integration. Integrating partially again to obtain:

$U = x^4(t + 1) + x\phi(t)$ where is (t) the second arbitrary constant.

To find the 2 arbitrary constants, we apply the secondary conditions at $x = 0$,

$\partial n = \sin t$ and $U = cos2t$ Substituting these into the relevant equation gives ∂n

$\phi(t) = sint + (t) = cos2t$

Because, $U = x^4(t+1) + x\,sint + \cos 2t$

Example 2

Solve the equation $U = x^4(t+1) = x\,sin\,(x+y)$ gives that at

$y = 0,\ \frac{\partial u}{\partial x} = 1$ and at $x = 0, U = (y-1)^2$

In just the same way in example 1,

$U =- \sin(x+y) + x + \sin x + (y-1)^2$

because, $\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y) \therefore \frac{\partial u}{\partial x} =- \cos(x+y) + \phi(x)$

at $y = 0, \quad \frac{\partial u}{\partial y} = 1 \quad \therefore 1 =- \cos x + \phi(x) \quad \therefore \phi(x) = 1 + \cos x$

$$\therefore \frac{\partial u}{\partial x} =- \cos(x+y) + 1 + \cos x$$

Integrating again partially, this time with respect to x, we have

$U =- \sin(u+y) + x + \sin y + (y-1)^2$

But at $x = 0,\ U = (y-1)^2 \quad \therefore (y-1)^2 =- \sin + \theta(y)$

$\therefore \theta(y) = (y-1)^2 \quad \therefore (y-1)^2 =- \sin y + \theta(y)$

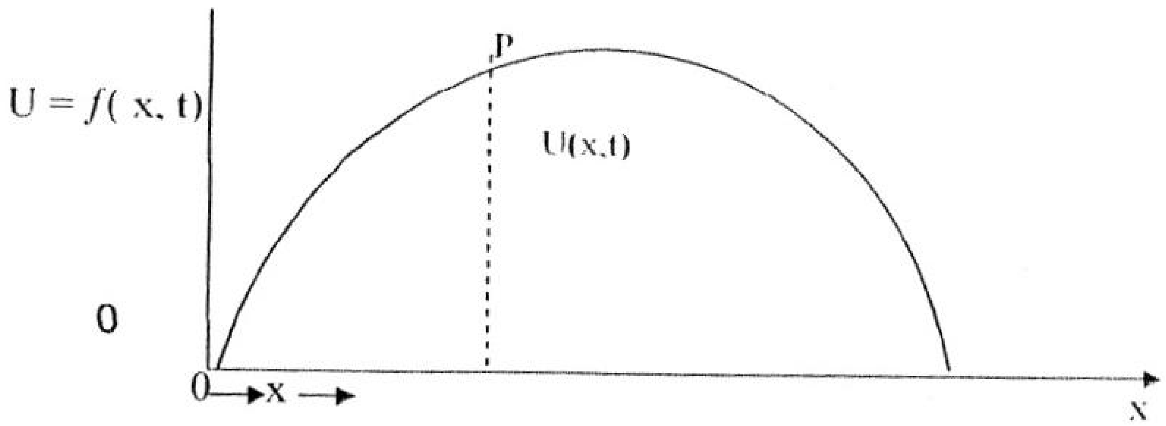
$\therefore \theta(y) = (y-1)^2 + \sin y$

$\therefore U =- \sin(x+y) + x + sin + \sin y + (y-1)^2$

3.2 INITIAL CONDITIONS AND BOUNDARY CONDITIONS

As with any differential equation, the arbitrary constants or arbitrary functions in any particular case are determined from the additional information given concerning the variables of the equation. These extra facts are called the initial conditions or Mine generally, the boundary conditions since they do not always refer to zero values of the independent variables.

3.3 THE WAVE EQUATION



Consider a perfectly flexible elastic stretched between 2 points at $x = 0$ and $x = L$ with uniform tension T. If the string is displaced slightly from its initial position of rest and released, with the end points remaining fixed, then the string will vibrate. The position of any point P in the string will then depend on its distance from one end and on the instant in time. Its displacement U at any time t can thus be exposed as $u = f(x, t)$, where x is the distance from the left end.

The equation of motion is given by $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \cdot \frac{\partial u}{\partial t^2}$

where $C^2 = \frac{T}{\rho}$ in which T is the tension in the string and the mass per unit length of the string.

The displacement of the string is regarded as 5m tall so that T and e remain constant.

Solution of the Wave Equation

The wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}$ has a solution

Solution U(x, t)

3.4 BOUNDARY CONDITIONS:

a) The string is fixed at both ends, i.e at $x = 0$ and $x = L$ for all values of time t. Therefore $u(x, L)$ becomes $U(0, t) = 0 \quad U(I, t) = 0$ for all values of $t \geq 0$

Initial condition

b) If the initial deflection of P at $t = 0$ is deviated by $f(x)$, then $U(x, 0) = f(x)$, then $x(x, 0) = f(x)$.

c) Let the initial velocity of P be $g(x)$, then

$$\left[\frac{\partial u}{\partial t}\right]_{t=0}^{=g(x)}$$

Solution by separating variables (3:6)

We assume a trial solution of the form $U(x, t)$

$= X(x)T(t)$ where

$X(x)$ is a function of x only

$T(t)$ is a function of t only

If we simplify the symbols to $U = XT$ and denote derivatives with respect to their own independent variables by primes, we have:

$$U = XT \quad \therefore \frac{\partial u}{\partial x} = X^{'}T \ and \frac{\partial^2 u}{\partial x^2} = X^{''}T$$

$$\frac{\partial u}{\partial t} = XT^{'} \ and \frac{\partial^2 u}{\partial t^2} = XT''$$

The wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 u}{\partial t^2}$ can be written as:

$$X''T = \frac{1}{C^2} XT''$$

And this can be transposed to $\frac{X''}{X} = \frac{1}{C^2}\frac{T''}{T}$

Notice that the left hand side expression involves function of x only and the right hand side expression involves function of t only.

∴ If these 2 expressions are to be equal for all values of the separate variables, then both expressions must be equal to a constant. Denote this arbitrary constant by K. then we have

$$\frac{X''}{X} = K \; and \frac{1}{C^2}\frac{T''}{T} = K$$

$$\therefore X'' - KX = 0 \;\; and \;\; T'' - C^2KT = 0$$

Let us consider the first of these 2 equation for different values of K.

1) If $K = 0, X'' = 0 \quad \therefore X' = a \quad \therefore X = ax + b$

But $X = 0 \;\; at \; x = 0 \quad \therefore b = 0 \quad \therefore X = ax$

$\therefore X = 0$ which is not oscillatory as the problem require it to be

2) If K is positive, let $K = P^2 \quad \therefore X'' - P^2X = 0$

The auxiliary equation is $\therefore m^2 - P^2 = 0 \quad \therefore m^2 = p^2 \; m =+$ or $-P$

$$\therefore X = Ae^{px} + Be^{-px}$$

But $X = 0 \; at \; x = 0 \quad \therefore 0 = A + B, \quad \therefore B =- A$

and $X = 0 \; at \; x = L \quad \therefore 0 = Ae^{pl} + Be^{-pl} \quad \therefore 0 = A$

$$\therefore A = 0 \quad \therefore A = B = 0$$

Here again, $X = 0$ which is not oscillatory

3) If k is negative, let $K = -P^2$ $\therefore X'' + p^2 X = 0$

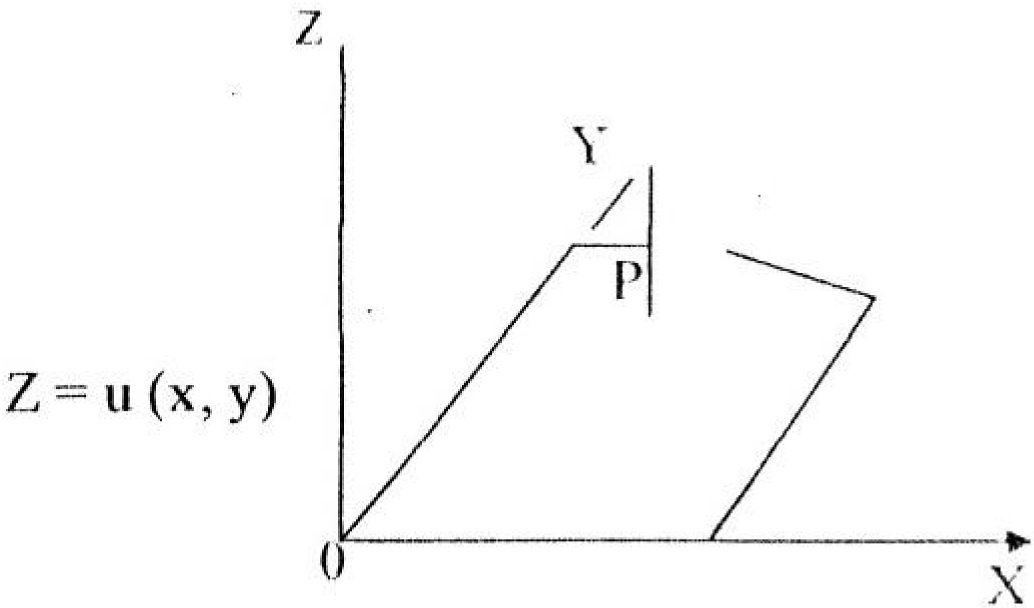
This is one of the standard equations listed at the beginning of this topics

$X = ACos\,px + BSin\,px$

Which fit the requirement of the solution of the separable variable.

3.5 THE LAPLACE EQUATION

The laplace equation concerns the distribution of a field e.g temperature, potential etc over a plane are subject to certain boundary conditions



The potential at a point P in a place can be indicator by an ordinate axis and is a function of its position, i.e $Z = u(x, y)$ where $u(x, y)$ is the solution of the laplace 2–dimensional equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Example 1

Determine a solution $u(x, y)$ of the laplce equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

to the following conditions:

$u = 0 \quad where\, x = 0, \;\; u = 0 \;\; when\, x = \pi$

$u = 0 \quad where\, y \to \infty, \;\; u = 3 \quad when\, y = 0$

as always, we begin with $u(x, y) = X(x)\, Y(y)$ rewrite the equation in term of x and y and separate variables. The equation then:

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Equating each side to $-P^2$ we have $X'' + p^2 X = 0 \quad and\, Y'' - P^2 Y = 0$

$X'' + P^2 X = 0$ has the solution

$X = ACos\, px + BSin\, Px$

The solution of $Y'' - P^2 Y = 0$ can be stated in three different forms

$T = (Cosh\, py + 1) Sinh\, py, Y = Ce^{py} + De^{-py}$

$Y = Sinhp(y + \phi)$ in this way, we will use the second one $Y = Ce^{py} + De^{-py}$.

Then $u(x, y) = \{ACos\, px + BSin\, Px\}\{Ce^{py} + De^{-py}\}$

Applying the first boundary condition

$U(0, y) = 0$ gives,

$A = 0 \; and \; u(x, y) = Sin \; px\{Pe^{py} + Qe^{-py}\}$

CHAPTER FOUR

4.0 SUMMARY AND CONCLUSION

This project work examines the concepts and applications of laplace transforms. Laplace may consist of many discrete functions e.g periodic, dirac- delta, wave equations, integral functions etc

A laplace system includes the transformation of a system of equation and its application to both odes and pdes (i.e system of linear equations).

Laplace theory studies laplace transformation system by formulating mathematical methods of their operations and then using the method to derive measures of solution. This analysis provides vital information for effectively designing laplace system. The concepts of laplace theory were thoroughly discussed. These concepts include the basic laplace system, the terms/theorem, pattern of solution.

Lastly, the applications of laplace theory were examined for the ordinary and partial differential equations by the application of initial conditions. Single method and other methods can be useful.

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