SECOND DERIVATIVE ADAMS TYPE FORMULAE FOR THE INTEGRATION OF STIFF STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS

BY

Ima Jacob JACOB
B.Sc. (Uyo), M.Sc. (Ibadan)
PG/PSC2016304

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BENIN,
BENIN CITY

DECEMBER 2025

SECOND DERIVATIVE ADAMS TYPE FORMULAE FOR THE INTEGRATION OF STIFF STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS

BY

Ima Jacob JACOB
B.Sc. (Uyo), M.Sc. (Ibadan)
PG/PSC2016304

A THESIS WRITTEN IN THE DEPARTMENT OF MATHEMATICS AND SUBMITTED TO THE COLLEGE OF POSTGRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF DEGREE OF DOCTOR OF PHILOSOPHY IN INDUSTRIAL MATHEMATICS OF THE UNIVERSITY OF BENIN, BENIN CITY, EDO STATE, NIGERIA.

DECEMBER 2025

CERTIFICATION

This is to certify that this thesis was written by Ima Jacob JACOB (MRS) with Matriculation Number PG/PSC2016304 in the Department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, Nigeria under the supervision of Prof. K. O. MUKA.

Sign & Date ____________________

Professor Muka, K. O., Ph.D.

Supervisor

Sign & Date ____________________

Professor D.U. Okuonghae, FNYA

Head of Department

DECLARATION

UNIVERSITY OF BENIN
SCHOOL OF POSTGRADUATE STUDIES
SECOND DERIVATIVE ADAMS TYPE FORMULAE FOR THE INTEGRATION OF STIFF STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS

BY

IMA JACOB JACOB
B.Sc (Uyo), M.Sc (Ibadan)
PG/PSC2016304

THE BOARD OF EXAMINERS DECLARE AS FOLLOWS, THAT THIS IS THE ORIGINAL WORK OF THE CANDIDATE, THAT THE THESIS IS ACCEPTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY (Ph.D) IN INDUSTRIAL MATHEMATICS.

____________________ ____________________

Prof. Muka, K. O., Ph.D.
Supervisor

Prof. D. U. OKUONGHAE, FNYA
Head of Department

CERTIFICATION OF THESIS ON PLAGIARISM

We the undersigned attest and declare that the thesis of Mrs. IMA JACOB JACOB (MRS) Titles: SECOND DERIVATIVE ADAMS TYPE FORMULAE FOR THE INTEGRATION OF STIFF STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS has successfully passed the anti-plagiarism test and does not violate any copy right regulation.

Sign & Date——————————————
Professor Muka, K. O., Ph.D.
Supervisor

Sign & Date——————————————
Professor D.U. OKUONGHAE, FNYA
Head of Department

DEDICATION

This thesis is dedicated to Almighty God.

ACKNOWLEDGMENT

First of all, i give thanks to my Creator and Messiah, I AM THAT I AM, the Controller of times and seasons, for seeing me through this research work. His grace has indeed been more than sufficient. My thanks also go to my supervisor, my boss, my mentor, an excellent reader and one of the most intelligent man I have ever met, Prof. K. O. Muka, for his academic and professional guidance necessary to make this work come to acceptable completion. May God bless you and elongate your life in Jesus mighty name, Amen. I am equally grateful to my Head of Department, Prof. D. U. Okuonghae for his advice and encouragement. I extend my appreciation to the Post Graduate Coordinator Dr. I. I. Ako for his continuous encouragement, Prof. C. I. Nkeki for your support, Dr. C. Ibe for your help, Dr. Mrs. R. Omoregie for your encouragement. I want to also thank Dr. Mrs. I. B. Aihie, Dr. Mrs. S. Akhigbe, Mrs. T. Batubo, Prof. Mrs. G. Nwachukwu, Prof. S. E. Omosigho, Prof. M. N. O. Ikhile, Prof. P. O. Ekoko, Prof. A. A. Osagiede, Prof. E. O. Oghre, Prof. S. E. Uwamusi, Prof. F. E. U. Osagiede, Prof. R. I. Okuonghae, Prof. D. E. A. Omorogbe, Prof. O. Izebizua, Dr. O. O. Olowu, Mr. H. O. Omokaro, Dr. K. E. Osawaru, Dr. S. E. Ogunfeyitimi, Mr. E. Igbinovia. And all non academic staff. I'm grateful for your efforts in making this Ph.D. programme come to a successful ending. I am also grateful to Dr. S.A. Osagie, Mrs. J. Adewole for their supports. God bless you all. I also want to thank my mentors, Prof. E. O. Ayoola for laying a foundation, Dr. M. E. Egwe, Dr. Z. Ongodiebi for their support and encouragement. I am indeed very grateful to my darling husband Dr. J. N. Jacob for the encouragement, care, love, support and prayers toward the success of this programme. To my lovely children, Awesome, Wonderful, Amazing, Reigner and Supremacy, thank you so much for your prayers and understanding. I also want to Thank my wonderful mother in-law, Pastor (Mrs) R. N. Jacob, for your support and prayers which have been major factors to the success of this research work. May God bless and preserve your life in Jesus name! Finally, I say thank you to my guardian angels and Spirit, the one that is in me, that makes me to be me. I salute your courage, faithfulness and enthusiasm. Keep on keeping on.

Table of Contents

Table of Figures

Table of Tables

ABSTRACT

Real life models that are characterized by randomness are best described using Stochastic Ordinary Differential Equations (SODEs). Most SODEs that satisfy existence and uniqueness theorem are often insoluble via the use of analytic methods. Numericals solution are derived and the complexity of generating approximate solutions to SODEs are heightened by the presence of the phenomenon called stiffness. Hence, A-stable numerical methods are desired. This is a strigent requirement that can only be met by implicit methods.

Two families of A-stable numerical methods for numerical approximation of SODEs are derived using Ito Taylor, Taylor's series and undetermined coefficients methods. The stability analysis of both families of methods are established using the Boundary locus method.

Families of methods developed are A-stable for $k \leq 12$. Mean-square stable and strong stable for order $p = 1$. The Numerical implementation generated on the standard test problems in the literature shows that the numerical solution to methods developed are in most cases better when compared to numerical solution generated by existing methods in the literature designed for stiff SODEs. The numerical solutions are also compared with exact solution where they are available. The numerical solution mimic the exact solution, hence the proposed methods are well suited for the treatment of stiff SODEs.

CHAPTER ONE

Introduction

1.1

Stochastic Ordinary Differential Equations (SODEs) represent a class of Mathematical models critical to describing systems evolving in time with inherent randomness. These equations integrate deterministic dynamics with stochastic processes, often modeled as Wiener processes or Brownian motion. In real-world systems, such equations capture uncertainties due to incomplete knowledge, environmental unpredictability, or intrinsic variability of the system under study. Among SODEs, a significant subclass exhibiting stiffness has attracted increasing attention due to both theoretical complexities and computational challenges posed by their combined features of rapidly varying and slowly evolving components.

Stiffness, originally a concept within deterministic Ordinary Differential Equations (ODEs), manifests when solutions exhibit multiple time scales wherein some modes decay or evolve much faster than others. Extending this concept to SODEs introduces additional complications because the stochastic terms can interfere with numerical stability and accuracy in unpredictable ways. The interplay between stiff drift components and stochastic perturbations leads to models that representative of many complex processes in science and engineering.

Mathematical models that are accounts for randomness are formulated using Stochastic Ordinary Differential Equation of the form:

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(t_0) = X_0, \quad t \in [t_o, T], \quad X : \mathbb{R} \to \mathbb{R}^n,$$
$$f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \quad g : \mathbb{R}^n \to \mathbb{R}^{n \times m}, dW(t) \in \mathbb{R}^m. \tag{1.1.1}$$

where $f(X(t))$ is the drift term, $g(X(t))$ is the diffusion (or the stochastic term) and $W(t)$ is the independent Wiener process, with increment $\Delta W(t) = W(t+\Delta t) - W(t)$ is a Guassian random variable, that is $\Delta W(t) \sim N(0, \Delta t)$. The SODE (1.1.1) can equivalently be written in integral form as:

$$X_t = X_0 + \int_{t_0}^{t} f(s, X_s)\, ds + \int_{t_0}^{t} g(s, X_s)\, dW_s. \tag{1.1.2}$$

where the first integral of (1.1.2) is Riemann sum integral and the second integral is called the Ito integral which cannot be understood as a Riemann-Stieltjes integral, because Wiener process is nowhere differentiable and unbounded, Kloeden and Platen (1992). The autonomous form of (1.1.1) is given as:

$$dX_t = f(X_t)\, dt + g(X_t)\, dW_t. \tag{1.1.3}$$

1.2 Stiff and non-stiff Stochastic Ordinary Differential Equations (SODEs)

An ODE is said to be stiff, if it contains components evolving on widely separated time scales Hairer et al (1993), Muka and Ikhile (2015). In the deterministic case (when $g \equiv 0$), stiffness is typically associated with rapidly decaying modes in the drift term, Muka and Ikhile (2015). When extending this idea to Stochastic Ordinary Differential Equations (SODEs), stiffness persists but becomes more complex due to the presence of noise. For stochastic systems, stiffness may arise due to the combined effects of both drift term $f(x, t)$ and diffusion term $g(x, t)$ in (1.1.1). This occurs particularly when the drift term $f(x, t)$ has eigenvalues with large negative real parts or when the diffusion term $g(x, t)$ introduces stochastic perturbations that amplify numerical instability. Both the drift and diffusion terms influence stability, and as a result, stiffness in SODEs is typically studied in terms of *mean-square stability* and *asymptotic stability* (Higham (2000) and Schurz (1999)). The SODE (1.1.1) is said to be **stiff** or **non-stiff** depending on the behaviour of its drift term and how it affects the numerical stability of the solution.

Definition 1.1. *(Burrage and Burrage (1996), Schurz (1999))*

A Stochastic Ordinary Differential Equation (SODE) is said to be stiff *if its drift and diffusion coefficients generate solution components that evolve on widely differing time scales.*

A simple test equation often used to study stiffness in SODEs is the *linear scalar stochastic test equation*:

$$dX_t = \lambda X_t\, dt + \mu X_t\, dW_t, \qquad X(0) = X_0, \tag{1.2.1}$$

where $\lambda, \mu \in \mathbb{C}$, and $\Re(\lambda) < 0$. The exact solution of (1.2.1) is

$$X_t = X_0 \exp\big(\big(\lambda - \tfrac{1}{2}\mu^2\big)\, t + \mu W_t\big)\,.$$

For this equation, stiffness arises when $|\Re(\lambda)| \gg 1$, that is, when the drift term causes rapid decay in the mean, while the diffusion term introduces slow stochastic variations.

Example:

$$dX_t = -100 X_t\, dt + 0.5 X_t\, dW_t, \qquad X_0 = 1. \tag{1.2.2}$$

Here, the deterministic part decays extremely fast, creating stiffness. Explicit Euler schemes require $h < \dfrac{2}{|\lambda|}$ for stability that is, $h < 0.02$. Such small h values make explicit integration inefficient.

Remark 1.1. *In the stochastic setting, stiffness can also be characterized through the spectral properties of the drift Jacobian and the diffusion matrix. If the eigenvalues of the drift Jacobian*

$\partial f/\partial x$ have large negative real parts and the diffusion term does not sufficiently dampen high-frequency stochastic modes, then the SODE is said to be stiff Schurz (1999).

Definition 1.2. *Non-Stiff SODE*

*An SODE (1.1.1) is **non-stiff** when its drift term does not vary rapidly and numerical methods can solve it accurately with a moderate step size h without loss of stability.*

Example:

$$dX_t = -2X_t\,dt + 0.5X_t\,dW_t, \qquad X_0 = 1. \tag{1.2.3}$$

Here, the deterministic decay rate (-2) is mild; thus, the numerical methods are stable even for moderately large time steps.

1.3 Stochastic Models

Stochastic models that describe real-life phenomena abound across different fields. In biochemical kinetics, certain reactions occur on molecular scales within nanoseconds, while others span minutes or longer, with random fluctuations from thermal agitation or molecular collisions driving stochasticity, Han (2019). Ecological and population models incorporate rapid birth-death events with slower environmental changes, all under uncertain environmental noise. Likewise, in engineering control systems, noise sources alter system trajectories in ways that can cause or mask instabilities, especially when fast feedback loops coexist with slow reference dynamics. Financial mathematics employs stochastic models for asset price evolution, often involving volatile and rapidly adjusting variables coupled with longer-term trends, creating stiff stochastic frameworks for option pricing and risk management. The challenge these stiff SODEs pose is multifaceted. Theoretically, the existence and uniqueness of solutions may require special constructions or restrictive assumptions. Practically, standard numerical methods can fail to converge or demand prohibitive computational effort to maintain stability. This complexity has made the study and numerical solution methods for stiff SODEs a major research area, linked inherently to modeling fidelity and efficient simulation for decision-making in critical systems.

1.3.1 Finance

Stock prices fluctuate randomly due to unpredictable market forces such as news, economic data, or investor sentiment. The Black–Scholes model describes how stock prices evolve over time as a geometric Brownian motion, Black and Scholes (1973).

The Black-Schole's model, given as:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.3.1}$$

where S_t is the asset price at time t, μ is the drift (rate of return), σ is the volatility (standard deviation of the stock's return), W_t is a Wiener process (Brownian motion).

The analytical solution of (1.3.1) is given by:

$$X_t = X_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right). \tag{1.3.2}$$

This model is used in option pricing, portfolio optimization, and is assessment to exemplify financial uncertainty in asset prices and forms the basis for option pricing and risk management.

1.3.2 Physics

A microscopic particle suspended in a fluid experiences random collisions with surrounding molecules, causing it to move irregularly, Langevin Equation, Gardiner (2009) is given as:

$$m\, dv_t = -\gamma v_t\, dt + \sqrt{2D}\, dW_t, \tag{1.3.3}$$

where v_t velocity of the particle at time t, $m :$ mass of the particle, $\gamma :$ friction (drag coefficient), $D :$ diffusion coefficient, $W_t :$ standard Wiener process (Brownian motion). $-\gamma v_t \, dt$ represents the frictional force slowing the particle. $\sqrt{2D} \, dW_t$ models the random molecular impacts from collisions with surrounding fluid molecules. This model is used in statistical physics, molecular dynamics, and nanoparticle tracking to check the behaviour in microscopic physical systems, bridging deterministic mechanics with random thermal effects.

1.3.3 Neuroscience

The stochastic Hodgkin-Huxley model describes the evolution of a neuron's membrane potential under the influence of ionic currents and random fluctuations. The stochastic component accounts for biological noise, such as variability in ion channel opening or synaptic input, which makes neuronal behaviour inherently probabilistic, Øksendal (2003).

$$dV(t) = C\Big[- g_K(V - V_K) - g_{Na}(V - V_{Na}) - g_L(V - V_L) + I(t)\Big] dt + \sigma \, dW_t, \quad (1.3.4)$$

where $V(t)$ membrane potential at time t, C is the membrane capacitance, g_K, g_{Na}, g_L is the conductances of potassium, sodium, and leak channels, respectively, V_K, V_{Na}, V_L is the corresponding reversal potentials, $I(t)$ input current, σ: intensity of stochastic fluctuations, W_t is the standard Wiener process representing random noise. The deterministic term models the classical Hodgkin-Huxley dynamics, where ionic currents drive the evolution of the membrane potential. The stochastic term $\sigma \, dW_t$ introduces randomness, capturing fluctuations due to synaptic input variability or random ion channel behaviour. The stochastic Hodgkin-Huxley model is widely used in computational neuroscience to simulate neuronal activity under realistic noisy conditions, including synaptic inputs and channel noise.

1.3.4 Engineering

(Kushner and Dupuis (2001)) Control systems in engineering often face random disturbances (e.g., sensor noise, wind gusts). Consider a stochastic linear control system :

$$dx_t = (Ax_t + Bu_t)dt + \sigma dW_t, \tag{1.3.5}$$

where x_t represents the system state, A and B are matrices defining system dynamics and control inputs, u_t is the control input, σ is the intensity of the noise, W_t is a Wiener process. This SODE describes the evolution of a system subject to both control inputs and random disturbances (e.g., sensor noise or mechanical vibrations). It is used in robotics, signal processing, aerospace control, and fault-tolerant systems to show the uncertainty in engineering systems and the need for robust control strategies.

1.3.5 Biology

(Allen (2007)) In nature, population growth is affected by random environmental changes (e.g., temperature, food supply, disease), Allen (2003). The stochastic logistic growth SODE for population P_t, is:

$$dP_t = rP_t \left(1 - \frac{P_t}{K}\right) dt + \sigma P_t dW_t, \tag{1.3.6}$$

where r is the growth rate, K is the carrying capacity of the environment, σ is the intensity of the environmental noise, W_t is a Wiener process. This model accounts for environmental fluctuations affecting population growth. It is used in ecology, epidemiology, and conservation biology to illustrate ecological variability and stochastic effects on population dynamics.

1.3.6 Environmental Science

Daily temperature variations follow deterministic seasonal trends with random fluctuations (weather noise). A stochastic climate model for temperature fluctuations T_t is described by the following SODE, Øksendal (2003):

$$dT_t = \alpha(T_{eq} - T_t)dt + \sigma dW_t, \tag{1.3.7}$$

where α is a rate of mean reversion, T_{eq} is the equilibrium temperature, σ is the noise strength, W_t is a Wiener process. The model is used in climate dynamics and energy demand forecasting to clearly show the stochastic environmental dynamics, particularly mean-reverting processes with noise.

1.3.7 Medicine

When a drug is administered, its concentration in the bloodstream fluctuates due to random physiological differences among patients, Higham (2001).

$$dC_t = -kC_t\, dt + \sigma C_t\, dW_t, \tag{1.3.8}$$

where C_t is the drug concentration in blood at time t, k is the rate constant of elimination, σ is the variability coefficient, W_t is the standard Wiener process (Brownian motion). The deterministic term $-kC_t\, dt$ models the drug elimination over time. The stochastic term $\sigma C_t\, dW_t$ represents random variability in metabolism or absorption between patients. This SODE is used in *clinical pharmacology* to predict drug concentration ranges and optimize dosing schedules.

1.4 Basic Concepts of Random Variables and Stochastic Processes

Stochastic calculus is concerned with the study of stochastic processes, which involve randomness or noise. A thorough understanding of random variables and probability theory is therefore required. In this section, we recall some key definitions and concepts that will be relevant in chapter 3 are presented.

1.4.1 Basic Definitions

Definition 1.3. *(Burrage (1999))*

*A σ-**algebra** $\mathcal{F}$ is a collection of subsets of a sample space Ω (the set of all possible outcomes of a random experiment) such that:*

(1) $\Omega \in \mathcal{F}$.

(2) If $A \in \mathcal{F}$*, then its complement* $A^c = \{\omega \in \Omega \mid \omega \notin A\} \in \mathcal{F}$.

(3) For any sequence $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$*, we have* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ *and* $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

*The elements of $\mathcal{F}$ are called **measurable sets**, and the pair $(\Omega, \mathcal{F})$ is a measurable space.*

Definition 1.4. *probability measure (Burrage (1999). Øksendal (2003))*

*A **probability measure** P on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that:*

(1) $P(\Omega) = 1$,

(2) $P(A) \geq 0$*, for all* $A \in \mathcal{F}$,

(3) For mutually disjoint sets $A_1, A_2, \cdots \in \mathcal{F}$,

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} P(A_n). \tag{1.4.1}$$

Definition 1.5. *probability space (Burrage (1999))*

A ***probability space*** *is a triple* $(\Omega, \mathcal{F}, P)$ *consisting of a sample space* Ω*, a* σ*-algebra* $\mathcal{F}$*, and a probability measure* P*.*

Definition 1.6. *(Øksendal (2003))*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A* ***random variable*** *is a measurable function*

$$X : \Omega \to \mathbb{R}$$

such that, for every real number $x \in \mathbb{R}$*,*

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}.$$

That is, the set of all outcomes ω *for which* $X(\omega) \leq x$ *is an event in* $\mathcal{F}$*. The* ***expected value*** *of* X *is*

$$\mu = \mathbb{E}[X] = \int_{\Omega} X \, dP, \tag{1.4.2}$$

provided the integral exists.

For a continuous random variable $x \in X$*, with density function* $f(x)$*,*

$$\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) \, dx. \tag{1.4.3}$$

Definition 1.7. *(Øksendal (2003)))*

The ***variance*** *of a random variable* X *is*

$$\sigma^2 = \text{Var}(X) = \mathbb{E}\big[(X - \mu)^2\big] = \mathbb{E}[X^2] - \mu^2, \tag{1.4.4}$$

where $\mu = \mathbb{E}[X]$*. The standard deviation is* $\sigma = \sqrt{\text{Var}(X)}$*.*

Definition 1.8. *(Burrage (1999))*

A random variable X is said to have a ***Gaussian distribution*** $X \sim N(\mu, \sigma^2)$ *if its probability density is*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big), \quad -\infty < x < \infty, \tag{1.4.5}$$

For $\mu = 0$, $\sigma^2 = 1$ is called the ***standard normal distribution****.*

1.4.2 Modes of Convergence of Random Variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{X_n\}_{n\geq 1}$ be a sequence of random variables defined on it. The sequence $\{X_n\}, n = 1, 2, \cdots$ is said to converge to a random variable X in different *modes of convergence* depending on how the probability or expectation behaves as $n \to \infty$.

Definition 1.9. *Almost sure convergence (Burrage (1999))*

The sequence of random variable $X_n, n = 1, 2, \cdots$ is said to converge to a random variable X. That is, $X_n \to X$ as $n \to \infty$ is almost sure convergent if

$$P\Big(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\Big) = 1. \tag{1.4.6}$$

Definition 1.10. *Convergence in mean square (Burrage (1999))*

The sequence of random variable $X_n, n = 1, 2, \cdots$ is said to converge to a random variable X. That is, $X_n \to X$ as $n \to \infty$ is mean square convergent if

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^2) = 0. \tag{1.4.7}$$

Definition 1.11. *Convergence in probability (Burrage (1999))*

The sequence of random variable $X_n, n = 1, 2, \cdots$ is said to converge to a random variable X. That is, $X_n \to X$ as $n \to \infty$ is convergent in probability if, for every $\varepsilon > 0$, the limit

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0 \tag{1.4.8}$$

holds.

Definition 1.12. *Convergence in Distribution (Burrage (1999))*

*The sequence of random variable $X_n, n = 1, 2, \cdots$ is said to converge **in distribution** to X, written as $X_n \overset{n}{\to} X$, if*

$$F_{X_n}(x) \longrightarrow F_X(x)$$

for all x where the cumulative distribution function $F_X(x)$ is continuous.

1.4.3 Basic Concepts of Stochastic Processes

A **stochastic process** is a family of random variables $\{X_t\}_{t\in I}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with index set $I \subseteq \mathbb{R}$. For each fixed t, X_t is a random variable; for each fixed $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is a sample path.

Definition 1.13. *Wiener process, Øksendal (2003)*

*A **Wiener process** (or Brownian motion) $\{W_t\}_{t\geq 0}$ is a stochastic process with the following properties:*

(1) $W_0 = 0$ almost surely,

(2) It has independent increments,

(3) $W_{t+h} - W_t \sim \mathcal{N}(0, h)$ for $h > 0$,

(4) Paths are continuous but almost surely nowhere differentiable.

Remark 1.2. *Burrage (1999)*

A standard Wiener process satisfies $\mathbb{E}[W_t] = 0$ *and* $\text{Var}(W_t - W_s) = t - s$ *for* $0 \leq s \leq t$.

Definition 1.14. *Burrage (1999)*

The ***white noise process*** ξ_t *is formally defined as the generalized derivative of the Wiener process:*

$$\xi_t dt = dW_t. \tag{1.4.9}$$

1.4.4 Existence and Uniqueness of Solution of SODEs

The Stochastic Ordinary Differential Equations (1.1.1), defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions of right-continuity and completeness, Kloeden and Platen (1992).

The coefficient functions f and g are assumed to satisfy the following conditions:

(1) Measurability: $f = f(t,x)$ and $g = g(t,x)$ are jointly L^2-measurable in $(t,x)\epsilon[t_0,T) \times R$.

(2) Lipschitz condition: $\exists$ a constant $k > 0$ such that, $\forall x, y\epsilon R$,

$$\|f(t,x) - f(t,y)\| \leq k \mid x - y \mid, \quad \text{and} \quad \|g(t,x) - g(t,y)\| \leq k \mid x - y \mid, t\epsilon[t_0,T] \tag{1.4.10}$$

(3) Linear Growth Bound: $\exists$ a constant $k > 0$, such that, $\forall x, y\epsilon R$,

$$\|f(t,x)\|^2 \leq k^2(1+ \mid x \mid^2), \quad \text{and} \quad \|g(t,x)\|^2 \leq k^2(1+ \mid x \mid^2),. \tag{1.4.11}$$

(4) $X_{t_0} = X_0$ is A_{t_0}-measurable with $E(\mid X_{t_0} \mid^2) < \infty$ where $A^+ = A_{t_0} : t \geq 0$ is the family of σ-algebra associated with the Wiener process and $X(t)$ is A_t-measurable. And it is said to have a unique solution if for a specified initial value $X(t_0)$.

Under the assumptions (1) -(4), there exists a unique, continuous, adapted process $\{X_t\}_{t\in[t_0,T]}$ that satisfies the SODE (1.1.1) almost surely, Kloeden and Platen (1992) and Mao (1997).

That is, the SODE admits a **unique strong solution**, meaning that for a given initial condition X_0, there exists exactly one adapted process X_t satisfying (1.1.2).

1.4.5 Stochastic Integrals

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{W_t, t \geq 0\}$ be a standard Wiener process (or Brownian motion). For a stochastic process $\{X_t, t \geq 0\}$ that is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies the condition

$$E\left[\int_0^T X_t^2\,dt\right] < \infty, \tag{1.4.12}$$

the **Itô stochastic integral** of X_t with respect to W_t over the interval $[0,T]$ is defined, Robler (2003) as:

$$\int_0^T X_t\,dW_t = \lim_{n\to\infty}\sum_{i=0}^{n-1} X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right), \tag{1.4.13}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of the interval $[0,T]$.

The Itô integral extends the concept of Riemann–Stieltjes integrals to stochastic processes where the integrator W_t has non-differentiable sample paths.

Definition 1.15. *Stratonovich integral (Robler (2003))*

*The **Stratonovich integral** is defined over the interval $[0,t]$ as;*

$$\int_0^t X_s \circ dW_s = \lim_{n\to\infty}\sum_{i=0}^{n-1} X_{\frac{t_i+t_{i+1}}{2}}(W_{t_{i+1}} - W_{t_i}). \tag{1.4.14}$$

Remark 1.3. *The relation between Itô and Stratonovich integrals can be expressed as*

$$\int_0^t X_s \circ dW_s = \int_0^t X_s\,dW_s + \frac{1}{2}\int_0^t \frac{\partial X_s}{\partial W_t}\,ds. \tag{1.4.15}$$

Theorem 1.1. *Ito Isometry, Oksendal (2003)*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and let $\{W_t\}_{t\geq 0}$ denote an m-dimensional Wiener process. Suppose $\phi(t)$ is an $\mathbb{R}^{n\times m}$-valued, square-integrable, and progressively measurable process such that

$$\mathbb{E}\left[\int_0^T \|\phi(t)\|^2\,dt\right] < \infty. \tag{1.4.16}$$

Then, the stochastic integral

$$I_T = \int_0^T \phi(t)\,dW_t \tag{1.4.17}$$

is a well-defined $\mathbb{R}^m$-valued random variable satisfying the ***Itô Isometry****:*

$$\mathbb{E}\left[\left\|\int_0^T \phi(t)\,dW_t\right\|^2\right] = \mathbb{E}\left[\int_0^T \|\phi(t)\|^2\,dt\right]. \tag{1.4.18}$$

Proof

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0, T]$ and let

$$\phi(t,\omega) = \sum_{k=0}^{n-1} \xi_k(\omega)\,\mathbf{I}_{(t_k,t_{k+1}]}(t), \tag{1.4.19}$$

where each ξ_k is $\mathcal{F}_{t_k}$-measurable and square-integrable. **I** is an Ito integral. The stochastic integral with respect to an m-dimensional Wiener process W_t is

$$\int_0^T \phi(t)\,dW_t = \sum_{k=0}^{n-1} \xi_k(W_{t_{k+1}} - W_{t_k}). \tag{1.4.20}$$

Compute the second moment:

$$\begin{aligned}
\mathbb{E}\left[\left\|\int_0^T \phi(t)\,dW_t\right\|^2\right] &= \mathbb{E}\left[\left\|\sum_{k=0}^{n-1}\xi_k(W_{t_{k+1}} - W_{t_k})\right\|^2\right] \\
&= \sum_{k=0}^{n-1}\mathbb{E}[\|\xi_k\|^2\,\|W_{t_{k+1}} - W_{t_k}\|^2] \\
&\quad + 2\sum_{0\le i<j\le n-1}\mathbb{E}[\langle \xi_i(W_{t_{i+1}} - W_{t_i}), \xi_j(W_{t_{j+1}} - W_{t_j})\rangle].
\end{aligned} \tag{1.4.21}$$

By independence of non-overlapping Wiener increments and the fact that ξ_i is $\mathcal{F}_{t_i}$-measurable, the cross terms vanish:

$$\mathbb{E}[\langle \xi_i(W_{t_{i+1}} - W_{t_i}),\, \xi_j(W_{t_{j+1}} - W_{t_j})\rangle] = 0 \quad \text{for } i < j. \tag{1.4.22}$$

Moreover, for each k,

$$\mathbb{E}[\|\xi_k\|^2\,\|W_{t_{k+1}} - W_{t_k}\|^2] = \mathbb{E}[\|\xi_k\|^2]\,\mathbb{E}[\|W_{t_{k+1}} - W_{t_k}\|^2] = \mathbb{E}[\|\xi_k\|^2]\,(t_{k+1} - t_k), \tag{1.4.23}$$

when $\|\cdot\|$ denotes the Euclidean norm and W is m-dimensional . Thus

$$\mathbb{E}\left[\left\|\int_0^T \phi(t)\,dW_t\right\|^2\right] = \sum_{k=0}^{n-1}\mathbb{E}[\|\xi_k\|^2]\,(t_{k+1} - t_k) = \mathbb{E}\left[\int_0^T \|\phi(t)\|^2\,dt\right]. \tag{1.4.24}$$

This completes the proof of the Itô isometry.

Lemma 1.1. *Ito Lemma ((Oksendal (2003))*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space with a filtration* $\{\mathcal{F}_t\}_{t\ge 0}$ *satisfying the usual conditions, and let* $\{W_t\}_{t\ge 0}$ *be a one-dimensional Wiener process. Let* X_t *be an Itô process of the form*

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \tag{1.4.25}$$

where the coefficients μ and σ are such that a unique strong solution X_t exists, and let $f \in C^{1,2}([0,T] \times \mathbb{R})$ (once continuously differentiable in t and twice in x). Then the process $Y_t = f(t, X_t)$ is also an Itô process and satisfies

$$\begin{aligned} df(t,X_t) &= \frac{\partial f}{\partial t}(t,X_t)\,dt + \frac{\partial f}{\partial x}(t,X_t)\,dX_t + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\,(dX_t)^2 \\ &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\right)(t,X_t)\,dt + \frac{\partial f}{\partial x}(t,X_t)\sigma(t,X_t)\,dW_t, \end{aligned} \tag{1.4.26}$$

where $(dX_t)^2$ is interpreted in the Itô calculus sense (so that $(dW_t)^2 = dt$, $dt \cdot dW_t = 0$ and $dt^2 = 0$).

Proof

Let $0 = t_0 < t_1 < \cdots < t_n = T$, be a partition of $[0,T]$ with mesh $\|\Pi\| = \max_k(t_{k+1} - t_k) \to 0$. Set $\Delta t_k = t_{k+1} - t_k$, $\Delta X_k = X_{t_{k+1}} - X_{t_k}$, and write

$$\Delta f_k = f(t_{k+1}, X_{t_{k+1}}) - f(t_k, X_{t_k}). \tag{1.4.27}$$

Apply the two-variable Taylor expansion with remainder to f around (t_k, X_{t_k}):

$$\begin{aligned} \Delta f_k = &\frac{\partial f}{\partial t}(t_k, X_{t_k})\,\Delta t_k + \frac{\partial f}{\partial x}(t_k, X_{t_k})\,\Delta X_k \\ &+ \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}(t_k, X_{t_k})\,(\Delta X_k)^2 + R_k, \end{aligned} \tag{1.4.28}$$

where the remainder R_k contains higher-order terms:

$$R_k = o(\Delta t_k) + o\big((\Delta X_k)^2\big) \tag{1.4.29}$$

in probability as $\|\Pi\| \to 0$, given the regularity of f and integrability properties of X. Summing over k,

$$f(T,X_T)-f(0,X_0)=\sum_{k=0}^{n-1}\frac{\partial f}{\partial t}(t_k,X_{t_k})\Delta t_k+\sum_{k=0}^{n-1}\frac{\partial f}{\partial x}(t_k,X_{t_k})\Delta X_k+\tfrac{1}{2}\sum_{k=0}^{n-1}\frac{\partial^2 f}{\partial x^2}(t_k,X_{t_k})(\Delta X_k)^2+\sum_{k=0}^{n-1}R_k. \tag{1.4.30}$$

Analyzing each term as the mesh goes to zero. The Riemann sum converges in probability (and in L^1 under integrability assumptions) to the time integral:

$$\sum_k \frac{\partial f}{\partial t}(t_k,X_{t_k})\Delta t_k \longrightarrow \int_0^T \frac{\partial f}{\partial t}(s,X_s)\,ds.$$

Using the definition of the Ito integral (1.4.14) as the limit of Riemann sums with left-endpoint evaluations (or progressively measurable approximations), gives:

$$\sum_k \frac{\partial f}{\partial x}(t_k,X_{t_k})\Delta X_k=\sum_k \frac{\partial f}{\partial x}(t_k,X_{t_k})\Big(\mu(t_k,X_{t_k})\Delta t_k+\sigma(t_k,X_{t_k})\Delta W_k\Big) \tag{1.4.31}$$

which splits into a Riemann sum converging to $\int_0^T \frac{\partial f}{\partial x}(s,X_s)\mu(s,X_s)\,ds$ and the stochastic Riemann sum converging in probability (and in L^2) to the Itô integral $\int_0^T \frac{\partial f}{\partial x}(s,X_s)\sigma(s,X_s)\,dW_s$.

Compute $(\Delta X_k)^2$. From (1.4.25),

$$\Delta X_k=\mu(t_k,X_{t_k})\Delta t_k+\sigma(t_k,X_{t_k})\Delta W_k+o(\Delta t_k), \tag{1.4.32}$$

so

$$(\Delta X_k)^2=\sigma^2(t_k,X_{t_k})(\Delta W_k)^2+\text{higher-order terms}. \tag{1.4.33}$$

Using the fundamental property of Wiener increments $(\Delta W_k)^2\approx\Delta t_k$ (more precisely, $(\Delta W_k)^2-$

$\Delta t_k \to 0$ in probability and sums of cross-terms vanish), yields:

$$\sum_k \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t_k, X_{t_k})(\Delta X_k)^2 \longrightarrow \frac{1}{2}\int_0^T \frac{\partial^2 f}{\partial x^2}(s, X_s)\sigma^2(s, X_s)\, ds. \tag{1.4.34}$$

Using regularity of f and the estimates for X that $\sum_k R_k \to 0$ in probability (or in L^1) as $\|\Pi\| \to 0$.

Collecting the limits yields the Ito formula (1.4.27). This completes the proof.

1.5 Stochastic Itô-Taylor Series Expansion

The Taylor expansion is a classical tool for deterministic functions, especially in numerical analysis. This concept extends naturally into the stochastic setting through the use of Itô's formula. Following the work of Platen and Wagner (1982), the *stochastic Taylor expansion*, is obtain which generalizes the deterministic Taylor series.

In deterministic numerical analysis, methods are often derived by matching the Taylor expansion of the method with that of the exact solution of the differential equation. The same principle applies in the stochastic case, where stochastic Taylor expansion is employed. The Itô-Taylor expansion was first introduced by Platen and Wagner (1982) and Kloeden and Platen (1995). It provides a systematic expansion of X_t (or functions of X_t) around the initial condition X_{t_0} in terms of multiple stochastic integrals and function evaluations.

The autonomous SODE (1.1.3) written in integral form as:

$$X_t = X_{t_0} + \int_{t_0}^t f(X_s)\, ds + \int_{t_0}^t g(X_s)\, dW_s. \tag{1.5.1}$$

Applying the stochastic chain rule (Itô formula) to a function $F(X_t)$ and $G(X_t)$ yields:

$$F(X_t) - F(X_{t_0}) = \int_{t_0}^t L_0 F(X_s)\, ds + \int_{t_0}^t L_1 G(X_s)\, dW_s, \tag{1.5.2}$$

where the operators L_0 and L_1 in the scalar case are defined as

$$L_0F(X) = \frac{dF}{dX}f(X) + \frac{1}{2}\frac{d^2F}{dX^2}g^2(X), \quad L_1G(X) = \frac{dG}{dX}g(X).$$

Substituting $F = f$ and $G = g$ in (1.5.2) and applying Itô's formula repeatedly gives the stochastic Taylor expansion of X_t:

$$\begin{aligned} X_t = X_{t_0} &+ \int_{t_0}^{t}\left[f(X_s) + \int_{t_0}^{s} L_0 f(X_u)\,du + \int_{t_0}^{s} L_1 f(X_u)\,dW_u\right] ds \\ &+ \int_{t_0}^{t}\left[g(X_s) + \int_{t_0}^{s} L_0 g(X_u)\,du + \int_{t_0}^{s} L_1 g(X_u)\,dW_u\right] dW_s. \end{aligned} \tag{1.5.3}$$

Remark 1.4. *The (1.5.1)-(1.5.3) is presented for one-dimensional autonomous SODEs. Extending the construction to non-autonomous equations. Let X_t be the solution of the SODE (1.1.2) and let $f \in C^{1,2}([0,T] \times \mathbb{R}^n, \mathbb{R})$. Applying Itô's formula to $X_t = f(t, X_t)$, yields:*

$$X_t = X_{t_0} + \int_{t_0}^{t} L_0 f(s, X_s)\,ds + \int_{t_0}^{t} L_1 g(s, X_s)\,dW_s,$$

where the operators are defined as:

$$L_0 = \frac{\partial}{\partial t} + f \cdot \nabla_x + \tfrac{1}{2}g^2\frac{\partial^2}{\partial x^2}, \qquad L_1 = g\frac{\partial}{\partial x}.$$

and

$$\nabla_x = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)^{\top}.$$

By applying these operators recursively to f and g, one obtains higher-order terms and thus the

Itô-Taylor expansion.

1.5.1 Multiple Stochastic Integrals and Multi-indices

To express the stochastic Taylor expansion in a concise way, one introduces the concept of multiple Itô integrals. For an m-dimensional Wiener process $W = (W^1, \dots, W^m)$ and a multi-index $\alpha = (j_1, j_2, \dots, j_L)$ with $j_i \in \{0, 1, \dots, m\}$, define

$$I_{\alpha,t} = \int_0^t \int_0^{s_L} \cdots \int_0^{s_2} dW_{s_1}^{j_1} \cdots dW_{s_L}^{j_L}, \tag{1.5.4}$$

where $dW_s^0 := ds$.

The set of all such multi-indices is denoted

$$M = \{\alpha = (j_1, j_2, \dots, j_L) : L \in \mathbb{N}, j_i \in \{0, 1, \dots, m\}\} \cup \{\nu\}, \tag{1.5.5}$$

with ν being the empty multi-index of length 0. For a multi-index α, let $L(\alpha)$ denote its length, and $n(\alpha)$ the number of zero components.

A set $H \subseteq M$ is called a *hierarchical set* if it is nonempty, bounded in length, and closed under removing the first element of α. The remainder set associated with H is

$$R(H) = \{\alpha \in M \setminus H : -\alpha \in H\}.$$

Theorem 1.2. *Itô-Taylor Expansion, Robler (2003)*

Let $H \subseteq M$ be a hierarchical set and let ρ, τ be stopping times such that $t_0 \le \rho(\omega) \le \tau(\omega) \le T < \infty$, $\mathbb{P}$-a.s. For the solution $(X_t)_{t\in J}$ of the Itô SODE (1.1.2), one has

$$f(\tau, X_\tau) = \sum_{\alpha \in H} I_{\alpha,\rho,\tau}[f_\alpha(\rho, X_\rho)] + \sum_{\alpha \in R(H)} I_{\alpha,\rho,\tau}[f_\alpha(\cdot, X_\cdot)], \tag{1.5.6}$$

provided that all derivatives of f and g, and all required multiple Itô integrals exist.

1.6 Stability of Stochastic Ordinary Differential Equations

Let X_t be the solution to an SODE (1.1.1). The **zero solution** (or **trivial solution**) of the SODE is the process

$$X_t \equiv 0, \quad \text{for all } t \geq 0,$$

which satisfies the SODE whenever

$$f(t,0) = 0 \quad \text{and} \quad g(t,0) = 0, \quad \text{for all } t \geq 0.$$

Definition 1.16 (Stability in Probability)**.** *, Schurz (1999)*

The zero solution X_0 is stable in probability*, if for every $\varepsilon > 0$ and $\delta > 0$, there exists $r = r(\varepsilon, \delta) > 0$ such that*

$$|x_0| < r \quad \textit{implies}\, \text{Pr}\{|X_t| > \varepsilon\} < \delta, \quad \forall t \geq t_0.$$

Definition 1.17 (Asymptotic Stability in Probability)**.** *, Schurz (1999)*

The zero solution X_0 is asymptotically stable in probability*, if it is stable in probability and additionally*

$$\lim_{t\to\infty} \text{Pr}\{|X_t| > \varepsilon\} = 0 \tag{1.6.1}$$

for any $\varepsilon > 0$, whenever $|x_0| < r(\varepsilon)$.

Definition 1.18 (Mean-Square Stability of the Exact Solution $X(t)$)**.** *The equilibrium solution $X_t = 0$ (or more generally, an equilibrium point $X_t = X^*$) of the SODE is said to be* mean-square stable *if, for every* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$\mathbb{E}[|X_t|^2] < \varepsilon, \quad \forall t \geq 0, \tag{1.6.2}$$

whenever $\mathbb{E}[|X_0|^2] < \delta$*. If, in addition,*

$$\lim_{t \to \infty} \mathbb{E}[|X_t|^2] = 0, \tag{1.6.3}$$

then the equilibrium $X_t = 0$ *is said to be* asymptotically mean-square stable.

Definition 1.19 (Mean-Square Stability of a Numerical Method X_n)**.** *Let* X_n *be the numerical approximation to the solution* $X(t)$ *obtained by a time-stepping method with step size* $h > 0$*. The method is said to be* mean-square stable*, if for the linear test equation (1.2.1), the numerical solution satisfies*

$$\lim_{n \to \infty} \mathbb{E}[|X_n|^2] = 0 \tag{1.6.4}$$

whenever the exact solution $X(t)$ *of the SODE satisfies (1.6.5). That is, the numerical scheme preserves the mean-square stability property of the continuous SODE.*

Remark 1.5.

(a) The mean-square stability condition can be analyzed using the linear test equation (1.2.1), whose exact solution satisfies

$$\mathbb{E}[|X_t|^2] = |X_0|^2 \exp\left[(2\Re(\lambda) + |\mu|^2)t\right]. \tag{1.6.5}$$

Hence, the zero solution is mean-square stable if and only if

$$2\Re(\lambda) + |\mu|^2 < 0. \tag{1.6.6}$$

(b) For a numerical method, mean-square stability is characterized by the stability function $S(\lambda h, \mu\sqrt{h})$ *such that*

$$\mathbb{E}[|X_{n+1}|^2] = |S(\lambda h, \mu\sqrt{h})|^2 \mathbb{E}[|X_n|^2]. \tag{1.6.7}$$

The method is mean-square stable if

$$|S(\lambda h, \mu\sqrt{h})| < 1. \tag{1.6.8}$$

1.7 Numerical Methods for SODEs

Methods for approximating (1.1.1) are few, and the uncontrolled fluctuation occasion by the Brownian motion that contribute to insolubility of solution of the system (1.1.1) via the analytical approach, hence there's need for numerical methods.

Numerical methods thus form the backbone of both theoretical investigations and practical applications involving stiff SODEs. Without dependable numerical solvers, modelers cannot simulate trajectories accurately, analyze parameter sensitivities, or optimize controls. Furthermore, numerical approximations enable the estimation of statistical properties, such as expectation, variance, or hitting times, which are crucial for interpreting stochastic models.

The complexity of the system (1.1.1) can be heightened with the phenomenon called stiffness, that happens when the transcient solution component decay at different time scale.

Moreover, the increasing prominence of large-scale stochastic simulations in finance, biology, and engineering necessitates efficient integration schemes that balance accuracy, stability, and computational cost. Hybrid and adaptive techniques extend the toolset but also amplify implementation complexity. Thus, tailored numerical methods designed explicitly for stiff stochastic problems are indispensable in environments where multidisciplinary demands and scale converge, Shampine and Reichelt(1979).

Numerical integration of stochastic differential equations has traditionally followed two principal paradigms: one-step and multi-step methods. Both have been extensively studied and generalized from deterministic ODEs to SODEs, with varied success in overcoming the challenges posed by stiffness and stochasticity.

1.7.1 One-Step method for SODEs

One-step methods calculate the solution at the next time step using information exclusively from the current or immediately preceding step Kloeden and Platen (1992). A discrete-time process $\{X_n\}_{n\geq 0}$ is called a *one-step numerical method* for the stochastic differential equation (1.1.1), if there exists a measurable mapping

$$\Phi : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^d$$

such that

$$X_{n+1} = \Phi(X_n, h, \Delta W_n), \quad X_0 = x_0, \tag{1.7.1}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$ represents the Wiener process increment over the interval $[t_n, t_{n+1}]$.

The (1.7.1) is said to be *consistent* if

$$\mathbb{E}\,|X_{n+1} - X(t_{n+1})| \to 0 \quad \text{as } h \to 0, \tag{1.7.2}$$

and *strongly convergent of order* $\gamma > 0$ if

$$\left(\mathbb{E}\,|X(t_n) - X_n|^2\right)^{1/2} = \mathcal{O}(h^\gamma). \tag{1.7.3}$$

Definition 1.20. *(Strong Convergence)*

A numerical approximation X_n *to the solution* $X(t_n)$ *of SODE (1.1.1) is said to converge strongly with order* $\gamma > 0$ *if, Kloeden and Platen (1992)*

$$\left(\mathbb{E}[\|X(t_n) - X_n\|^2]\right)^{1/2} \leq Ch^{\gamma}, \tag{1.7.4}$$

for step size $h \to 0$*, where* $C > 0$ *is independent of* h*.*

Definition 1.21. *(Weak Convergence)*

A numerical method to SODE (1.1.1) is said to converge weakly with order β *if for any sufficiently smooth test function* ϕ*, Kloeden and Platen (1992)*

$$\left|\mathbb{E}[\phi(X(t_n))] - \mathbb{E}[\phi(X_n)]\right| \leq Ch^{\beta}. \tag{1.7.5}$$

1.7.1.1 Euler–Maruyama Method

The Euler-Maruyama method which is an extension of the popular classical Euler method is the simplest and most widely used one-step scheme:

$$X_{n+1} = X_n + f(X_n)h + g(X_n)\Delta W_n, \tag{1.7.6}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$ and h is the step-size. It converges strongly with order 0.5 under global Lipschitz conditions (Kloeden and Platen (1992)).

1.7.1.2 Milstein Method

To improve accuracy of (1.7.6), the Milstein method includes a correction term involving the derivative of g:

$$X_{n+1} = X_n + f(X_n)h + g(X_n)\Delta W_n + \tfrac{1}{2}g(X_n)g'(X_n)\left((\Delta W_n)^2 - h\right). \tag{1.7.7}$$

This method achieves strong order 1.0 convergence (Milstein (1975).

1.7.1.3 Runge-Kutta-Type Methods

Runge-Kutta schemes extend deterministic Runge Kutta methods to SODEs. A general s-stage stochastic Runge-Kutta method can be written as

$$X_{n+1} = X_n + \sum_{i=1}^{s} \alpha_i f(X_i)h + \sum_{i=1}^{s} \beta_i g(X_i)\Delta W_n, \tag{1.7.8}$$

$$X_i = X_n + \sum_{j=1}^{s} a_{ij} f(X_j)h + \sum_{j=1}^{s} b_{ij} g(X_j)\Delta W_n, \quad i = 1, \dots, s \tag{1.7.9}$$

These methods can be tailored for weak or strong convergence (Burrage et al (2003), Debrabant et al (2022)).

1.7.2 Linear Multistep Methods (LMM)

Linear Multistep Method (LMM) is a class of Numerical method for integrating Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs) given as:

$$\rho(E)X_n = h\sigma(E)f_n, \tag{1.7.10}$$

where $\rho(E)$ and $\sigma(E)$ and h are the first, second characteristic polynomials and steplength respectively. $\rho(E) = \sum_{j=0}^{k} \alpha_j E^j$, $\sigma(E) = \sum_{j=0}^{k} \beta_j E^j$, and E is the shift operator and k the step number.

A subclass of (1.7.10) is LMM whose first characteristics polynomial is:

$$\rho(E) = \alpha_k E^k - \alpha_{k-1} E^{k-1}. \qquad (1.7.11)$$

The class of method whose $\rho(E)$ is in the form (1.7.11) is called Adams-Type method.

The polynomial $\rho(E)$ in (1.7.11) has all its roots in the unit disk and the root unit is the multiplicity one , hence Adams type method is zero stable apriori. The Second Derivative Adams Type method is of the form:

$$\rho(E) X_n = h\sigma(E) f_n + h^2 \theta(E) G_n, \qquad (1.7.12)$$

where $G = \frac{d}{dt} f(X(t))$, $g' = \frac{d}{dt} g(X(t))$, $\theta(E)$ is the third characteristic polynomial given as: $\gamma(E) = \sum_{j=0}^{k} \theta_j E^j$

1.7.3 Stochastic Linear Multistep Methods (SLMM)

A **stochastic linear multistep method (SLMM)** extends the concept of deterministic linear multistep methods (1.7.10) to the numerical approximation of **Stochastic Ordinary Differential Equations (SODEs)** of the form (1.1.1).

The general k-step stochastic linear multistep method is expressed as

$$\sum_{j=0}^{k} \alpha_j X_{n+j} = h \sum_{j=0}^{k} \beta_j f(t_{n+j}, X_{n+j}) + \sum_{j=1}^{k} \gamma_j \, g(t_{n+j-1}, X_{n+j-1}) \, \Delta W_{n+j-1}, \qquad (1.7.13)$$

where the coefficients α_j, β_j, and γ_j are real constants, and h is the time step size, $\alpha_k = 1$, without loss of generality. The initial values

$$X_0, X_1, \dots, X_{k-1} \in L^2(\Omega, \mathbb{R}^n)$$

are assumed to be $\mathcal{F}_{t_n}$-measurable random variables for $n = 0, 1, \dots, k-1$ Buckwar and Winkler (2007). The SLMM to the autonomous SODE (1.1.3) becomes

$$\sum_{j=0}^{k} \alpha_j X_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=1}^{k} \gamma_j \, g_{n+j-1} \, \Delta W_{n+j-1}, \tag{1.7.14}$$

This is the standard form of the stochastic linear multistep method used to approximate the solution of autonomous Itô SDEs, Buckwar and Winkler (2007).

1.7.4 Classification SLMM

The SLMM is classified into explicit and implicit depending on whether the new approximation X_n (or equivalently X_{n+1}) appears on the right-hand side of the method.

Explicit SLMM: The method is called *explicit* if the coefficients β_j and γ_j associated with the current step ($j = 0$) vanish, that is.

$$\beta_0 = 0, \quad \gamma_0 = 0.$$

In this case, the computation of X_{n+1} involves only previously known values $X_n, X_{n-1}, \dots, X_{n-k+1}$. An example is the *stochastic Adams–Bashforth method*:

$$X_{n+1} = X_n + hf(X_n) + g(X_n)\Delta W_n, \tag{1.7.15}$$

which corresponds to the explicit Euler–Maruyama scheme Kloeden and Platen (1992).

Implicit SLMM: The method is called *implicit* if β_k or γ_k is nonzero, that is.

$$\beta_k \neq 0 \quad \text{or} \quad \gamma_k \neq 0.$$

In this case, X_{n+1} appears on both sides of the formula and must be determined implicitly. A common example is the *stochastic Adams–Moulton method*:

$$X_{n+1} = X_n + hf(X_{n+1}) + g(X_{n+1})\Delta W_n, \tag{1.7.16}$$

which requires solving a nonlinear equation for X_{n+1} at each step (Higham, 2001).

The coefficients $\{\alpha_j\}_{j=0}^k$, $\{\beta_j\}_{j=0}^k$, and $\{\gamma_j\}_{j=0}^k$ determine the order and stability of the SLMM. They are chosen to ensure:

1. **Consistency** with the stochastic Taylor expansion of the exact solution;
2. **Mean-square stability** for stiff systems;
3. **Desired order of strong convergence** γ (typically $\gamma = 0.5$ or 1.0).

In the analysis of stochastic numerical schemes, particularly stochastic linear multistep methods (SLMMs), it is essential to examine the different sources of error that influence the accuracy and stability of the computed solution. The main sources of numerical error are the *local truncation error* and the *global error*. Each of these is defined and discussed below.

Definition 1.22. *: Local Error, (Kloeden and Platen (1992), Higham (2001), Burrage et al. (2003).)*

The ***local error*** *(or local truncation error) of an SLMM measures the deviation made during a single time step, assuming all previous values are exact. For the SODE (1.1.1), and a discrete* *numerical scheme of the general form:*

$$\Psi(X_n, X_{n-1}, \dots, X_{n-k}, h, \Delta W_n) = 0, \tag{1.7.17}$$

the local error at time t_{n+1} is defined as;

$$\tau_{n+1} = \mathbb{E}\left[\frac{1}{h}\left(X(t_{n+1}) - \Psi(X(t_n), X(t_{n-1}), \dots, X(t_{n-k}), h, \Delta W_n)\right)\right]. \tag{1.7.18}$$

This quantity represents how well the method reproduces the exact solution over a single step.

For a method of strong order p, the local mean-square error satisfies

$$\left(\mathbb{E}\left[|X(t_{n+1}) - X_{n+1}|^2\right]\right)^{1/2} = \mathcal{O}(h^{p+\frac{1}{2}}). \tag{1.7.19}$$

*An SLMM is said to be **consistent** if the local error tends to zero as $h \to 0$.*

As an example, the Euler–Maruyama method has a local mean-square error of order $\mathcal{O}(h^{3/2})$, corresponding to a global strong order of $\mathcal{O}(h)$.

Definition 1.23. *: Global Error, (Kloeden and Platen (1992), Mao (1997), Higham (2001).*

*The **global error** at time t_n quantifies the total accumulated deviation between the true solution $X(t_n)$ and the numerical approximation X_n:*

$$e_n = \left(\mathbb{E}\left[|X(t_n) - X_n|^2\right]\right)^{1/2}. \tag{1.7.20}$$

This error accounts for the combined effects of the local truncation errors and their propagation through successive steps of the numerical scheme.

If the SLMM is stable and the local error is of order $\mathcal{O}(h^{p+\frac{1}{2}})$, then the global mean-square *error satisfies*

$$\left(\mathbb{E}[|X(t_n) - X_n|^2]\right)^{1/2} = \mathcal{O}(h^p), \tag{1.7.21}$$

where p is the strong order of convergence.

Definition 1.24. *Robler (2003)*

A process $(X_t)_{t\in I}$ with values in $\mathbb{R}^d$ is called a ***strong solution*** *of the SODE (1.1.1) with respect to a fixed Wiener process $(W_t)_{t\in I}$ and initial value X_0, if the following conditions are satisfied:*

(a) X_t *is adapted to the filtration* $(\mathcal{F}_t)_{t\in I}$.

(b) The sample paths of X_t are continuous almost surely.

(c) For the multidimensional case, it holds that for all $i = 1, 2, \dots, d$, $j = 1, 2, \dots, m$, and $t \in I$,

$$\int_0^t \left(|f(s, X_s)|^2 + \|g(s, X_s)\|^2\right) ds < \infty, \quad \textit{almost surely.}$$

(d) With probability 1, X_t satisfies the stochastic integral equation (1.4.14)

Definition 1.25. *:Absolute Error (AE) (Sauer (2012))*

The ***absolute error*** *(AE) measures the direct difference between the numerical approximation and the exact value at a specific time.*

If $X(T)$ is the exact solution at final time T, and X_N is the numerical approximation,

$$\textit{Absolute Error} = |X(T) - X_N|.$$

Definition 1.26. *: Pathwise Error (PE) (Milstein and Tretyakov (2004))*

The ***pathwise error*** *(PE) refers to the difference between the numerical and exact solution along individual sample paths.*

For each realization of Brownian motion (each path), the error is:

$$\textit{Pathwise Error} = \sup_{0\le n\le N} |X(t_n,\omega) - X_n(\omega)|.$$

This emphasizes sample-wise convergence, rather than in expectation.

Definition 1.27. *: Global Mean Square Error (GMSE) (Kloeden and Platen (1992))*

*The **global mean square error** (GMSE) is the root-mean-square error accumulated over the interval* $[0,T]$. *Let* $X(t_n)$ *be the exact solution and* X_n *the numerical solution. Then,*

$$\textit{Global Mean Square Error} = \left(\mathbb{E}\left(|X(t_n) - X_n|^2\right)\right)^{1/2}.$$

If this converges like $\mathcal{O}(h^\gamma)$, *then* γ *is the strong order.*

Definition 1.28. *: Strong Error (SE) (Higham (2001))*

*The **strong error**(SE) measures how well the numerical method approximates the exact solution in a pathwise (sample-wise) sense, averaged over all paths.*

$$\textit{Strong Error} = \left(\mathbb{E}\left(\|X(T) - X_N\|^p\right)\right)^{1/p}, \quad \textit{commonly with } p = 2.$$

This quantifies how closely the numerical path follows the true stochastic path.

Definition 1.29. *: Mean Square Error (MSE)(Milstein and Tretyakov (2004))*

*The **mean square error** (MSE) (in the context of SODEs) is a specific case of strong error where* $p = 2$.

$$\textit{Mean Square Error} = \mathbb{E}\left(\|X(T) - X_N\|^2\right).$$

Its square root is the global mean square error. It measures expected squared distance between exact and numerical solutions.

Definition 1.30. *:Relative Mean Square Error (RMSE) (Higham (2001))*

The Relative Mean Square Error (RMSE) divides the MSE by the mean square norm of the exact solution, giving a scale-independent measure:

$$RMSE = \frac{\mathbb{E}(\,|X_n - X(t_n)|^2)}{\mathbb{E}(\,|X(t_n)|^2)}. \tag{1.7.22}$$

1.7.5 Numerical Stability of SLMM

In the numerical analysis of Stochastic Ordinary Differential Equations (SODEs), the concept of stability plays a fundamental role in determining whether a multistep method can produce a stable sequence of approximations when the step size tends to zero.

Definition 1.31 (Zero-Stability of a Numerical Method for SODEs). *Consider a k-step stochastic linear multistep method (SLMM) of the form*

$$\sum_{j=0}^{k} \alpha_j Y_{n-j} = h \sum_{j=0}^{k} \beta_j f(Y_{n-j}) + \sum_{j=0}^{k} \gamma_j g(Y_{n-j}) \Delta W_{n-j}, \tag{1.7.23}$$

where h is the step size, $\Delta W_{n-j} = W_{t_{n-j+1}} - W_{t_{n-j}}$, and Y_n is the numerical approximation to the true solution $X(t_n)$ of the SODE.

The method is said to be zero-stable *if small perturbations in the initial data or round-off errors do not cause the numerical solution to diverge as $n \to \infty$ when $h \to 0$.*

Definition 1.32 (Root Condition for Zero-Stability). *Let the coefficients α_j in* (1.7.23) *define the* first characteristic polynomial $\rho(\xi) = \alpha_0 \xi^k + \alpha_1 \xi^{k-1} + \cdots + \alpha_k$. *The stochastic linear multistep method is* zero-stable *if and only if all roots ξ_i of $\rho(\xi) = 0$ satisfy $|\xi_i| \leq 1,$ and any root with $|\xi_i| = 1$ is simple.*

This condition ensures that perturbations or random fluctuations do not grow uncontrollably in successive steps of the numerical solution.

Remark 1.6.

1. *Zero-stability ensures that the numerical method reacts mildly to small perturbations, such as local truncation errors and stochastic fluctuations.*

2. *It is a necessary condition for the convergence of any consistent stochastic multistep method.*

3. *For stochastic Runge–Kutta or one-step methods, zero-stability is automatically satisfied since each step depends only on the immediately preceding value.*

Definition 1.33 (Mean-Square A-Stability)**.** *A numerical method applied to the linear stochastic test equation (1.2.1) is said to be* ***mean-square A-stable*** *if the numerical approximation* $\{X_n\}_{n\geq 0}$ *satisfies*

$$\lim_{n\to\infty} \mathbb{E}[|X_n|^2] = 0, \qquad \textit{for all step sizes } h > 0, \tag{1.7.24}$$

whenever the exact solution also decays in the mean-square sense, i.e.,

$$2\Re(\lambda) + |\mu|^2 < 0. \tag{1.7.25}$$

Equivalently, if the amplification factor $R(h, \lambda, \mu)$ *of the scheme satisfies*

$$\mathbb{E}\big[\,|R(h,\lambda,\mu)|^2\,\big] < 1, \qquad \forall h > 0, \quad \textit{whenever } 2\Re(\lambda) + |\mu|^2 < 0. \tag{1.7.26}$$

Lemma 1.2 (Arnold (1974); Hasminskii (1980); Mao (1997); Buckwar and Sickenberger (2011))**.** *The equilibrium solution of (1.2.1) is globally asymptotically mean-square stable if and only if*

$$\Re(\lambda) + \tfrac{1}{2}\sum_{j=1}^{m} |\mu_j|^2 < 0. \tag{1.7.27}$$

1.8 Numerical Methods for Stiff SODEs

Stiff stochastic differential equations (SODEs) arise in various applications such as chemical kinetics, control theory, finance, and biological systems, where the system exhibits both slow and fast dynamics. The presence of stiffness often leads to numerical instability when using explicit methods unless an extremely small step size is adopted. Therefore, specialized numerical schemes are required to efficiently approximate the solution of stiff SODEs.

According to Schurz (1999) and Saito (1996), stiffness in SODEs is characterized by the rapid decay of some components of the solution or by widely separated eigenvalues in the drift term. The test equation (1.2.1) is said to be *stiff* if $\mathrm{Re}(\lambda) < \alpha$ for $\alpha 0$ (i.e., the drift term is strongly dissipative), such that the deterministic part decays rapidly, making explicit methods unstable for practical step sizes.

To overcome this, several implicit and semi-implicit schemes have been developed to approximate stiff SODEs while maintaining stability and computational efficiency.

1.8.1 Implicit Euler-Maruyama Method

The *implicit Euler–Maruyama method* is one of the most commonly used methods for stiff SODEs. It is given by

$$X_{n+1} = X_n + hf(X_{n+1}) + g(X_n)\Delta W_n, \tag{1.8.1}$$

where h is the step size and $\Delta W_n = W_{t_{n+1}} - W_{t_n}$.

Unlike the explicit Euler–Maruyama method, where $f(Y_n)$ is used, this implicit form evaluates the drift term at the unknown point X_{n+1}. This approach improves numerical stability for stiff systems.

For the linear test equation (1.2.1), the implicit Euler–Maruyama scheme satisfies the mean-square stability condition

$$\mathbb{E}\left[|X_{n+1}|^2\right] \leq \mathbb{E}\left[|X_n|^2\right], \tag{1.8.2}$$

for all step sizes $h > 0$ if $\mathrm{Re}(\lambda) < 0$. This property makes it suitable for stiff SODEs, Higham (2001), Schurz (1999), Saito (1996).

1.8.2 Implicit Milstein Method

The implicit Milstein method is an extension of the classical Milstein scheme, designed to enhance stability when approximating stiff SODEs. It modifies the drift term to be evaluated implicitly, thereby allowing larger time steps without numerical instability. The scheme is expressed as

$$X_{n+1} = X_n + hf(X_{n+1}) + g(X_n)\Delta W_n + \tfrac{1}{2}g(X_n)g'(X_n)\left((\Delta W_n)^2 - h\right), \tag{1.8.3}$$

where h denotes the step size and $\Delta W_n = W_{t_{n+1}} - W_{t_n}$.

The implicit Milstein method maintains a strong order of convergence equal to one in the mean-square sense. It also provides improved accuracy for systems with multiplicative noise compared to the implicit Euler–Maruyama scheme. However, the method requires solving nonlinear systems at each time step when the drift term is nonlinear, thereby increasing computational cost.

In summary, the implicit Milstein method offers a reliable and stable numerical approach for solving stiff SDEs, balancing stability and accuracy at the expense of higher computational effort.

1.8.3 Semi-Implicit (Split-Step) Methods

Semi-implicit or *split-step* methods combine explicit and implicit components to achieve both stability and computational efficiency. A general form is

$$X_{n+\frac{1}{2}} = X_n + hf(X_{n+\frac{1}{2}}), \tag{1.8.4}$$

$$X_{n+1} = X_{n+\frac{1}{2}} + g(X_{n+\frac{1}{2}})\Delta W_n. \tag{1.8.5}$$

The drift term is treated implicitly while the diffusion term remains explicit.

Split-step methods maintain strong stability for stiff drift terms while avoiding fully implicit nonlinear solves. They are widely applied in stiff SODEs with multiplicative noise, Mao and Szpruch (2013); Schurz (1999).

1.8.4 Backward Differentiation Formula (BDF) Methods

Backward Differentiation Formula (BDF) methods are multistep implicit schemes that are suitable for stiff deterministic and stochastic systems. A k-step stochastic BDF method has the form

$$\sum_{j=0}^{k} \alpha_j X_{n-j} = h\beta_0 f(X_n) + \sum_{j=0}^{k} \gamma_j g(X_{n-j})\Delta W_{n-j}. \tag{1.8.6}$$

These methods are particularly effective when the deterministic part dominates the stochastic term.

BDF methods can achieve A-stability or A-mean-square stability depending on the choice of coefficients α_j and β_j, Buckwar and Winkler (2007); Schurz (1999); Tang and Xiong (2018)..

1.8.5 Stochastic Theta Method

The stochastic θ-method is a one-parameter family of methods that bridges the gap between explicit and implicit schemes. It is defined as

$$X_{n+1} = X_n + h\left[(1-\theta)f(X_n) + \theta f(X_{n+1})\right] + g(X_n)\Delta W_n, \tag{1.8.7}$$

where $\theta \in [0, 1]$.

$\theta = 0$ gives the explicit Euler–Maruyama method. $\theta = 1$ gives the fully implicit Euler–Maruyama method. $\theta = \frac{1}{2}$ gives the semi-implicit Crank–Nicolson scheme.

The method is mean-square A-stable for $\theta \geq \frac{1}{2}$, making it appropriate for stiff problems, Higham (2001); Saito (1996); Mao (2015).

1.8.6 Stochastic Runge–Kutta Methods for Stiff SODEs

Implicit and diagonally implicit Runge-Kutta (DIRK) schemes have been extended to SODEs to handle stiffness efficiently. A general implicit stochastic Runge-Kutta (SRK) method is written as

$$X_{n+1} = X_n + h\sum_{i=1}^{s} \alpha_i f(X_{n,i}) + \sum_{i=1}^{s} \beta_i g(X_{n,i})\Delta W_{n,i} \tag{1.8.8}$$

$$X_{n,i} = X_n + h\sum_{j=1}^{s} a_{ij} f(X_{n,j}) + \sum_{j=1}^{s} b_{ij} g(X_{n,j})\Delta W_{n,j}. \tag{1.8.9}$$

These methods achieve higher order accuracy and strong mean-square stability for stiff SODEs when a_{ij} and b_{ij} are chosen to satisfy certain implicit stability conditions, Li et al. (2023).

1.8.7 Adams-Type Method

The numerical method is classified as an **Adams-type method** because it is derived from the fundamental idea of the Adams family of linear multistep methods. In Adams-type methods, the numerical approximation of the solution is obtained by integrating the differential equation over a time interval and approximating the integral using previously computed values of the drift function and, where applicable, its derivatives.

For a general ordinary differential equation (ODE)

$$dX(t) = f(t, X(t)), \quad X(t_0) = X_0, \tag{1.8.10}$$

the exact solution can be written in the integral form

$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} f(t, X(t))\, dt. \tag{1.8.11}$$

Adams-type methods approximate this integral by interpolating the function $f(t, X(t))$ using known past values $f_n, f_{n-1}, \dots, f_{n-k+1}$.

Depending on whether the interpolation involves only past points or includes the future point t_{n+1}, the method can be either:

- **Explicit Adams-type (Adams–Bashforth):**

$$X_{n+k} = X_{n+k-1} + h \sum_{j=0}^{k-1} \beta_j f_{n+j}. \tag{1.8.12}$$

- **Implicit Adams-type (Adams–Moulton):**

$$X_{n+k} = X_{n+k-1} + h \sum_{j=0}^{k} \beta_j f_{n+j}. \tag{1.8.13}$$

The **second derivative Adams-type method** extends this concept by incorporating derivative information of the drift term to enhance accuracy and stability, particularly for stiff problems. The general form of such a method is given by

$$X_{n+1} = X_n + h\sum_{j=0}^{k-1}\beta_j f_{n-j} + h^2\sum_{j=0}^{k-1}\theta_j f'_{n-j}, \tag{1.8.14}$$

where $f'(t, X) = f_t(t, X) + f_X(t, X)f(t, X)$ represents the total derivative of f with respect to time.

When extended to the stochastic case, yields SODE (1.1.1) the Adams-type stochastic scheme takes the form

$$X_{n+1} = X_n + h\sum_{j=0}^{k}\beta_j f_{n-j} + h^2\sum_{j=0}^{k}\theta_j f'_{n-j} + \sum_{j=0}^{k}\gamma_j g_{n-j}\Delta W_{n-j}. \tag{1.8.15}$$

This form maintains the Adams framework by integrating both drift and diffusion terms using previously computed information. The implicit form of (1.8.15) is,

$$X_{n+1} = X_n + h\sum_{j=0}^{k}\beta_j f_{n+1-j} + h^2\sum_{j=0}^{k}\theta_j f'_{n+1-j} + \sum_{j=0}^{k}\gamma_j g_{n+1-j}\Delta W_{n-j}. \tag{1.8.16}$$

Properties of Adams-Type Methods

Adams-type methods, including their second derivative and stochastic variants, possess the following properties:

(a) **Multistep nature:** They depend on multiple past approximations $(X_n, X_{n-1}, \dots, X_{n-k+1})$, rather than a single step.

(b) **Predictor–corrector formulation:** The explicit (Adams–Bashforth) and implicit (Adams–Moulton) variants can be combined to form predictor–corrector pairs for improved accuracy.

(c) **Higher-order accuracy:** Inclusion of derivative terms (e.g., second derivative or total derivative of *f*) increases the local truncation order.

(d) **Stability improvement:** Second derivative terms enhance the stability region, making the method more suitable for stiff systems.

(e) **Consistency and zero-stability:** Adams-type schemes satisfy the standard linear multistep method conditions for consistency and zero-stability.

(f) **Extension to stochastic systems:** By including the diffusion term $g(t, X_t)$ and its Wiener increment ΔW_t, the Adams framework naturally extends to Stochastic Ordinary Differential Equations.

(g) **Adaptability:** The coefficients $\beta_j, \theta_j, \gamma_j$ can be tuned to achieve desired strong or weak convergence orders.

1.9 Statement of the problem

Stochastic differential equations (SODEs) are widely used to model systems that are subject to random perturbations, with applications in physics, biology, finance, and engineering. Many such systems are stiff, meaning that their solutions evolve on widely separated time scales. Stiffness poses a major difficulty in numerical approximation, since explicit schemes require prohibitively small time steps to maintain stability, making them computationally inefficient (Schurz (1999), Higham (2000)).

Although several numerical approaches for SODEs exist, including Milstein-type and stochastic Runge–Kutta methods (Kloeden and Platen (1992)), they are often either of low order, unstable in stiff regimes, or computationally expensive when implicit formulations are employed. Furthermore, while deterministic multistep methods are well understood in terms of accuracy and stability, their stochastic counterparts are not yet fully developed for stiff problems (Buckwar and Winkler (2007)).

This situation highlights a gap in the literature, the need to construct and analyze *high-order, mean-square stable methods* that can efficiently approximate stiff SODEs while ensuring accuracy.

1.10 Aim and Objectives

The aim of this study are to develop families of numerical methods for approximating stiff Stochastic Ordinary Differential Equations.

The following are the objectives of the study:

1. derive high-order numerical methods for integrating stiff SODEs.
2. determine the order of accuracy of the methods.
3. derive mean-square stable methods.

1.11 Motivation of study

The numerical approximation of stiff Stochastic Ordinary Differential Equations (SODEs) remains a central challenge in stochastic numerical analysis. Stiffness arises in systems involving multiple time scales, where rapidly decaying modes force explicit methods to employ prohibitively small step sizes for stability. Such restrictions render standard schemes such as the Euler–Maruyama method inefficient in practice (Schurz (1999), Buckwar and Winkler (2007), Higham (2000)).

The need for stable and accurate numerical methods is evident in applications such as chemical reaction networks, population dynamics under uncertainty, financial mathematics, and stochastic control problems. In these settings, higher-order accuracy and stability are crucial for obtaining reliable approximations without excessive computational cost.

Although methods such as stochastic Runge–Kutta and Milstein-type schemes have been widely studied for non-stiff systems (Kloeden and Platen (1992)), their direct application to stiff SODEs often suffers from instability or inefficiency. This motivates the development of families of numerical methods specifically tailored for stiff SODEs that are mean-square stable, A-stable methods and rigorously analyze their accuracy and stability properties.

1.12 Limitations of the Study

Some limitations are recognized:

1. **Scope of Models:** The analysis is restricted to Ito-type SODEs with Lipschitz continuous drift and diffusion coefficients. Extensions to non-globally Lipschitz systems or rough path frameworks remain outside the scope.

2. **Class of Methods:** The study emphasizes Adams-type families of methods. Other numerical approaches, such as adaptive stochastic Runge–Kutta or exponential integrators, are not treated in depth.

3. **Test Equations:** Stability analysis is based on linear test equations.

4. **Computational Experiments:** Numerical experiments are limited to low- and moderate-dimensional examples. Large-scale or high-dimensional stiff systems are not addressed due to computational constraints.

1.13 Summary

Stiff SODEs are central to modeling systems with multiple time scales and randomness. One-step methods are simple but face stability limits, while multi-step and second derivative Adams type schemes offer better stability and accuracy potential. Significant gaps remain in theory, computation, and software, motivating the research in this thesis.

This thesis is organized into five chapters. Chapter One covers General introduction of where the problem exist in real life, need for numerical methods, one step and multistep, gaps identified statement of problem, Aim and objectives, limitation of study, basic definitions of terms, and illustrative examples from stochastic calculus, the theory of SODEs, linear multi-step methods, and the stochastic Ito-Taylor expansion, all of which are included to provide the necessary theoretical background.

Chapter two covers Review of literature in a chronological ordering, Linear multistep method for SODEs, Second derivative LMM some types of stochastic linear multi-step Maruyama methods (SLMMM's) for solving SODEs are studied and derived analytically.

Chapter three covers, Derivation of methods, order and stability analysis of the methods, proposition and theorems.

Chapter four covers, Numerical experiment, Description of test problems, description of existing method for comparison, Discussion of results.

Chapter five covers, Findings, contribution to knowledge, area of future research, summary. Finally, the computer programs used in this thesis are coded in MATHEMATICA 11.1 computer software.

CHAPTER TWO

Literature Review

2.1 Introduction

The development of numerical schemes for SODE has a direct bearing from existing methods for numerical integration of IVPs in ODEs. This chapter reviews the evolution and advancement of numerical schemes for SODE with a cursory look at possible upgrade via the frontier of higher derivative scheme for IVPs in ODEs.

2.2 One-Step Methods

One-step methods for the treatment of SODE (1.1.1) has over the years advanced and still enjoys research attention. Historically, Euler Maruyama method, developed in Maruyama (1955) is one of the foremost method for the integration of SODE (1.1.1).

In Kloeden and Platen (1992), the convergence theory of Euler Maruyama method, proving that under global Lipschitz conditions was carried out. Euler-Maruyama achieves strong order 0.5 and weak order 1. The stochastic Taylor expansion framework revealed why higher accuracy required more sophisticated approaches.

In Schurz (1999), stability, stationarity, and boundedness of numerical methods for stiff Stochastic Ordinary Differential Equations (SODEs) was investigated, focusing on whether discrete approximations preserve qualitative properties of the exact solutions. This was achieved by analyzing implicit and semi-implicit numerical schemes, including linear multistep and

Runge-Kutta methods, and deriving conditions under which these methods maintain mean-square stability and boundedness for stiff SODEs. The study demonstrated that implicit methods generally provide superior stability compared to explicit schemes, especially for stiff systems, while also highlighting the role of stepsize restrictions. A limitation is that the analysis primarily covers specific classes of stiff SODEs, and the applicability to highly nonlinear or high-dimensional problems may require further adjustments.

In Higham (2000), a systematic framework for translating deterministic stability theory into the stochastic domain was provided. The analysis unifies mean-square and asymptotic stability theory for SODEs and gives practical criteria for stability in numerical simulations. However, the reliance on linearization, real parameters, and numerical estimation limits its direct application to highly nonlinear or high-dimensional stochastic systems.

In Hutzenthaler et al. (2011), it was exposed that a more fundamental limitation for SODEs with superlinearly growing coefficients, even implicit Euler-Maruyama could diverge in strong and weak senses. This result demonstrated that standard discretizations could fail catastrophically for physically relevant systems.

In Mao (2015), the truncated Euler-Maruyama scheme, which bounded coefficients outside a growing domain was developed. This method restored moment bounds and strong convergence for SODEs with non-globally Lipschitz coefficients, demonstrating that careful coefficient modification could overcome stability issues.

In Ukuedojor et al. (2023), Euler–Maruyama method was extended to the domain of uncertain stochastic systems, providing a useful link between stochastic and uncertain modeling under the framework of chance theory. However, the method's low-order accuracy, absence of theoretical stability analysis, and limited numerical validation restrict its effectiveness when applied to complex or stiff uncertain stochastic systems.

In Dai and Wang (2023), adaptive Euler-Maruyama methods with step-size control strategies that adapted to local behavior of drift and diffusion terms was developed. While improving efficiency, the computational cost of adaptivity could offset gains for large-scale systems.

In He et al. (2024), explicit Euler methods for McKean-Vlasov SODEs driven by fractional Brownian motion was developed, addressing memory effects in the noise process. The method maintained convergence while allowing larger step sizes, though efficiency decreased for certain Hurst parameter values.

In Haghighi (2023), partially truncated Euler-Maruyama for SODEs with super-linear piecewise continuous drift and Hölder diffusion coefficients was introduced. This method addressed divergence problems while maintaining accuracy for smooth problems where truncation wasn't necessary.

The Euler-Maruyama method's journey demonstrates a clear evolutionary pattern. Each improvement addressed specific limitations while introducing new challenges. From basic discretization to stability enhancements to handling non-standard conditions, this development path created the foundation for higher-order methods while highlighting the fundamental tradeoffs between accuracy, stability, and computational efficiency that persist in contemporary SODE numerical analysis.

In Milstein (1974), the ground breaking insight that including the derivative of the diffusion coefficient could dramatically improve accuracy was introduced. This method achieved strong order 1.0, doubling the convergence rate of Euler-Maruyama by capturing the essential second-order stochastic term missing in simpler discretizations.

In Gaines and Lyons (1994), the method efficiently simulate SODEs by introducing a variable step-size method to improve computational performance over traditional fixed-step schemes. The method enabled adaptive refinement and coarsening of time steps while maintaining the pathwise consistency of Brownian motion, leading to more accurate and efficient simulations. The method successfully demonstrated how adaptive step control could balance accuracy and cost in stochastic integration. However, it was limited to non-stiff, low-dimensional SODEs, lacked rigorous theoretical convergence analysis, and involved complex implementation due to the handling of Brownian increments.

In Wiktorsson (2001) developed practical simulation methods for these iterated integrals, making Milstein-type schemes more feasible for moderate-dimensional problems. However, the computational complexity grew quadratically with noise dimension, limiting practical application to low-dimensional systems.

Kruse (2019) developed drift-randomized Milstein methods that exploited random quadrature to improve convergence with non-smooth drift coefficients. This approach demonstrated that randomization techniques could extend the applicability of higher-order methods beyond classical smoothness assumptions.

In Mrungowius (2021), new simulation approaches that significantly reduced computational overhead for iterated integrals was introduced. These developments made Milstein and related methods increasingly feasible for multidimensional problems, though implementation complexity remained substantial compared to Euler-type methods.

In Liu et al. (2023), Milstein-type schemes for highly nonlinear non-autonomous time-changed SODEs was developed, proving strong convergence for drift terms with superlinear growth under Hölder continuity conditions. This extension broadened applicability but introduced restrictive regularity requirements.

In Huang (2023), a modified Milstein schemes for fractional diffusions driven by fractional Brownian motion was proposed, achieving optimal convergence rates through rough path theory. The method's efficiency decreased as the Hurst parameter approached critical values, demonstrating context-dependent performance.

In, Zhang et al. (2023), an adaptive time-stepping Milstein methods for SODEs with piecewise continuous arguments was developed, demonstrating strong L_p convergence while improving stability under superlinear growth conditions. The adaptive strategy increased implementation complexity but provided better stability control.

In Wu and Yan (2025), Milstein schemes for stochastic semilinear subdiffusion equations was analyzed, combining finite element spatial discretization with Milstein time stepping. The method achieved higher temporal accuracy than Euler schemes but required smooth noise and substantial computational resources.

In Lateef et al. (2025), antithetic multilevel Monte Carlo Milstein schemes for SPDEs with non-commutative noise was introduced, achieving improved efficiency through variance reduction techniques. The method's complexity and truncation-induced bias limited application to high-dimensional problems.

The Milstein method's development illustrates a classic pattern in numerical analysis, achieving higher accuracy requires confronting exponentially increasing computational complexity. While the method fundamentally improved upon Euler-Maruyama's accuracy limitations, its practical implementation challenges—particularly for multi-dimensional systems—motivated the development of Runge-Kutta approaches that could achieve similar accuracy without derivative calculations.

The development of Stochastic Runge-Kutta (SRK) methods represented a pivotal evolution in SODE numerical analysis, addressing the computational bottlenecks of Milstein-type methods while maintaining higher-order accuracy through multi-stage formulations.

Kloeden and Platen (1992) established the foundational framework for SRK methods, demonstrating how deterministic Runge-Kutta techniques could be integrated with stochastic calculus. Their work showed that carefully designed multi-stage methods could achieve weak order two without requiring explicit derivative calculations, thus overcoming a major limitation of Milstein schemes.

In Burrage and Burrage (1999), high-strong-order numerical methods for SODEs was constructed, particularly for non-commutative multiple Wiener processes where standard methods fail. The method incorporate commutator terms, allowing strong convergence order up to 1.5 for general SODEs. The methods significantly improve accuracy compared to traditional explicit schemes of order 0.5. However, the approach is limited by increased computational complexity, additional stochastic integrals, and practical implementation challenges in non-commutative or large-scale systems.

In Rößler (2006), stochastic Runge–Kutta (SRK) methods that achieve high weak-order convergence for Itô stochastic differential equations (SODEs) was constructed, particularly in systems driven by multi-dimensional Wiener processes. The method introduced a general class of SRK methods and employed rooted tree analysis to derive general order conditions for their coefficients, enabling the design of methods with arbitrarily high weak-order accuracy. However, the methods presented are primarily applicable to Itô SODEs and may not directly extend to Stratonovich equations or systems with non-commutative noise, limiting their broader applicability.

In Debrabant and Rößler (2009), efficient second-order stochastic Runge–Kutta (SRK) methods for the weak approximation of Itô stochastic differential equations (SODEs) driven by multi-dimensional Wiener processes was constructed. The method introduced a class of SRK methods that significantly reduce the number of function evaluations compared to traditional methods, which typically scale quadratically with the number of Wiener processes. The study provided a full classification of the coefficients for explicit SRK methods with minimal stage numbers and presented coefficients for optimal schemes that satisfy higher-order conditions. However, these methods are primarily applicable to Itô SODEs and may not directly extend to Stratonovich equations or systems with non-commutative noise, limiting their broader applicability.

In Li et al. (2023), conservative continuous-stage SRKs that blended geometric integration principles with stochastic analysis was proposed. These methods excelled at preserving invariants such as energy and mass but achieved higher order only in restrictive settings with single integrand cases, demonstrating the ongoing trade-off between generality and specialized performance.

Kruse et al. (2018) introduced randomized SRK methods that employed random quadrature techniques to relax smoothness assumptions on the drift. This innovation extended SRK applicability to problems with weaker regularity conditions, though it introduced additional statistical variability that required careful error analysis.

In Debrabant et al. (2022), balanced SRK schemes that extended the concept of balanced Euler-type methods to higher-order Runge-Kutta settings was developed. These methods ensured controlled moment growth and long-time stability while retaining weak order accuracy, particularly valuable for stiff systems with dissipative properties.

In Hallern et al. (2024), Exponential Stochastic Runge-Kutta methods for SPDEs of Nemytskii type, achieving strong temporal convergence order up to 1.5 was introduced. The scheme combined exponential integrator ideas with SRK methodology, eliminating derivative requirements in certain components. However, the method demanded spatial discretization and exhibited higher computational complexity, with implementation challenges for highly irregular SPDEs.

In Zhang et al. (2024), Structure-Preserving Implicit Runge-Kutta methods for stochastic Poisson systems with multiple noises was developed. The diagonal implicit SRK and transformed RK methods preserved Poisson structure, Casimir functions, and quadratic Hamiltonians, demonstrating excellent performance for systems like the stochastic rigid body. The limitation emerged in the requirement to solve algebraic systems at each step, increasing computational cost while restricting applicability to specific structural frameworks.

In D'Giovacchino and D'Ambrosio (2024), comprehensive contractivity analysis of SRK methods applied to mean-square dissipative SODEs was conducted. The method demonstrated that SRK methods derived from algebraically stable deterministic RK methods could inherit contractivity properties when applied to SODEs with globally one-sided Lipschitz drift. This provided crucial tools for analyzing SRK stability in stiff or dissipative contexts, though it required specific structural conditions that limited broad applicability.

In Arif et al. (2025), Hybrid Exponential Runge-Kutta schemes for stochastic SIQR disease models with spatial diffusion was proposed . The method demonstrated superior accuracy compared to Euler-Maruyama for nonlinear epidemiological dynamics but was primarily validated in specific application contexts rather than general SODE/SPDE classes, highlighting the ongoing challenge of balancing specialized performance with general applicability.

The development of Stochastic Runge-Kutta methods illustrates a clear evolutionary trajectory from derivative-laden higher-order methods to sophisticated multi-stage approaches that maintain accuracy while reducing computational complexity. Each innovation addressed specific limitations, order conditions provided theoretical foundation, B-series enabled systematic construction, geometric methods addressed structure preservation, and randomized variants expanded applicability. However, the fundamental trade-offs between computational cost, implementation complexity, and general applicability persist, motivating continued development of more efficient and robust SRK families.

In Burrage and Burrage (1996), Stochastic Linear Multistep Methods (SLMM) were introduced to extend classical linear multistep methods for the numerical solution of SODEs. The formulated SLMM achieve both strong and weak convergence, providing a systematic framework for approximating SODE solutions over multiple steps. The methods' stability and consistency was analyzed, highlighting their efficiency for non-stiff problems but noting potential instability when applied to stiff SODEs.

In Denk and Schäffler (1997), the problem of developing stable and convergent multistep methods for the numerical solution of SODs was addressed. This was achieved by analyzing the stability and convergence of implicit and semi-implicit linear multistep schemes, deriving conditions under which the methods are mean-square stable, and providing examples of schemes that satisfy these conditions. The study also highlighted limitations, noting that explicit schemes may fail to maintain stability for stiff problems and that achieving high strong order convergence is restricted by the structure of the stochastic terms.

In Brugnano et al. (2000), strongly convergent numerical solutions for stochastic ordinary differential equations (SODEs) using linear multistep methods was obtained. It was achieved by deriving conditions on the coefficients of Adams-type methods to ensure strong local and global convergence, considering both commutative and non-commutative noise, and proposing predictor-corrector schemes for efficiently handling implicit stochastic terms. It was shown that explicit Adams-type methods cannot reach strong order 1, and in general, methods cannot exceed strong order 1; higher orders are only achievable in special cases such as additive noise or linear drift.

In Buckwar and Winkler (2006), weak second-order stochastic Runge–Kutta (SRK) methods for Itô stochastic ordinary differential equations (SODEs) with small noise was constructed, aiming to improve computational efficiency without sacrificing accuracy. A class of SRK methods that approximate the drift term using deterministic ODE methods and discretize the diffusion term using a Maruyama scheme, with the option to incorporate mixed classical-stochastic integrals for enhanced accuracy was introduced. The study demonstrated that the effectiveness of these methods is contingent upon a careful balance between the step size and the magnitude of the noise, and provided simulation results to illustrate their theoretical findings. However, the methods are primarily applicable to Itô SODEs with small noise and may not directly extend to systems with large noise or non-commutative noise structures, limiting their broader applicability.

In Stamatiou (2017), comprehensive asymptotic mean-square stability theory for two-step methods was developed , providing necessary and sufficient algebraic criteria in terms of associated stability polynomials. The method established stochastic analogues of deterministic root locus analysis, enabling method selection based on stability properties rather than solely accuracy considerations.

In D'Ambrosio and Buckwar (2021), exponential mean-square stability for nonlinear stochastic linear multistep methods (SLMM) was analzed, focusing on whether numerical solutions can reproduce the contractive behavior of the exact SODE solution. This was achieved by deriving sufficient conditions on the stepsize and method coefficients for one-step and two-step SLMM to ensure exponential mean-square contractivity, supported by theoretical inequalities and numerical experiments using methods like Adams-Moulton, Milne-Simpson, and BDF2. The study showed that while some methods, like BDF2, are nearly unconditionally stable, others, such as Milne-Simpson, fail to maintain exponential mean-square stability under practical stepsizes. A limitation is that stability depends strongly on the chosen stepsize and method, requiring careful selection and computation of bounds to preserve contractive behavior.

In Zhao et al. (2014), multistep methods to coupled forward-backward SODEs, producing high-order accurate schemes for financial and control applications was extended. The method demonstrated that carefully designed multistep methods could achieve both accuracy and efficiency gains in these coupled systems, though the computational burden of evaluating iterated stochastic integrals remained significant.

In Fu et al. (2018), multistep methods for forward-backward SODEs with jumps, incorporating Poisson increments and jump integrals into the multistep framework was developed. This extension demonstrated SLMM adaptability to more general stochastic dynamics but introduced additional complexity in handling discontinuous processes.

In Fang and Zhao (2023), ODE-based multistep schemes for Backward SODEs by reformulating the problem via the nonlinear Feynman-Kac formula was proposed. The strong stability preserving (SSP) multistep discretizations was employed and proved consistency, stability, and convergence. The approach demonstrated how multistep methods could be adapted to backward problems with improved numerical performance, though conditional expectation approximations remained challenging.

Bussell and García-Trillos (2023), a deep multistep mixed algorithm for solving high-dimensional nonlinear PDEs and associated Backward SODEs was introduced. The method integrated neural networks with multistep time discretizations to address the curse of dimensionality, demonstrating that multistep formulations could be valuable beyond traditional analysis frameworks, though training accuracy and network generalization introduced new sources of error.

The development of Stochastic Linear Multistep Methods illustrates a consistent focus on computational efficiency through information reuse, with each evolutionary stage addressing emerging stability and implementation challenges. From initial formulations through rigorous theoretical foundations to modern computational implementations, SLMMs have maintained their core advantage of efficiency while gradually overcoming stability limitations through sophisticated analysis and method design. The persistent challenge remains balancing the inherent stability constraints of multistep approaches with the demanding requirements of stiff stochastic systems.

Second-derivative methods represent a potentially transformative but largely unexplored evolutionary direction in SODE numerical analysis, building upon their demonstrated success in deterministic contexts while addressing the unique challenges of stochastic systems.

Enright (1974) pioneered second-derivative multistep methods for stiff ODEs, demonstrating that incorporating second-derivative information could substantially enlarge absolute stability regions. The method established the fundamental advantage: second-derivative methods could achieve higher accuracy and superior stability properties compared to first-derivative counterparts, particularly valuable for stiff systems.

Mehdizadeh Khalsaraei et al. (2012) comprehensively reviewed families of higher-order A-stable multistep methods based on second derivatives, systematically analyzing their advantages for stiff initial value problems. Their work demonstrated that second-derivative methods could achieve orders of accuracy unreachable by conventional methods while maintaining excellent stability properties.

Nwachukwu and Mokwunye (2018) developed generalized Adams-type second-derivative methods (SDGAMs) that achieved remarkable stability together with accuracy orders as high as $2k + 2$. The method demonstrated that carefully designed coefficient sets could leverage second-derivative information to simultaneously address accuracy and stability requirements, though implementation required efficient computation of second derivatives.

Mehdizadeh Khalsaraei et al. (2012) systematically analyzed families of A-stable SD-LMMs, proving that methods with k steps can achieve order $2k+1$ while maintaining absolute stability. Their work established that the additional degrees of freedom from second-derivative terms enable simultaneous optimization of both accuracy and stability constraints.

Skwame et al. (2018) introduced a two-step second-derivative Adams-Moulton method specifically designed for stiff second-order ODEs. Their method demonstrated excellent performance for highly oscillatory and stiff problems, providing a template for potential stochastic extensions while highlighting the computational challenges of derivative evaluation.

The absence of systematic SD-LMM development for SODEs represents not merely a literature gap but a significant opportunity to leverage proven deterministic advantages in the stochastic context. This thesis directly addresses this opportunity by developing second-derivative Adams-type formulae specifically designed for stiff stochastic ordinary differential equations, with careful attention to the unique mathematical, theoretical, and computational challenges of the stochastic setting. Despite the progress in ODE contexts, there is no established literature on second-derivative Adams-type methods for stiff SODEs. Existing studies on stiff stochastic problems have primarily focused on Euler-type, Milstein-type, and first-derivative multistep or Runge–Kutta methods schurz (1999) stability, Buckwar and Winkler (2007). The absence of systematic investigations into second-derivative Adams-type methods for SODEs represents a clear gap in the current body of knowledge. This gap forms the central motivation of the present study, which aims to develop families of such methods, analyze their accuracy and stability, and establish their suitability for approximating stiff stochastic systems.

2.3 Summary

This chapter reviewed the development of numerical methods for SODEs, from early one-step schemes to advanced Adams-type and second-derivative methods. While second-derivative methods have proven effective in ODEs, their adaptation to SODEs, particularly stiff cases remains unexplored. This gap forms the basis for the research presented in chapter 3.

CHAPTER THREE

Second Derivative Adams Type Formulae

3.1 Introduction

Higher Derivative methods for the treatment of stiff IVPs in ODEs were borne out of the desire to improve the order of the numerical methods used in the approximation of solution to stiff IVPs. This chapter is on the development of Second Derivative methods for integration of SODe (1.1.1).

The numerical treatment of SODEs is more delicate than that of ordinary differential equations (ODEs), owing to the presence of stochastic forcing terms represented by Wiener processes. In particular, the numerical integration of stiff SODEs requires schemes that combine both accuracy and robust stability properties. Classical linear multistep methods have been extended to the stochastic context; however, their performance on stiff problems is often limited. The inclusion of second derivative terms in Adams-type methods has been shown, in the ODE context, to enhance stability and efficiency. This motivates the extension of such methods to the stochastic setting.

3.2 Construction of Method

The second derivative LMM for SODE (1.1.1) is given as:

$$\begin{aligned} X_{n+1} &= \bar{X}_{n+1} + \left(1 - hf'(\bar{X}_{n+1})\right)^{-1} h\left(f(\bar{X}_{n+1}) - f(X_n)\right), \\ \bar{X}_{n+1} &= X_n + hf(X_n) + g(X_n)\,\Delta W_n + \frac{1}{2}g(X_n)g'(X_n)\left((\Delta W_n)^2 - h\right). \end{aligned} \tag{3.2.1}$$

By setting $\theta_j = 0, j = 0, 1, 2, ...k-1$, in the spirit of (Enright(1974), yields:

$$\begin{aligned} Y_{n+1} &= \bar{Y}_{n+1} + \left(1 - hf'(\bar{Y}_{n+1})\right)^{-1} h\left(f(\bar{Y}_{n+1}) - f(Y_n)\right), \\ \bar{Y}_{n+1} &= Y_n + hf(Y_n) + g(Y_n)\,\Delta W_n + \frac{1}{2}g(Y_n)g'(Y_n)\left((\Delta W_n)^2 - h\right). \end{aligned} \tag{3.2.2}$$

The (3.2.2) is the proposed method in this study. The method (3.2.2) is called Second Derivative Adams Type Formula 1 ($SDATF_1$) for reference purpose. Here $f_{n+j-k+1}$ is the drift term, $g_{n+j-k+1}$ is the diffusion term, $g' = \frac{\partial g}{\partial t}$ and $\beta_j, \gamma_j, \eta_j, \theta_k$ are constants to be determined.

The parameters $\beta_j, \theta_k, \gamma_j, \eta_j$ are obtained using Taylor series expansion, Ito Taylor expansion about (t_n, X_n) and method of undetermined coefficients using the following procedure:

1. Expand the exact solution of the SODE using the stochastic Taylor formula.
2. Expand the numerical scheme in powers of h and ΔW_n.
3. Equate coefficients of like stochastic terms ($h, \Delta W_n, (\Delta W_n)^2$, etc.) between the two expansions.
4. Solve the resulting system of algebraic equations to determine the sets $\{\alpha_j\}$, $\{\beta_j\}$, and $\{\gamma_j\}$.

This procedure ensures that the SLMM reproduces the exact stochastic behaviour of the SODE up to the desired order of strong or weak convergence.

Let $X(t_n)$ denote the exact solution of the SODE (1.1.1). The Linear difference operator associated with (3.2.2) with $\tau_j = (j-k)h$ and $t_n + h + \tau_j = t_n + jh$. where $\Delta W(t_n + h) = W(t_n + h) - W(t_n)$ and $L_1 g = gg'$.

Denote values at the base point t_n by

$$f_n := f(X(t_n)), \quad G_n := f'(X(t_n)), \quad H_n := f''(X(t_n)), \quad J_n := f'''(X(t_n)).$$

Taylor expansions of $X(t_n + h)$ about (t_n, X_n) is given as:

$$X(t_n + h) = X(t_n) + hf_n + \tfrac{h^2}{2} G_n + \tfrac{h^3}{6} H_n + \tfrac{h^4}{24} J_n + O(h^5), \tag{3.2.3}$$

also, $f(X(t_n + h + \tau_j))$ about (t_n, X_n) yields:

$$f(X(t_n + h + \tau_j)) = f_n + (jh)\, G_n + \tfrac{(jh)^2}{2} H_n + \tfrac{(jh)^3}{6} J_n + O(h^4), \tag{3.2.4}$$

Substituting (3.2.4) and (3.2.5) into (3.2.3) and collect powers of h, and match independent derivatives, yields the deterministic moment (consistency) conditions:

$$\sum_{j=0}^{k} \beta_j = 1, \tag{3.2.5}$$

$$\sum_{j=0}^{k} \beta_j j = \tfrac{1}{2} - \theta_k, \tag{3.2.6}$$

$$\sum_{j=0}^{k} \beta_j j^2 = \tfrac{1}{3}, \tag{3.2.7}$$

and in general, for $m \geq 0$,

$$\sum_{j=0}^{k} \beta_j j^m = \begin{cases} \frac{1}{2} - \theta_k, & m = 1, \\ \frac{1}{m+1}, & m \neq 1. \end{cases} \tag{3.2.8}$$

For the stochastic terms, the Linear difference operator associated with the stochastic part of (3.2.3) is:

$$L(X(t_n); h) = -\Delta W\Big(\sum_{j=0}^{k} \gamma_j\Big) g(t_n) - \tfrac{1}{2}((\Delta W)^2 - h)\Big(\sum_{j=0}^{k} \eta_j\Big)(L_1 g(t_n)) + O(h^{3/2}). \tag{3.2.9}$$

Ito Taylor expansion of (3.2.10) about (t_n, X_n) gives:

$$g(X(t_n + h + \tau_j)) = g(t_n) + \tau_j (L_0 g)(t_n) + O(h^2), \tag{3.2.10}$$

$$(L_1 g)(X(t_n + h + \tau_j)) = (L_1 g(t_n)) + O(h). \tag{3.2.11}$$

Substituting (3.2.11) and (3.2.12) into (3.2.10) and matching the Ito–Taylor yields:

$$\sum_{j=0}^{k} \gamma_j = 1, \qquad \sum_{j=0}^{k} \eta_j = 1. \tag{3.2.12}$$

For fully implicit choices concentrated at $j = k$, (3.2.13) holds automatically (since $j - k = 0$ for $j = k$) (Dai and Wang (2025)).

$$\gamma_j = \begin{cases} 0, & j < k, \\ 1, & j = k, \end{cases} \qquad \eta_j = \begin{cases} 0, & j < k, \\ 1, & j = k, \end{cases}. \tag{3.2.13}$$

3.2.1 Coefficients of $SDATF_1$

For $k = 1$

The nodes $j = 0, j = 1$. Unknowns: $\beta_0, \beta_1, \theta_1$. Moment equations (from $m = 0, 1, 2$) (3.2.6),(3.2.7) and (3.2.8) yields:

$$\beta_0 + \beta_1 = 1, \tag{3.2.14}$$

$$0 \cdot \beta_0 + 1 \cdot \beta_1 = \tfrac{1}{2} - \theta_1, \tag{3.2.15}$$

$$0^2\beta_0 + 1^2\beta_1 = \tfrac{1}{3}. \tag{3.2.16}$$

Solving the linear system (3.2.15)-(3.2.17) (Hairer and Wanner (2010)) yields:

$$\beta_0 = \tfrac{2}{3}, \quad \beta_1 = \tfrac{1}{3}, \quad \theta_1 = \tfrac{1}{6}.$$

From (3.2.13),

$$\gamma_0 + \gamma_1 = 1 \tag{3.2.17}$$

$$\eta_0 + \eta_1 = 1 = 1 \tag{3.2.18}$$

Solving (3.2.18) and (3.2.19) gives: $\gamma_0 = 0,\ \gamma_1 = 1,\ \eta_0 = 0,\ \eta_1 = 1$, the resulting scheme is

For $k = 2$

The nodes are $j = 0, 1, 2$ $(t_n, t_n{+}h, t_n{+}2h)$. Unknowns: $\beta_0, \beta_1, \beta_2, \theta_2$. Moment equations (from $m = 0, 1, 2, 3$):

$$\beta_0 + \beta_1 + \beta_2 = 1, \tag{3.2.19}$$

$$0 \cdot \beta_0 + 1 \cdot \beta_1 + 2 \cdot \beta_2 = \tfrac{1}{2} - \theta_2, \tag{3.2.20}$$

$$0^2\beta_0 + 1^2\beta_1 + 2^2\beta_2 = \tfrac{1}{3}, \tag{3.2.21}$$

$$0^3\beta_0 + 1^3\beta_1 + 2^3\beta_2 = \tfrac{1}{4}. \tag{3.2.22}$$

Solving the system of equation (3.2.21)- (3.2.24) yields:

$$\beta_0 = \tfrac{29}{48}, \quad \beta_1 = \tfrac{5}{12}, \quad \beta_2 = -\tfrac{1}{48}, \quad \theta_2 = \tfrac{1}{8}.$$

From (3.2.16),

$$\gamma_0 + \gamma_1 + \gamma_2 = 1 \tag{3.2.23}$$

,

$$\eta_0 + \eta_1 + \eta_2 = 1 \tag{3.2.24}$$

Solving the system of equation (3.2.25) and (3.2.26) gives: $\gamma_0 = \gamma_1 = 0,\ \gamma_2 = 1,\ \eta_0 = \eta_1 = 0,\ \eta_2 = 1$, the scheme becomes;

For $k = 3$

Nodes are $j = 0, 1, 2, 3$ (nodes $t_n, \dots, t_n + 3h$). Unknowns: $\beta_0, \beta_1, \beta_2, \beta_3, \theta_3$. Moment equations (from $m = 0, 1, 2, 3, 4$):

$$\sum_{j=0}^{3} \beta_j j^m = \frac{1}{m+1} \quad (m \neq 1), \qquad \sum_{j=0}^{3} \beta_j j = \tfrac{1}{2} - \theta_3. \tag{3.2.25}$$

Solving the linear system (3.2.29) gives the coefficients:

$$\beta_0 = \frac{307}{540},\ \beta_1 = \frac{19}{40},\ \beta_2 = -\frac{1}{20},\ \beta_3 = \frac{7}{1080},\ \theta_3 = \frac{19}{180}.$$

From (3.2.16),

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = 1 \tag{3.2.26}$$

$$\eta_0 + \eta_1 + \eta_2 + \eta_3 = 1 \tag{3.2.27}$$

Solving the system of equation (3.2.29) and (3.2.30) gives: $\gamma_0 = \gamma_1 = \gamma_2 = 0, \gamma_3 = 1$ and $\eta_0 = \eta_1 = \eta_2 = 0, \eta_3 = 1$, the scheme is:

k	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	θ_k	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}	η_0	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}
1	$\frac{1}{3}$	$\frac{2}{3}$										$\frac{-1}{6}$	0	1										0	1									
2	$\frac{-1}{48}$	$\frac{5}{12}$	$\frac{29}{48}$									$\frac{-1}{8}$	0	0	1									0	0	1								
3	$\frac{7}{1080}$	$\frac{-1}{20}$	$\frac{19}{40}$	$\frac{307}{540}$								$\frac{-19}{180}$	0	0	0	1								0	0	0	1							
4	$\frac{-17}{5760}$	$\frac{1}{45}$	$\frac{-41}{480}$	$\frac{47}{90}$	$\frac{3133}{5760}$							$\frac{-3}{32}$	0	0	0	0	1							0	0	0	0	1						
5	$\frac{41}{25200}$	$\frac{-529}{40320}$	$\frac{373}{7560}$	$\frac{-1271}{10080}$	$\frac{2837}{5040}$	$\frac{317731}{604800}$						$\frac{-863}{10080}$	0	0	0	0	0	1						0	0	0	0	0	1					
6	$\frac{-731}{725760}$	$\frac{179}{20160}$	$\frac{-5771}{161280}$	$\frac{8131}{90720}$	$\frac{-13823}{80640}$	$\frac{12079}{20160}$	$\frac{247021}{483840}$					$\frac{-275}{3456}$	0	0	0	0	0	0	1					0	0	0	0	0	0	1				
7	$\frac{8563}{12700800}$	$\frac{-35453}{5443200}$	$\frac{86791}{3024000}$	$\frac{-2797}{36288}$	$\frac{157513}{1088640}$	$\frac{-133643}{604800}$	$\frac{1147051}{1814400}$	$\frac{1758023}{3528000}$				$\frac{-33953}{453600}$	0	0	0	0	0	0	0	1				0	0	0	0	0	0	0	1			
8	$\frac{-27719}{58060800}$	$\frac{9143}{1814400}$	$\frac{-88313}{3628800}$	$\frac{648439}{9072000}$	$\frac{-417793}{2903040}$	$\frac{391877}{1814400}$	$\frac{-995891}{3628800}$	$\frac{1202489}{1814400}$	$\frac{141646727}{290304000}$			$\frac{-8183}{115200}$	0	0	0	0	0	0	0	0	1			0	0	0	0	0	0	0	0	1		
9	$\frac{190073}{538876800}$	$\frac{-517157}{127733760}$	$\frac{2984887}{139708800}$	$\frac{-8236373}{119750400}$	$\frac{3023179}{19958400}$	$\frac{-38944747}{159667200}$	$\frac{3650797}{11975040}$	$\frac{-1890811}{5702400}$	$\frac{6898799}{9979200}$	$\frac{57812684753}{120708403200}$		$\frac{-3250433}{47900160}$	0	0	0	0	0	0	0	0	0	1		0	0	0	0	0	0	0	0	0	1	
10	$\frac{-516149}{1916006400}$	$\frac{1602701}{479001600}$	$\frac{-9810863}{510935040}$	$\frac{18873499}{279417600}$	$\frac{-5785817}{35481600}$	$\frac{4586369}{15966720}$	$\frac{-27345007}{70963200}$	$\frac{3294113}{7983360}$	$\frac{-250482007}{638668800}$	$\frac{344046077}{479001600}$	$\frac{25271227027}{53648179200}$	$\frac{-4671}{71680}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1

Table 3.1: $SDATF_1$ coefficients for orders $k = 1, \ldots, 10$.

3.3 Order and Stability of $SDATF_1$

The Ito differential operators (Kloeden and Platen (1992)) are defined as follows:

$$L^0 = f\frac{\partial}{\partial t} + \tfrac{1}{2}g(t)^2\frac{\partial^2}{\partial t^2}, \tag{3.3.1}$$

$$L^1 = g\frac{\partial}{\partial x}. \tag{3.3.2}$$

Itô integrals Over the interval $[t_n, t_{n+1}]$ with step h and Wiener increment ΔW_n, (Kloeden and Platen (1992))

$$\begin{aligned}
I_{(0)} &= h, \\
I_{(1)} &= \Delta W_n, \\
I_{(1,1)} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u \, dW_s = \tfrac{1}{2}((\Delta W_n)^2 - h), \\
I_{(1,0)} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u \, ds, \\
I_{(0,1)} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} du \, dW_s.
\end{aligned} \tag{3.3.3}$$

The expectation of Ito integrals satisfy the following relation:

$$\mathbb{E}|I_{(1,0)}|^2 = O(h^3), \mathbb{E}|I_{(0,1)}|^2 = O(h^3), \tag{3.3.4}$$

i.e. they are of size $O(h^{3/2})$ in mean-square, (Kloeden and Platen (1992)).

The Ito's Taylor expansion of $X(t_{n+1})$ of the solution (1.1.1) at the point (t_{n+1}) is given as follows:

$$
\begin{aligned}
X(t_{n+1}) = X(t_n) + hL^0 + L^1\Delta W_n + \tfrac{1}{2}L^0L^0h^2 \\
+ \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dW_u\, ds, L^1L^0 + \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} du\, dW_s L^0L^1 \\
+ \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dW_u\, dW_s L^1L^1 + R_{n+1},
\end{aligned}
\tag{3.3.5}
$$

using (3.3.3),

$$
\begin{aligned}
X(t_{n+1}) = X(t_n) + L^0\, I_{(0)} + L^1\, I_{(1)} + \tfrac{1}{2}L^0L^0\, I_{(0,0)} \\
+ L^1L^0\, I_{(1,0)} + L^0L^1\, I_{(0,1)} + L^1L^1\, I_{(1,1)} + R_{n+1},
\end{aligned}
\tag{3.3.6}
$$

where R_{n+1} is the remainder and satisfies the relation $\mathbb{E}|R_{n+1}|^2 = O(h^3)$.

where

$$
\begin{aligned}
L^0 f &= ff' + \tfrac{1}{2}g^2 f''; \\
L^1 f &= gf'; \\
L^0 g &= fg' + \tfrac{1}{2}g^2 g''; \\
L^1L^1 x &= gg'.
\end{aligned}
\tag{3.3.7}
$$

Substituting (3.3.7) into (3.3.6) readily gives:

$$
\begin{aligned}
X(t_{n+1}) = X(t_n) + f_n h + g_n \Delta W_n \\
+ \tfrac{1}{2}(f_n f_n' + \tfrac{1}{2}g_n^2 f_n'')h^2 \\
+ (g_n f_n')I_{(1,0)} + (f_n g_n' + \tfrac{1}{2}g_n^2 g_n'')I_{(0,1)} \\
+ (g_n g_n')\tfrac{1}{2}((\Delta W_n)^2 - h) + R_{n+1},
\end{aligned}
\tag{3.3.8}
$$

Proposition 3.1. *$SDATF_1$ (3.2.2) is of strong order 1.*

proof

Assume $f, g \in C^3(\mathbb{R}^n)$. Also assume f, g and satisfy global Lipschitz and polynomial growth bounds,

Then the one step map of $SDATF_1$ (3.2.2) is given by:

The local truncation error (lte),

$$\tau_{n+1} = X(t_{n+1}) - X_{n+1}. \tag{3.3.9}$$

obtained from equating (3.3.8) and (3.3.9) as:

Applying the consistency conditions:

$$\sum_{j=0}^{k} \beta_j = 1, \quad \sum_{j=0}^{k} \gamma_j = 1, \quad \sum_{j=0}^{k} \eta_j = 1, \tag{3.3.10}$$

(3.3.12) reduces to:

$$\tau_{n+1} = \Big(\tfrac{1}{2}(L^0 f_n) - \theta_k G_{n+1}\Big)h^2 + (g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} + \widetilde{R}_{n+k}. \tag{3.3.11}$$

Taking the norm of both sides of (3.3.14) and squaring the resultant gives:

$$\|\tau_{n+1}\|^2 = \|\Big(\tfrac{1}{2}(L^0 f_n) - \theta_k G_{n+1}\Big)h^2 + (g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} + \widetilde{R}_{n+k}\|^2. \tag{3.3.12}$$

$$\|\tau_{n+1}\|^2 \leq \|\Big(\tfrac{1}{2}(L^0 f_n) - \theta_k G_{n+1}\Big)h^2\|^2 + \|(g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)}\|^2 + \|\widetilde{R}_{n+k}\|^2. \tag{3.3.13}$$

Taking expectation of both sides of (3.3.16) to obtain:

$$\mathbb{E}\|\tau_{n+1}\|^2 \le \mathbb{E}\|\Big(\tfrac{1}{2}(L^0 f_n) - \theta_k G_{n+1}\Big)h^2\|^2 + \mathbb{E}\|(g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)}\|^2 + \mathbb{E}\|\widetilde{R}_{n+k}\|^2. \tag{3.3.14}$$

Recall (3.3.4), (3.3.17) becomes:

$$\mathbb{E}\|\tau_{n+1}\|^2 \le Ch^3. \tag{3.3.15}$$

The global error e_{n+1}, Incured by approximating the solution (1.1.1) using $SDATF_1$ is obtained as:

$$e_{n+1} = \tau_{n+1} + X(t_n) - X_n. \tag{3.3.16}$$

Taking expectation of both sides of (3.3.19) yield:

$$\mathbb{E}|e_{n+1}|^2 \le (1 + Ch)\, \mathbb{E}|e_n|^2 + C\,\mathbb{E}|\tau_{n+1}|^2. \tag{3.3.17}$$

Substituting (3.3.18) into (3.3.20) gives:

$$\mathbb{E}|e_n|^2 \le (1 + Ch)\, \mathbb{E}|e_{n-1}|^2 + Ch^3. \tag{3.3.18}$$

By discrete Grönwall's lemma (Kloeden and Platen (1992)), uniformly for $t_n \le T$,

$$\mathbb{E}|e_n|^2 \le Ch^2. \tag{3.3.19}$$

Therefore, $SDATF_1$ is of strong order 1 and the proof is established.

Remark 3.1. *Let Φ_h be the numerical one-step increment produced by SDATF$_1$ applied using* *exact solution values on the right-hand side, so that*

$$\mathbb{E}\left[\varphi(x+\Phi_h)\right] = \varphi(x) + h\mathcal{L}\varphi(x) + \tfrac{h^2}{2}\mathcal{L}^2\varphi(x) + \mathcal{O}(h^3), \tag{3.3.20}$$

for every smooth test φ; this gives local weak error $O(h^3)$. then lifts local $O(h^3)$ to global weak order 2 under zero-stability (Talay and Tubaro (1990)). Using standard Brownian moment identities

$$\mathbb{E}[\Delta W] = 0, \quad \mathbb{E}[(\Delta W)^2] = h, \quad \mathbb{E}[(\Delta W)^3] = 0, \quad \mathbb{E}[(\Delta W)^4] = 3h^2, \tag{3.3.21}$$

and corresponding moments for the multistep Brownian combinations. Boundedness / moment assumptions on g, gg' ensure all needed expectations are finite and uniformly bounded.

Proposition 3.2. *SDATF1 (3.2.2) is of weak order 2.*

proof Let $\varphi \in C_P^{2(p+1)}(\mathbb{R}^n, \mathbb{R})$ be a smooth test function

By Ito–Taylor expansion (Kloeden and Platen, 1992; Talay and Tubaro, 1990),

$$\mathbb{E}[\varphi(X(t_{n+1})) \mid X_n] = \varphi(X_n) + hL\varphi(X_n) + \tfrac{h^2}{2}L^2\varphi(X_n) + O(h^3), \tag{3.3.22}$$

where $L\varphi = f\varphi' + \frac{1}{2}g^2\varphi''$.

Let $\Xi = X_{n+1} - X_n$. Using (3.3.24), we compute

$$\mathbb{E}[\Xi] = hS_\beta^{(0)}f_n + h^2\big(S_\beta^{(1)}(Lf)_n + \theta_k G_n\big) + O(h^3), \tag{3.3.23}$$

$$\mathbb{E}[\Xi^2] = h(S_\gamma^{(0)})^2g_n^2 + h^2\big((S_\beta^{(0)})^2f_n^2 + \tfrac{1}{2}(S_\eta^{(0)})^2(g'g)_n^2 + 2S_\gamma^{(0)}S_\gamma^{(1)}g_n(Lg)_n\big) + O(h^3), \tag{3.3.24}$$

$$\mathbb{E}[\Xi^3] = 3h^2(S_\gamma^{(0)})^2g_n^2\big(S_\beta^{(0)}f_n + S_\eta^{(0)}g_ng_n'\big) + O(h^{5/2}), \tag{3.3.25}$$

$$\mathbb{E}[\Xi^4] = 3h^2(S_\gamma^{(0)})^4 g_n^4 + O(h^{5/2}). \tag{3.3.26}$$

Substituting into (3.3.25)- (3.3.28) into:

$$\mathbb{E}[\varphi(X_{n+1}) \mid X_n] = \sum_{m=0}^{4} \frac{\varphi^{(m)}(X_n)}{m!} \mathbb{E}[\Xi^m] + O(h^3), \tag{3.3.27}$$

gives the numerical expansion up to $O(h^2)$.

Comparing with the exact expansion, (3.3.13) and

$$\sum_{j=0}^{k} \gamma_j (j-k+1) = \tfrac{1}{2}, \tag{3.3.28}$$

$$\sum_{j=0}^{k} \beta_j (j-k+1) + \theta_k = \tfrac{1}{2}. \tag{3.3.29}$$

are required. If the coefficients satisfy (3.3.13) and (3.3.20), then

$$\mathbb{E}[\varphi(X(t_{n+1}))] - \mathbb{E}[\varphi(X_{n+1})] = O(h^3), \tag{3.3.30}$$

so by Talay and Tubaro (1990) the global weak error is $O(h^2)$, which implies that:

$$\mathbb{E}\big[\varphi(X(T))\big] - \mathbb{E}\big[\varphi(X_N)\big] = \mathcal{O}(h^2). \tag{3.3.31}$$

Hence $SDATF_1$ achieves weak order 2.

Lemma 3.1. *$SDATF_1$ is mean-square stable*

Proof:

Applying (3.2.2) to (1.2.1),

$$f(X) = \lambda X, \quad g(X) = \mu X, \quad f'(X) = \lambda, \quad g'(X) = \mu, \quad G(X) = \lambda^2 X.$$

Substituting into the $ATSDF_1$ (3.2.5) yields:

Isolating X_{n+1} terms:

Let ρ be the shift operator, that is

$$X_{n+1} = \rho^j X_n, \quad j = 0, 1, \dots, k.$$

Then (3.3.34), becomes

Factorizing X_n, and let $\xi = \Delta W_n$.

$$\rho X_n \Big[1 - h\lambda\beta_k - h^2\theta_k\lambda^2 - \mu\gamma_k\,\xi - \tfrac{1}{2}\mu^2\eta_k(\xi^2 - h)\Big] \\ = \Big[1 + h\lambda \sum_{j=0}^{k-1} \beta_j\rho + \mu\,\xi \sum_{j=0}^{k-1} \gamma_j\rho + \tfrac{1}{2}\mu^2(\xi^2 - h) \sum_{j=0}^{k-1} \eta_j\rho\Big] X_n. \tag{3.3.32}$$

(3.3.37) reduces to

$$\rho\Big[1 - h\lambda\beta_k - h^2\theta_k\lambda^2 - \mu\gamma_k\,\xi - \tfrac{1}{2}\mu^2\eta_k(\xi^2 - h)\Big] \\ = 1 + h\lambda \sum_{j=0}^{k-1} \beta_j\rho + \mu\,\xi \sum_{j=0}^{k-1} \gamma_j\rho + \tfrac{1}{2}\mu^2(\xi^2 - h) \sum_{j=0}^{k-1} \eta_j\rho. \tag{3.3.33}$$

Bring all terms in (3.3.38) to the left-hand side to obtain

$$P(\rho;\xi) = \rho a_k(\xi) - \sum_{j=0}^{k-1} b_j(\xi)\rho = 0, \tag{3.3.34}$$

where

$$a_k(\xi) = 1 - h\lambda\beta_k - h^2\theta_k\lambda^2 - \mu\gamma_k\,\xi - \tfrac{1}{2}\mu^2\eta_k(\xi^2 - h), \tag{3.3.35}$$

$$b_j(\xi) = h\lambda\beta_j + \mu\xi\gamma_j + \tfrac{1}{2}\mu^2(\xi^2 - h)\eta_j, \quad 0 \le j \le k-2, \tag{3.3.36}$$

$$b_{k-1}(\xi) = 1 + h\lambda\beta_{k-1} + \mu\xi\gamma_{k-1} + \tfrac{1}{2}\mu^2(\xi^2 - h)\eta_{k-1}. \tag{3.3.37}$$

For each fixed realization of ξ, the method is mean square stable if all roots of $P(\rho;\xi) = 0$ satisfy $|\rho| < 1$.

When $k = 1$, the equation reduces to

$$\rho\, a_1(\xi) = b_0(\xi), \tag{3.3.38}$$

so the *stability function* is

$$R(\xi) = \frac{b_0(\xi)}{a_1(\xi)}, \tag{3.3.39}$$

with

$$a_1(\xi) = 1 - h\lambda\beta_1 - h^2\theta_1\lambda^2 - \mu\gamma_1\xi - \tfrac{1}{2}\mu^2\eta_1(\xi^2 - h), \tag{3.3.40}$$

$$b_0(\xi) = 1 + h\lambda\beta_0 + \mu\gamma_0\xi + \tfrac{1}{2}\mu^2\eta_0(\xi^2 - h). \tag{3.3.41}$$

Taking the square of the expectation of (3.3.42) and applying (3.3.24) gives:

$$\mathbb{E}[\,|R(\xi)|^2\,] < 1. \tag{3.3.42}$$

Hence $SDATF_1$ is mean-square stable whenever (3.3.61) holds.

Lemma 3.2. *$SDATF_1$ (3.2.2) is Mean-square zero-stable*

proof:

Set the error $e_n = X_n - \widetilde{X}_n$. Subtract the two instances of (3.2.2) to obtain the error recursion. Using the notation (for the two solutions the same coefficients apply), Buckwar and Winkler (2007)

$$A_n = \sum_{j=0}^{k} \beta_j f_{n+j-k+1}, \quad B_n = \sum_{j=0}^{k} \gamma_j g_{n+j-k+1}, \quad C_n = \tfrac{1}{2}\sum_{j=0}^{k} \eta_j (g'g)_{n+j-k+1},$$

and similarly $\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n$ for $\widetilde{X}$, yields:

$$e_{n+1} = e_n + h(A_n - \widetilde{A}_n) + (B_n - \widetilde{B}_n)\Delta W_n + (C_n - \widetilde{C}_n)((\Delta W_n)^2 - h) + h^2\theta_k(G_{n+1} - \widetilde{G}_{n+1}). \tag{3.3.43}$$

Square the norm and take expectations yields:

$$\mathbb{E}\|e_{n+1}\|^2 \leq 4\mathbb{E}\|e_n\|^2 + 4h^2\mathbb{E}\|A_n - \widetilde{A}_n\|^2 + 4\mathbb{E}[\|B_n - \widetilde{B}_n\|^2(\Delta W_n)^2] + 4\mathbb{E}[\|C_n - \widetilde{C}_n\|^2((\Delta W_n)^2 - h)^2] + 4h^4\theta_k^2\mathbb{E} \tag{3.3.44}$$

Applying (3.3.24) yields:. Thus

$$\begin{aligned}\mathbb{E}\|e_{n+1}\|^2 \leq 4\mathbb{E}\|e_n\|^2 + 4h^2\mathbb{E}\|A_n - \widetilde{A}_n\|^2 + 4h\,\mathbb{E}\|B_n - \widetilde{B}_n\|^2 + 8h^2\,\mathbb{E}\|C_n - \widetilde{C}_n\|^2 \\ + 4h^4\theta_k^2\mathbb{E}\|G_{n+1} - \widetilde{G}_{n+1}\|^2.\end{aligned} \tag{3.3.45}$$

Apply Lipschitz and linear growth bound component wise to the delayed combinations: because each A_n, B_n, C_n, G_{n+1} is a finite linear combination of $f, g, g'g$, and (if present) Lf evaluated at a finite fixed set of past times, there is a constant L_1 (depending only on the stencil coefficients and on L) such that for all n,

$$\mathbb{E}\|A_n - \widetilde{A}_n\|^2 + \mathbb{E}\|B_n - \widetilde{B}_n\|^2 + \mathbb{E}\|C_n - \widetilde{C}_n\|^2 + \mathbb{E}\|G_{n+1} - \widetilde{G}_{n+1}\|^2 \leq L_1 \max_{0\leq j\leq k-1} \mathbb{E}\|e_{n+j-k+1}\|^2. \tag{3.3.46}$$

(The maximum over the relevant past indices appears because the combinations use values from at most the last k steps.)

Hence, there exists constants $C_1, C_2 > 0$ such that

$$\mathbb{E}\|e_{n+1}\|^2 \leq (4 + C_1 h)\mathbb{E}\|e_n\|^2 + C_2 h \max_{0\leq j\leq k-1} \mathbb{E}\|e_{n+j-k+1}\|^2. \tag{3.3.47}$$

Rearrange to a standard companion inequality on windows of length k.

$$E_n = \max_{n-k+1 \le m \le n} \mathbb{E}\|e_m\|^2 \qquad (n \ge k-1). \tag{3.3.48}$$

From (3.3.65) there exists $h_0 > 0$ and constant $C > 0$ such that for all $0 < h \le h_0$,

$$E_{n+1} \le (1 + Ch)E_n \qquad \text{for all } n \ge k-1. \tag{3.3.49}$$

Iterating gives

$$E_n \le (1 + Ch)^{n-k+1} E_{k-1}. \tag{3.3.50}$$

For any fixed final time T with $n \le T/h$, this yields

$$E_n \le \exp(CT)\, E_{k-1}, \tag{3.3.51}$$

so the error remains bounded uniformly in n (for n up to T/h) and in particular

$$\mathbb{E}\|e_n\|^2 \le C' \max_{0 \le j \le k-1} \mathbb{E}\|e_j\|^2, \tag{3.3.52}$$

where $C' = \exp(CT)$ is independent of h for h small. This proves mean-square zero-stability.

Lemma 3.3. *$SDATF_1$ is Mean-Square A-Stable*

Proof

Applying the $SDATF_1$ to the test equation (1.2.1), yields the stability function (3.3.42). The plots of stability domain are done with the aid of boundary locus method, Lambert (1991). The stability regions for $SDATF_1$ (3.2.2) for $k = 2, 3, ..., 12$ are presented in fig 3.1 - 3.9. The unbounded regions in the fig 3.1-3.11 indicate the region of absolute stability of (3.2.2).

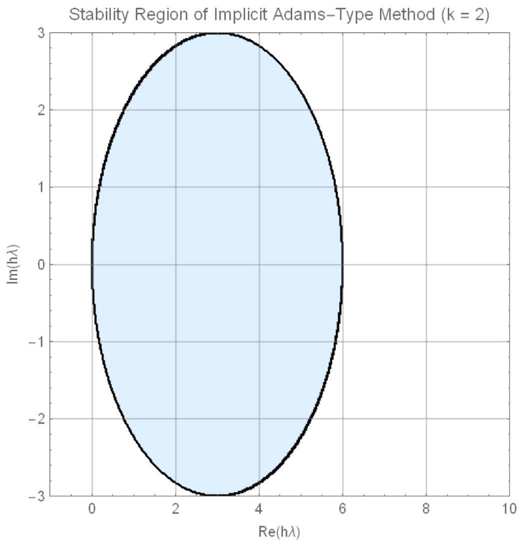

Figure 3.1: Stability plot of SDATF1 for k=2

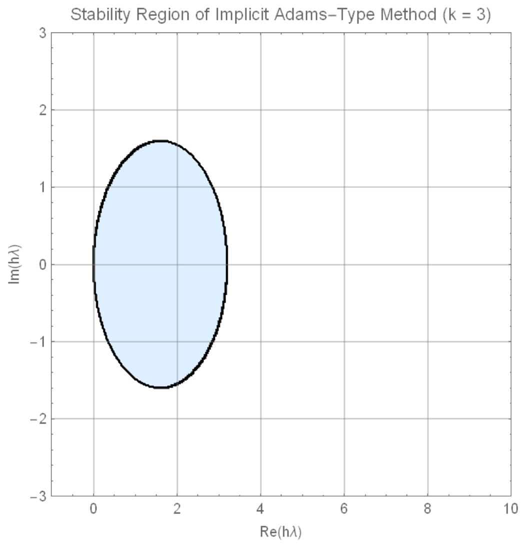

Figure 3.2: Stability plot of SDATF1 for k=3

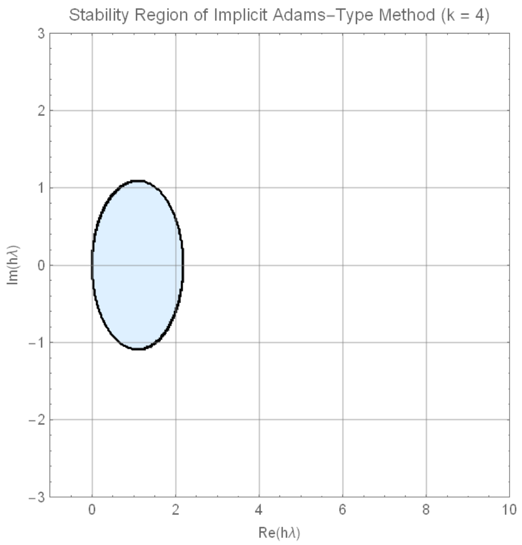

Figure 3.3: Stability plot of SDATF1 for k=4

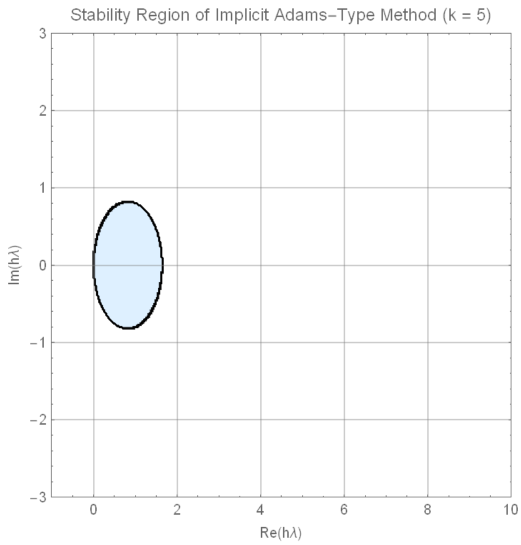

Figure 3.4: Stability plot of SDATF1 for k=5

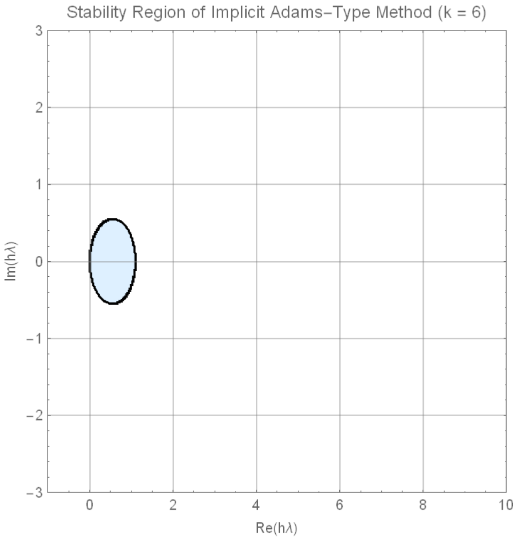

Figure 3.5: Stability plot of SDATF1 for k=6

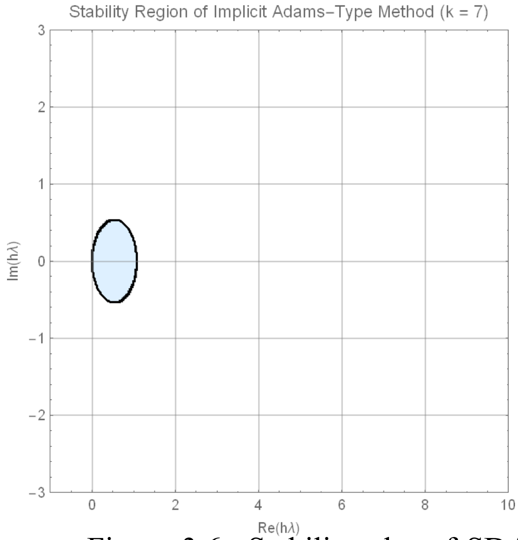

Figure 3.6: Stability plot of SDATF1 for k=7

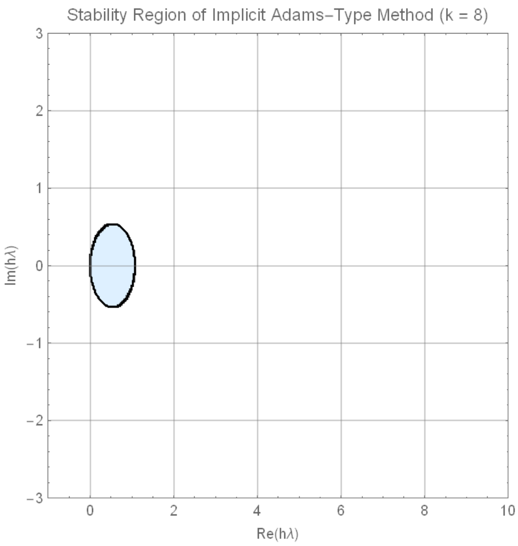

Figure 3.7: Stability plot of SDATF1 for k=8

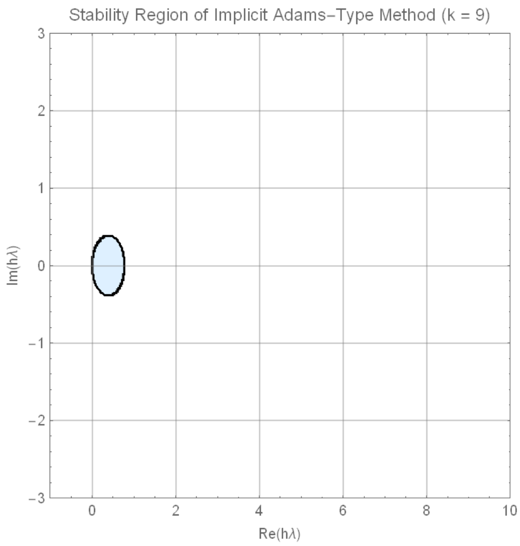

Figure 3.8: Stability plot of SDATF1 for k=9

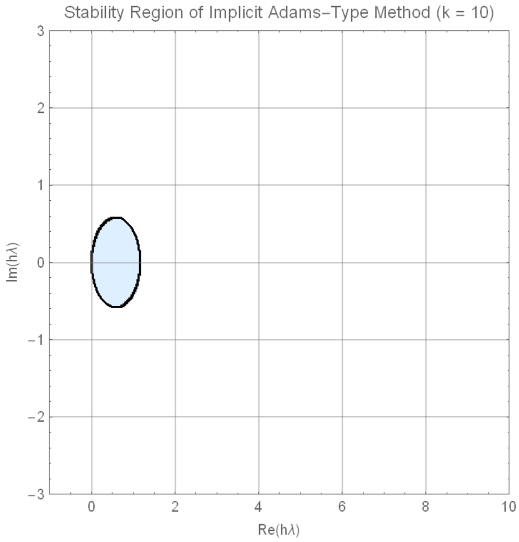

Figure 3.9: Stability plot of SDATF1 for k=10

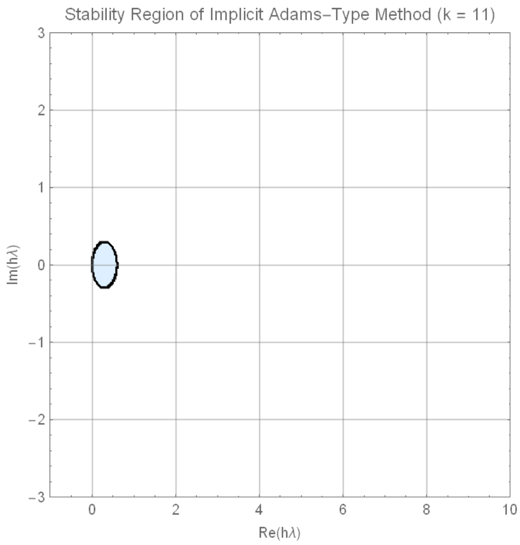

Figure 3.10: Stability plot of SDATF1 for k=11

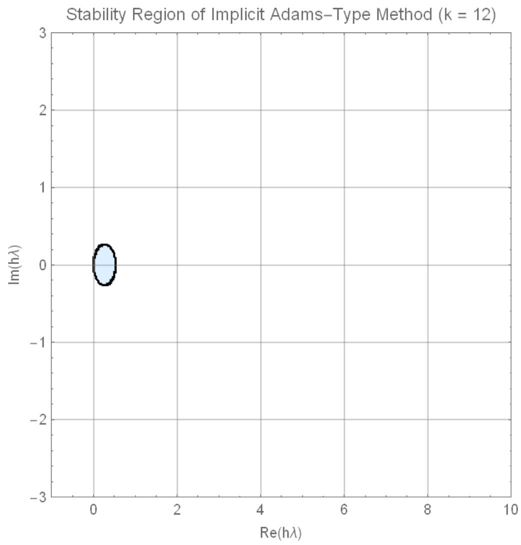

Figure 3.11: Stability plot of SDATF1 for k=12

The boundary locus plots in fig. (3.1)-(3.11), shows that $SDATF_1$ for step size $k = 2$ to $k = 12$, the stability region includes the entire left of the complex plane, meaning that $SDATF_1$ is A-stable. The bounded region is the region of instability. In fig (3.1), the bounded region is between $[0-6]$, in fig. (3.2), the bounded region is between $[0-3]$, in fig. (3.3), the bounded region is between $[0-2.1]$, in fig (3.4), the bounded region is between $[0-1.8]$, in fig. (3.5), the bounded region is between $[0-1]$. As the step size increases, the instability(bounded) region shrinks, and the stability region increases.

3.4 Construction of Second Derivative Adams Type Formula 2 ($SDATF_2$)

If in (3.2.2), $\theta_j = 0$, for all $j = 0, 1, \cdots$. Then the second family of method is given as:

The Linear difference operator associated with (3.4.1) is given by:

$$\begin{aligned} L(X(t_n), h) := X(t_n + h) - X(t_n) - h \sum_{j=0}^{k} \beta_j f(X(t_n + h + \tau_j)) \\ - \Delta W(t_n + h) \sum_{j=0}^{k} \gamma_j g(X(t_n + h + \tau_j)) \\ - \tfrac{1}{2}\Big[(\Delta W(t_n + h))^2 - h\Big] \sum_{j=0}^{k} \eta_j (L_1 g)(X(t_n + h + \tau_j)), \end{aligned} \tag{3.4.1}$$

Taylor expansions about (t_n, X_n) is as follows:

$$X(t_n + h) = X(t_n) + hf_n + \tfrac{h^2}{2} G_n + \tfrac{h^3}{6} H_n + \tfrac{h^4}{24} J_n + O(h^5), \tag{3.4.2}$$

$$f(X(t_n + h + \tau_j)) = f_n + (j)h\, G_n + \tfrac{(jh)^2}{2} H_n + \tfrac{(jh)^3}{6} J_n + O(h^4). \tag{3.4.3}$$

Substituting (3.4.3) and (3.4.4) into (3.4.2), collect powers of h, and match independent derivatives. This yields the deterministic moment (consistency) conditions:

$$\beta_0 + \beta_1 = 1, \qquad -\beta_0 = -\tfrac{1}{2}. \tag{3.4.4}$$

For the stochastic terms, Ito Taylor expansion of (3.4.2) about (t_n, X_n) gives:

$$g(X(t_n + h + \tau_j)) = g(t_n) + \tau_j (L_0 g)(t_n) + O(h^2), \tag{3.4.5}$$

$$(L_1 g)(X(t_n + h + \tau_j)) = (L_1 g(t_n)) + O(h). \tag{3.4.6}$$

Then the Linear operator of the stochastic part is:

$$L(X(t_n); h) = \Delta W \Big(\sum_{j=0}^{k} \gamma_j \Big) g(t_n) + \tfrac{1}{2}((\Delta W)^2 - h) \Big(\sum_{j=0}^{k} \eta_j \Big) (L_1 g(t_n)) + O(h^{3/2}). \tag{3.4.7}$$

Matching Ito–Taylor yields:

$$\sum_{j=0}^{k} \gamma_j = 1, \qquad \sum_{j=0}^{k} \eta_j = 1.$$

For fully implicit choices concentrated at $j = k$ this condition holds automatically (since $j - k = 0$ for $j = k$).

$$\gamma_j = \begin{cases} 0, & j < k, \\ 1, & j = k, \end{cases} \qquad \eta_j = \begin{cases} 0, & j < k, \\ 1, & j = k, \end{cases}. \tag{3.4.8}$$

3.5 Coefficients of $SDATF_2$

For $k = 1$,

The moment equation $m = 0, 1$ from (3.4.5) gives: $\beta_0 = \frac{1}{2}, \beta_1 = \frac{1}{2}$,

With $\gamma_0 = 0,\ \gamma_1 = 1,\ \eta_0 = 0,\ \eta_1 = 1$, the resulting scheme is

For $k = 2$. The moment equations $m = 0, 1, 2$ are

$$\beta_0 + \beta_1 + \beta_2 = 1,$$

$$-2\beta_0 - \beta_1 = -\tfrac{1}{2},$$

$$4\beta_0 + \beta_1 = \tfrac{1}{3}.$$

Solving the system of equation yields:

$$\beta_0 = \tfrac{1}{2} - 2\beta_0, \quad 4\beta_0 + (\tfrac{1}{2} - 2\beta_0) = \tfrac{1}{3} \;\Rightarrow\; 2\beta_0 = -\tfrac{1}{6} \;\Rightarrow\; \beta_0 = -\tfrac{1}{12}.$$

Then $\beta_1 = \frac{2}{3}$, and $\beta_2 = 1 - \beta_2 - \beta_1 = \frac{5}{12}$. Hence

$$\beta_0 = -\tfrac{1}{12}, \quad \beta_1 = \tfrac{2}{3}, \quad \beta_2 = \tfrac{5}{12}$$

With $\gamma_0 = \gamma_1 = 0,\ \gamma_2 = 1,\ \eta_0 = \eta_1 = 0,\ \eta_2 = 1$, the scheme becomes:

$$\begin{aligned} X_{n+1} &= X_n + h\big(-\tfrac{1}{12} f_{n-1} + \tfrac{2}{3} f_n + \tfrac{5}{12} f_{n+1}\big) \\ &\quad + g_{n+1}\,\Delta W_n + \tfrac{1}{2} g'_{n+1} g_{n+1}\big((\Delta W_n)^2 - h\big). \end{aligned} \tag{3.5.1}$$

For $k = 3$,

$$\beta_0 = \tfrac{1}{24},\ \beta_1 = -\tfrac{5}{24},\ \beta_2 = \tfrac{19}{24},\ \beta_3 = \tfrac{3}{8},$$

Diffusion coefficients

$\gamma_0 = \gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1,$

$\eta_0 = \eta_1 = \eta_2 = 0, \quad \eta_3 = 1$.

The scheme is:

$$X_{n+1} = X_n + h\left(\tfrac{1}{24}f_{n-2} - \tfrac{5}{24}f_{n-1} + \tfrac{19}{24}f_n + \tfrac{3}{8}f_{n+1}\right) + g_{n+1}\,\Delta W_n + \tfrac{1}{2}g'_{n+1}g_{n+1}\left[(\Delta W_n)^2 - h\right]. \tag{3.5.2}$$

For k=4

$$\begin{aligned} X_{n+1} = X_n + h\Big(&-\tfrac{19}{720}f_{n-3} + \tfrac{53}{360}f_{n-2} - \tfrac{11}{30}f_{n-1} + \tfrac{323}{360}f_n + \tfrac{251}{720}f_{n+1}\Big) \\ &+ g_{n+1}\,\Delta W_n + \tfrac{1}{2}\,g'_{n+1}g_{n+1}\big((\Delta W_n)^2 - h\big), \end{aligned} \tag{3.5.3}$$

For k=5

$$\begin{aligned} X_{n+1} = X_n + h\Big(&\tfrac{3}{160}f_{n-4} - \tfrac{173}{1440}f_{n-3} + \tfrac{241}{720}f_{n-2} - \tfrac{133}{240}f_{n-1} + \tfrac{1427}{1440}f_n + \tfrac{95}{288}f_{n+1}\Big) \\ &+ g_{n+1}\,\Delta W_n + \tfrac{1}{2}\,g'_{n+1}g_{n+1}\big((\Delta W_n)^2 - h\big), \end{aligned} \tag{3.5.4}$$

For k=6

$$\begin{aligned} X_{n+1} = X_n + h\Big(&-\tfrac{863}{60480}f_{n-5} + \tfrac{263}{2520}f_{n-4} - \tfrac{6737}{20160}f_{n-3} + \tfrac{586}{945}f_{n-2} \\ &- \tfrac{15487}{20160}f_{n-1} + \tfrac{2713}{2520}f_n + \tfrac{19087}{60480}f_{n+1}\Big) \\ &+ g_{n+1}\,\Delta W_n + \tfrac{1}{2}\,g'_{n+1}g_{n+1}\big((\Delta W_n)^2 - h\big), \end{aligned} \tag{3.5.5}$$

For k=7

$$\begin{aligned} X_{n+1} = X_n + h\Big(&\tfrac{275}{24192}f_{n-6} - \tfrac{11351}{120960}f_{n-5} + \tfrac{1537}{4480}f_{n-4} - \tfrac{88547}{120960}f_{n-3} \\ &+ \tfrac{123133}{120960}f_{n-2} - \tfrac{4511}{4480}f_{n-1} + \tfrac{139849}{120960}f_n + \tfrac{5257}{17280}f_{n+1}\Big) \\ &+ g_{n+1}\,\Delta W_n + \tfrac{1}{2}\,g'_{n+1}g_{n+1}\big((\Delta W_n)^2 - h\big), \end{aligned} \tag{3.5.6}$$

For k=8

$$X_{n+1} = X_n + h\Big(-\tfrac{33953}{3628800} f_{n-7} + \tfrac{156437}{1814400} f_{n-6} - \tfrac{645607}{1814400} f_{n-5} + \tfrac{1573169}{1814400} f_{n-4} - \tfrac{31457}{22680} f_{n-3} + \tfrac{2797679}{1814400} f_{n-2} - \tfrac{2302297}{1814400} f_{n-1} + \tfrac{2233547}{1814400} f_n + \tfrac{1070017}{3628800} f_{n+1}\Big) + g_{n+1}\,\Delta W_n + \tfrac{1}{2}\, g'_{n+1} g_{n+1}\big((\Delta W_n)^2 - h\big), \tag{3.5.7}$$

For k=9

$$X_{n+1} = X_n + h\Big(\tfrac{8183}{1036800} f_{n-8} - \tfrac{116687}{1451520} f_{n-7} + \tfrac{335983}{907200} f_{n-6} - \tfrac{462127}{453600} f_{n-5} + \tfrac{6755041}{3628800} f_{n-4} - \tfrac{8641823}{3628800} f_{n-3} + \tfrac{200029}{90720} f_{n-2} - \tfrac{1408913}{907200} f_{n-1} + \tfrac{9449717}{7257600} f_n + \tfrac{25713}{89600} f_{n+1}\Big) + g_{n+1}\,\Delta W_n + \tfrac{1}{2}\, g'_{n+1} g_{n+1}\big((\Delta W_n)^2 - h\big), \tag{3.5.8}$$

For k=10

$$X_{n+1} = X_n + h\Big(-\tfrac{3250433}{479001600} f_{n-9} + \tfrac{9071219}{119750400} f_{n-8} - \tfrac{12318413}{31933440} f_{n-7} + \tfrac{23643791}{19958400} f_{n-6} - \tfrac{21677723}{8870400} f_{n-5} + \tfrac{2227571}{623700} f_{n-4} - \tfrac{33765029}{8870400} f_{n-3} + \tfrac{12051709}{3991680} f_{n-2} - \tfrac{296725183}{159667200} f_{n-1} + \tfrac{164046413}{119750400} f_n + \tfrac{26842253}{95800320} f_{n+1}\Big) + g_{n+1}\,\Delta W_n + \tfrac{1}{2}\, g'_{n+1} g_{n+1}\big((\Delta W_n)^2 - h\big). \tag{3.5.9}$$

k	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
1	$\frac{1}{2}$	$\frac{1}{2}$									
2	$-\frac{1}{12}$	$\frac{2}{3}$	$\frac{5}{12}$								
3	$\frac{1}{24}$	$-\frac{5}{24}$	$\frac{19}{24}$	$\frac{3}{8}$							
4	$-\frac{19}{720}$	$\frac{53}{360}$	$-\frac{11}{30}$	$\frac{323}{360}$	$\frac{251}{720}$						
5	$\frac{3}{160}$	$-\frac{173}{1440}$	$\frac{241}{720}$	$-\frac{133}{240}$	$\frac{1427}{1440}$	$\frac{95}{288}$					
6	$-\frac{863}{60480}$	$\frac{263}{2520}$	$-\frac{6737}{20160}$	$\frac{586}{945}$	$-\frac{15487}{20160}$	$\frac{2713}{2520}$	$\frac{19087}{60480}$				
7	$\frac{275}{24192}$	$-\frac{11351}{120960}$	$\frac{1537}{4480}$	$-\frac{88547}{120960}$	$\frac{123133}{120960}$	$-\frac{4511}{4480}$	$\frac{139849}{120960}$	$\frac{5257}{17280}$			
8	$-\frac{33953}{3628800}$	$\frac{156437}{1814400}$	$-\frac{645607}{1814400}$	$\frac{1573169}{1814400}$	$-\frac{31457}{22680}$	$\frac{2797679}{1814400}$	$-\frac{2302297}{1814400}$	$\frac{2233547}{1814400}$	$\frac{1070017}{3628800}$		
9	$\frac{8183}{1036800}$	$-\frac{116687}{1451520}$	$\frac{335983}{907200}$	$-\frac{462127}{453600}$	$\frac{6755041}{3628800}$	$-\frac{8641823}{3628800}$	$\frac{200029}{90720}$	$-\frac{1408913}{907200}$	$\frac{9449717}{7257600}$	$\frac{25713}{89600}$	
10	$-\frac{3250433}{479001600}$	$\frac{9071219}{119750400}$	$-\frac{12318413}{31933440}$	$\frac{23643791}{19958400}$	$-\frac{21677723}{8870400}$	$\frac{2227571}{623700}$	$-\frac{33765029}{8870400}$	$\frac{12051709}{3991680}$	$-\frac{296725183}{159667200}$	$\frac{164046413}{119750400}$	$\frac{26842253}{95800320}$

k	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}
1	0	1									
2	0	0	1								
3	0	0	0	1							
4	0	0	0	0	1						
5	0	0	0	0	0	1					
6	0	0	0	0	0	0	1				
7	0	0	0	0	0	0	0	1			
8	0	0	0	0	0	0	0	0	1		
9	0	0	0	0	0	0	0	0	0	1	
10	0	0	0	0	0	0	0	0	0	0	1

k	η_0	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}
1	0	1									
2	0	0	1								
3	0	0	0	1							
4	0	0	0	0	1						
5	0	0	0	0	0	1					
6	0	0	0	0	0	0	1				
7	0	0	0	0	0	0	0	1			
8	0	0	0	0	0	0	0	0	1		
9	0	0	0	0	0	0	0	0	0	1	
10	0	0	0	0	0	0	0	0	0	0	1

Table 3.2: $SDATF_2$ coefficients for $k = 1 \dots 10$.

3.6 Order and Stability of $SDATF_2$

Proposition 3.3. *$SDATF_2$ (3.4.1) is of strong order 1.*

proof

Assume $f, g \in C^3(\mathbb{R}^n)$. Also assume f, g and satisfy Lipschitz and linear growth condition,

Then the one step map of $SDATF_2$ (3.4.1) is given by:

$$\begin{aligned} X_{n+1} = X_n + h \sum_{j=0}^{k} \beta_j \, f(X_{n+j}) + \sum_{j=0}^{k} \gamma_j \, g(X_{n+j}) \, \Delta W_n \\ + \tfrac{1}{2} \sum_{j=0}^{k} \eta_j \, g'(X_{n+j})((\Delta W_n)^2 - h) \end{aligned} \tag{3.6.1}$$

The local truncation error (lte), is obtained by equating (3.3.8) and (3.6.1) as:

$$\begin{aligned} \tau_{n+1} = \Big(f_n - \sum_{j=0}^{k} \beta_j f_{n+j}\Big)h + \Big(g_n - \sum_{j=0}^{k} \gamma_j g_{n+j}\Big)\Delta W_n \\ + \Big(\tfrac{1}{2}(g_n g_n') - \tfrac{1}{2} \sum_{j=0}^{k} \eta_j g_{n+j}'\Big)((\Delta W_n)^2 - h) \\ + (g_n f_n') \, I_{(1,0)} + (L^0 g_n) \, I_{(0,1)} + \widetilde{R}_{n+k}. \end{aligned} \tag{3.6.2}$$

$$\begin{aligned} \tau_{n+1} = \Big(1 - \sum_{j=0}^{k-1} \beta_j\Big) f_n h + \Big(1 - \sum_{j=0}^{k-1} \gamma_j\Big) g_n \Delta W_n \\ + \Big((1 - \sum_{j=0}^{k-1} \eta_j\Big) \tfrac{1}{2} g_n g_n'((\Delta W_n)^2 - h) \\ + (g_n f_n') \, I_{(1,0)} + (L^0 g_n) \, I_{(0,1)} + \widetilde{R}_{n+k}, \end{aligned} \tag{3.6.3}$$

applying (3.3.13), (3.6.3) reduces to:

$$\tau_{n+1} = (g_n f_n') \, I_{(1,0)} + (L^0 g_n) \, I_{(0,1)} + \widetilde{R}_{n+k}. \tag{3.6.4}$$

Taking the norm of both sides of (3.6.4) and squaring the resultant gives:

$$\|\tau_{n+1}\|^2 = \|(g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)} + \widetilde{R}_{n+k}\|^2. \tag{3.6.5}$$

$$\|\tau_{n+1}\|^2 \leq \|(g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)}\|^2 + \|\widetilde{R}_{n+k}\|^2. \tag{3.6.6}$$

Taking expectation of both sides of (3.6.6) to obtain:

$$\mathbb{E}\|\tau_{n+1}\|^2 \leq \mathbb{E}\|\Big(\tfrac{1}{2}(L^0 f_n) - \theta_k G_{n+1}\Big)h^2\|^2 + \mathbb{E}\|(g_n f_n')\, I_{(1,0)} + (L^0 g_n)\, I_{(0,1)}\|^2 + \mathbb{E}\|\widetilde{R}_{n+k}\|^2. \tag{3.6.7}$$

Recall (3.3.4), (3.6.7) becomes:

$$\mathbb{E}\|\tau_{n+1}\|^2 \leq Ch^3. \tag{3.6.8}$$

The global error e_{n+1} Incured by approximating the solution (1.1.1) using $SDATF_1$ is obtained as:

$$e_{n+1} = \tau_{n+1} + X(t_n) - X_n. \tag{3.6.9}$$

Taking expectation of both sides of (3.6.9) yield:

$$\mathbb{E}|e_{n+1}|^2 \leq (1 + Ch)\, \mathbb{E}|e_n|^2 + C\, \mathbb{E}|\tau_{n+1}|^2. \tag{3.6.10}$$

Substituting (3.6.8) into (3.6.10) gives:

$$\mathbb{E}|e_n|^2 \leq (1 + Ch)\, \mathbb{E}|e_{n-1}|^2 + Ch^3. \tag{3.6.11}$$

By discrete Grönwall's lemma (Platen and Kloeden (1992)), uniformly for $t_n \le T$,

$$\mathbb{E}|e_n|^2 \le Ch^2.$$

Therefore, $SDATF_2$ is of strong order 1 and the proof is established.

Proposition 3.4. *$SDATF_2$ is of weak order 2*

Proof:

Assuming $f, g \in C_P^6(\mathbb{R})$ (that is, they are sufficiently smooth with polynomial growth). Let $\varphi \in C_P^4(\mathbb{R})$ be an arbitrary test function. We denote $t_{n+1} = t_n + h$, $\Delta W_n = W(t_{n+1}) - W(t_n)$, and $X_n \approx X(t_n)$. For convenience, set $f_n = f(X_n)$, $g_n = g(X_n)$, etc.

$$\begin{aligned} S_\beta(r) &= \sum_{j=0}^{k} \beta_j (j-k+1)^r, \\ S_\gamma(r) &= \sum_{j=0}^{k} \gamma_j (j-k+1)^r, \\ S_\eta(0) &= \sum_{j=0}^{k} \eta_j. \end{aligned} \tag{3.6.12}$$

In particular,

$$S_\beta(0) = \sum_{j=0}^{k} \beta_j, \quad S_\gamma(0) = \sum_{j=0}^{k} \gamma_j. \tag{3.6.13}$$

By the Itô–Taylor expansion, Kloeden and Platen (1992), Talay and Tubaro (1990),

$$\mathbb{E}[\varphi(X(t_{n+1})) \,|\, X_n] = \varphi(X_n) + hL\varphi(X_n) + \frac{h^2}{2} L^2 \varphi(X_n) + \mathcal{O}(h^3), \tag{3.6.14}$$

where the infinitesimal generator L acts as

$$L\psi = f\psi' + \frac{1}{2} g^2 \psi''. \tag{3.6.15}$$

This represents the exact one–step expansion to be matched up to order $\mathcal{O}(h^2)$.

The numerical increment

$$\Xi = X_{n+1} - X_n. \tag{3.6.16}$$

From the numerical scheme,

$$\Xi = h\sum_{j=0}^{k}\beta_j f_{n+j-k+1} + \sum_{j=0}^{k}\gamma_j g_{n+j-k+1}\Delta W_n + \frac{1}{2}\sum_{j=0}^{k}\eta_j g'_{n+j-k+1}g_{n+j-k+1}[(\Delta W_n)^2 - h]. \tag{3.6.17}$$

The Wiener increment satisfies

$$\mathbb{E}[\Delta W_n] = 0, \quad \mathbb{E}[(\Delta W_n)^2] = h, \quad \mathbb{E}[(\Delta W_n)^3] = 0, \quad \mathbb{E}[(\Delta W_n)^4] = 3h^2.$$

We require the conditional moments $\mathbb{E}[\Xi^m \mid X_n]$ up to terms contributing through $\mathcal{O}(h^2)$.

Expanding coefficients around X_n:

$$\begin{aligned} f_{n+j-k+1} &= f_n + (j-k+1)h(Lf)_n + \mathcal{O}(h^2), \\ g_{n+j-k+1} &= g_n + (j-k+1)h(Lg)_n + \mathcal{O}(h^2), \\ g'_{n+j-k+1}g_{n+j-k+1} &= (g'g)_n + \mathcal{O}(h). \end{aligned} \tag{3.6.18}$$

$$\begin{aligned} \mathbb{E}[\Xi \mid X_n] &= h\sum_{j=0}^{k}\beta_j(f_n + (j-k+1)h(Lf)_n) + \frac{1}{2}\sum_{j=0}^{k}\eta_j(g'g)_n\,\mathbb{E}[(\Delta W_n)^2 - h] + \mathcal{O}(h^3) \\ &= hS_\beta(0)f_n + h^2S_\beta(1)(Lf)_n + \mathcal{O}(h^3), \end{aligned}$$

since $\mathbb{E}[(\Delta W_n)^2 - h] = 0$.

$$\begin{aligned} \mathbb{E}[\Xi^2 \mid X_n] &= \mathbb{E}\left[\left(\sum_{j=0}^{k}\gamma_j g_{n+j-k+1}\Delta W_n\right)^2\right] + \frac{1}{4}\mathbb{E}\left[\left(\sum_{j=0}^{k}\eta_j(g'g)_{n+j-k+1}[(\Delta W_n)^2 - h]\right)^2\right] + \mathcal{O}(h^3) \\ &= h\,(S_\gamma(0))^2g_n^2 + h^2\left[(S_\beta(0))^2f_n^2 + 2S_\gamma(0)S_\gamma(1)g_n(Lg)_n + \frac{1}{2}(S_\eta(0))^2(g'g)_n^2\right] + \mathcal{O}(h^3). \end{aligned}$$

Using standard moment identities,

$$\mathbb{E}[\Xi^3 \mid X_n] = 3h^2(S_\gamma(0))^2 g_n^2 (S_\beta(0) f_n + S_\eta(0)(g'g)_n) + \mathcal{O}(h^{5/2}),$$
$$\mathbb{E}[\Xi^4 \mid X_n] = 3h^2(S_\gamma(0))^4 g_n^4 + \mathcal{O}(h^{5/2}). \tag{3.6.19}$$

Expanding $\varphi(X_n + \Xi)$ about X_n and taking expectation:

$$\mathbb{E}[\varphi(X_{n+1}) \mid X_n] = \sum_{m=0}^{4} \frac{1}{m!} \varphi^{(m)}(X_n)\, \mathbb{E}[\Xi^m \mid X_n] + \mathcal{O}(h^3). \tag{3.6.20}$$

Substituting the moment expansions and collecting terms gives

$$\mathbb{E}[\varphi(X_{n+1}) \mid X_n] = \varphi(X_n) + hA_1(X_n) + h^2 A_2(X_n) + \mathcal{O}(h^3), \tag{3.6.21}$$

where A_1, A_2 are linear combinations of φ', φ'', $\varphi^{(3)}$, and $\varphi^{(4)}$ with coefficients depending on f_n, g_n and the sums $S_\beta, S_\gamma, S_\eta$.

Matching with the exact Itô–Taylor expansion up to order $\mathcal{O}(h^2)$ requires that the coefficients of φ' and φ'' coincide with those in $L\varphi$ and $\frac{1}{2}L^2\varphi$, respectively. This leads to the algebraic order conditions

$$S_\gamma(1) = \frac{1}{2}, \qquad S_\beta(1) = \frac{1}{2}. \tag{3.6.22}$$

If the order conditions above are satisfied, then

$$\mathbb{E}[\varphi(X(t_{n+1})) \mid X_n] - \mathbb{E}[\varphi(X_{n+1}) \mid X_n] = \mathcal{O}(h^3), \tag{3.6.23}$$

so that the local weak error is of order $\mathcal{O}(h^3)$. By the Talay–Tubaro (1990), the global weak error after $N = T/h$ steps satisfies

$$\left|\mathbb{E}[\varphi(X(T))] - \mathbb{E}[\varphi(X_N)]\right| = \mathcal{O}(h^2). \tag{3.6.24}$$

Hence, $SDATF_2$ achieves weak order 2

Lemma 3.4. *The Adams-type second derivative formula $SDATF_2$ (3.4.1) is mean-square stable.*

Proof:

appling (3.4.1) to test equation (1.2.1) yields:

$$\begin{aligned} X_{n+1} &= X_n + h\lambda \sum_{j=0}^{k} \beta_j X_{n+j-k+1} \\ &\quad + \mu \Delta W_n \sum_{j=0}^{k} \gamma_j X_{n+j-k+1} + \frac{1}{2}\mu^2 \left[(\Delta W_n)^2 - h \right] \sum_{j=0}^{k} \eta_j X_{n+j-k+1}. \end{aligned} \tag{3.6.25}$$

isolating X_{n+1} terms in (3.6.17),

$$\begin{aligned} X_{n+1} &\left[1 - h\lambda\beta_k - \mu\gamma_k \Delta W_n - \frac{1}{2}\mu^2 \eta_k \left((\Delta W_n)^2 - h \right) \right] \\ &= X_n + h\lambda \sum_{j=0}^{k-1} \beta_j X_{n+j-k+1} + \mu \Delta W_n \sum_{j=0}^{k-1} \gamma_j X_{n+j-k+1} \\ &\quad + \frac{1}{2}\mu^2 \left[(\Delta W_n)^2 - h \right] \sum_{j=0}^{k-1} \eta_j X_{n+j-k+1}. \end{aligned} \tag{3.6.26}$$

Characteristic equation associated with the difference equation (3.6.18) gives:

$$\begin{aligned} \rho &\Big[1 - h\lambda\beta_k - \mu\gamma_k \, \xi - \tfrac{1}{2}\mu^2 \eta_k (\xi^2 - h) \Big] \\ &= 1 + h\lambda \sum_{j=0}^{k-1} \beta_j \rho + \mu \, \xi \sum_{j=0}^{k-1} \gamma_j \rho + \tfrac{1}{2}\mu^2 (\xi^2 - h) \sum_{j=0}^{k-1} \eta_j \rho. \end{aligned} \tag{3.6.27}$$

The stability function $P(\rho;\xi)$ associated with $SDATF_2$ (3.4.1) is:

$$P(\rho, \xi) = \rho a_k(\xi) - \sum_{j=0}^{k-1} b_j(\xi) \rho,$$

where,

$$a_k(\xi) = 1 - h\lambda\beta_k - \mu\gamma_k\,\xi - \tfrac{1}{2}\mu^2\eta_k(\xi^2 - h), \tag{3.6.28}$$

$$b_j(\xi) = h\lambda\beta_j + \mu\xi\gamma_j + \tfrac{1}{2}\mu^2(\xi^2 - h)\eta_j, \quad 0 \le j \le k-2, \tag{3.6.29}$$

$$b_{k-1}(\xi) = 1 + h\lambda\beta_{k-1} + \mu\xi\gamma_{k-1} + \tfrac{1}{2}\mu^2(\xi^2 - h)\eta_{k-1}. \tag{3.6.30}$$

The method is mean square stable, if all roots of $P(\rho, \xi) = 0$ satisfy $|\rho(\xi)| < 1$.

When $k = 1$, the equation reduces to

$$\rho\, a_1(\xi) = b_0(\xi),$$

so the *stability function* is

$$R(\xi) = \frac{b_0(\xi)}{a_1(\xi)}, \tag{3.6.31}$$

with

$$a_1(\xi) = 1 - h\lambda\beta_1 - \mu\gamma_1\xi - \tfrac{1}{2}\mu^2\eta_1(\xi^2 - h), \tag{3.6.32}$$

$$b_0(\xi) = 1 + h\lambda\beta_0 + \mu\gamma_0\xi + \tfrac{1}{2}\mu^2\eta_0(\xi^2 - h). \tag{3.6.33}$$

Taking the square expectation of (3.6.23) and applying (3.3.24)gives:

$$\mathbb{E}[\,|R(\xi)|^2\,] < 1, \quad \xi \sim \mathcal{N}(0, h).$$

Hence $SDATF_2$ is mean-square stable whenever (3.6.24) holds.

Lemma 3.5. *$SDATF_2$ is Zero-Stability in Mean-Square Sense*

Proof:

Let X_n and Y_n be two sequences generated by the (3.4.1). Using the same realizations of the Wiener increments $\{\Delta W_n\}$, but possibly with different initial data on the first k starting steps. The error

$$e_n = X_n - Y_n. \tag{3.6.34}$$

$$\begin{aligned} e_{n+1} = e_n &+ h\sum_{j=0}^{k}\beta_j\big(f(X_{n+j-k+1}) - f(Y_{n+j-k+1})\big) \\ &+ \sum_{j=0}^{k}\gamma_j\big(g(X_{n+j-k+1}) - g(Y_{n+j-k+1})\big)\Delta W_n \\ &+ \frac{1}{2}\sum_{j=0}^{k}\eta_j\big(g'g(X_{n+j-k+1}) - g'g(Y_{n+j-k+1})\big)[(\Delta W_n)^2 - h]. \end{aligned}$$

Take conditional expectation (given the sigma-field up to time t_n) of the squared norm and then total expectation. Using $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$ and linearity, we get

$$\begin{aligned} \mathbb{E}[|e_{n+1}|^2] \leq 4\mathbb{E}[|e_n|^2] &+ 4\,\mathbb{E}\Big|h\sum_{j=0}^{k}\beta_j\Delta f_{n+j-k+1}\Big|^2 + 4\,\mathbb{E}\Big|\sum_{j=0}^{k}\gamma_j\Delta g_{n+j-k+1}\Delta W_n\Big|^2 \\ &+ 4\,\mathbb{E}\Big|\frac{1}{2}\sum_{j=0}^{k}\eta_j\Delta(g'g)_{n+j-k+1}((\Delta W_n)^2 - h)\Big|^2, \end{aligned}$$

where

$$\Delta f_m = f(X_m) - f(Y_m), \qquad \Delta g_m = g(X_m) - g(Y_m), \qquad \Delta(g'g)_m = g'g(X_m) - g'g(Y_m).$$

By Lipschitz continuity and boundedness of β_j,

$$\Big|h\sum_{j=0}^{k}\beta_j\Delta f_{n+j-k+1}\Big|^2 \leq h^2\Big(\sum_{j=0}^{k}|\beta_j|\Big)^2 L^2 \max_{0\leq r\leq k-1}|e_{n+r-k+1}|^2.$$

Taking expectations,

$$\mathbb{E}\Big|h\sum_{j=0}^{k}\beta_j\Delta f_{n+j-k+1}\Big|^2 \le C_1h^2 \max_{n-k+1\le m\le n}\mathbb{E}[|e_m|^2].$$

Using $\mathbb{E}[(\Delta W_n)^2]=h$ and independence of ΔW_n from the past,

$$\mathbb{E}\Big|\sum_{j=0}^{k}\gamma_j\Delta g_{n+j-k+1}\Delta W_n\Big|^2 = \mathbb{E}\Big[\Big(\sum_{j=0}^{k}\gamma_j\Delta g_{n+j-k+1}\Big)^2\Big]\mathbb{E}[(\Delta W_n)^2] \le C_2h \max_{n-k+1\le m\le n}\mathbb{E}[|e_m|^2].$$

Since $\mathbb{E}[((\Delta W_n)^2-h)^2]=2h^2$ and $g'g$ is Lipschitz,

$$\mathbb{E}\Big|\frac{1}{2}\sum_{j=0}^{k}\eta_j\Delta(g'g)_{n+j-k+1}((\Delta W_n)^2-h)\Big|^2 \le C_3h^2 \max_{n-k+1\le m\le n}\mathbb{E}[|e_m|^2].$$

Combining and renaming constants,

$$\mathbb{E}[|e_{n+1}|^2] \le 4\mathbb{E}[|e_n|^2] + 4(C_1h^2+C_2h+C_3h^2)\max_{n-k+1\le m\le n}\mathbb{E}[|e_m|^2].$$

Then there exist constants $C>0$ and $h_0>0$ (depending only on Lipschitz constants, coefficient bounds and final time T) such that for all step-sizes $0<h\le h_0$ and all $0\le n\le N=T/h$,

$$\max_{0\le m\le n}\mathbb{E}[|e_m|^2] \le C\max_{0\le m\le k-1}\mathbb{E}[|e_m|^2].$$

$$\mathbb{E}[|e_{n+1}|^2] \le (1+Ch)\max_{n-k+1\le m\le n}\mathbb{E}[|e_m|^2],$$

for some constant C independent of h, provided h is sufficiently small.

Let

$$M_n = \max_{0\le m\le n}\mathbb{E}[|e_m|^2].$$

Then it implies that

$$M_{n+1} \leq (1 + Ch)M_n.$$

Iterating from $n = k - 1$ to N gives

$$M_N \leq (1 + Ch)^{N-k+1} M_{k-1} \leq e^{CT} M_{k-1}.$$

Hence, for all $n \leq N$,

$$\max_{0 \leq m \leq n} \mathbb{E}[|e_m|^2] \leq e^{CT} \max_{0 \leq m \leq k-1} \mathbb{E}[|e_m|^2],$$

which proves the mean-square zero-stability of the method. □

Lemma 3.6. *$SDATF_2$ is A-stable in the mean-square sense*

Proof

Using Boundary locus plot, the stability plots of $SDATF_2$ for k=2 to 10 are given in fig 3.12-3.21, showing the scheme is A-stable.

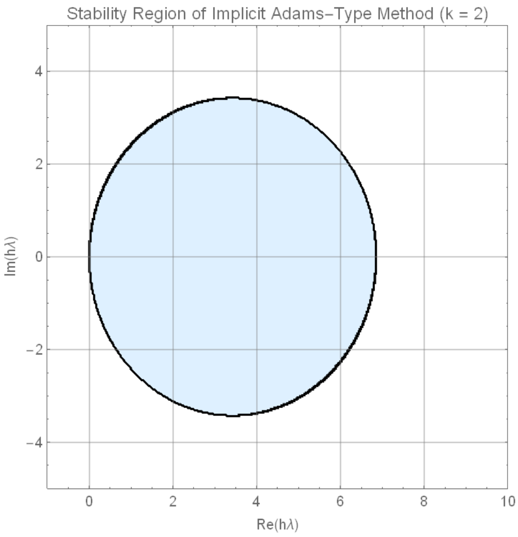

Figure 3.12: Stability plot of SDATF2 for k=2

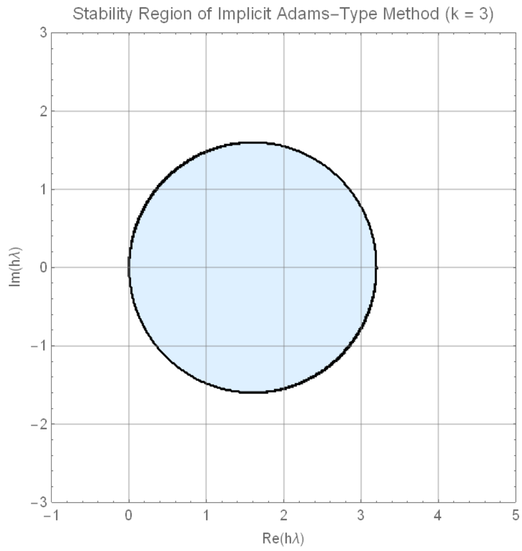

Figure 3.13: Stability plot of SDATF2 for k=3

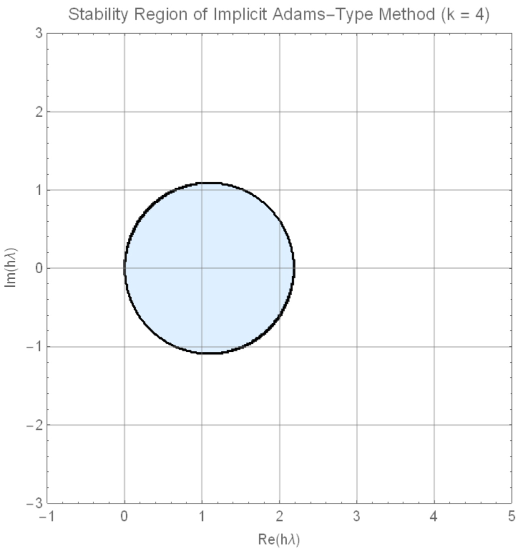

Figure 3.14: Stability plot of SDATF2 for k=4

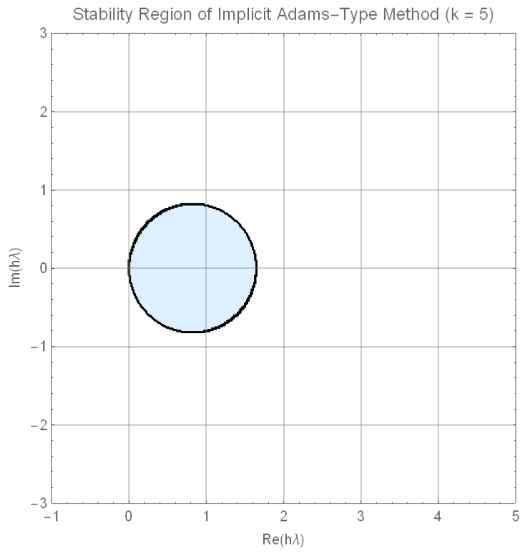

Figure 3.15: Stability plot of SDATF2 for k=5

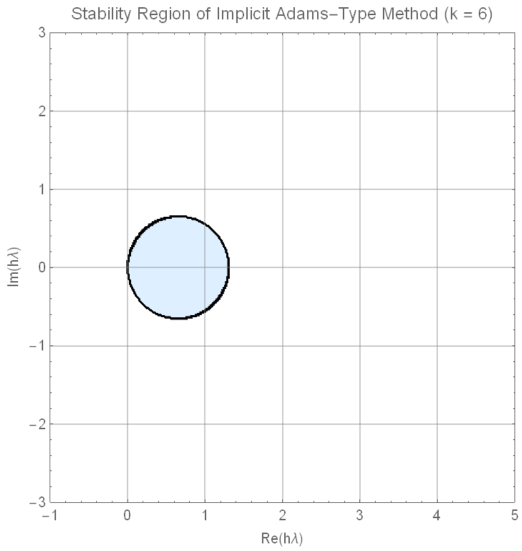

Figure 3.16: Stability plot of SDATF2 for k=6

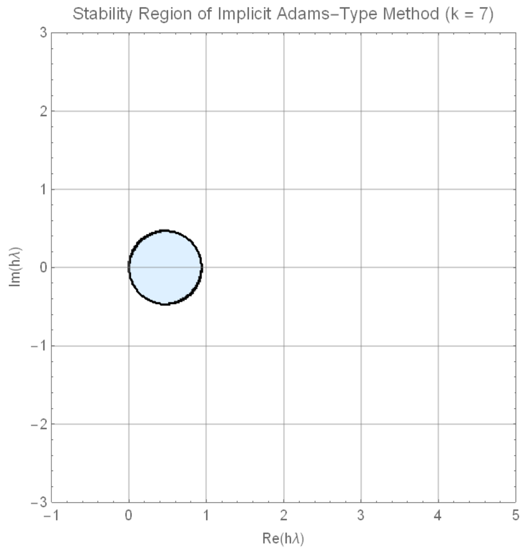

Figure 3.17: Stability plot of SDATF2 for k=7

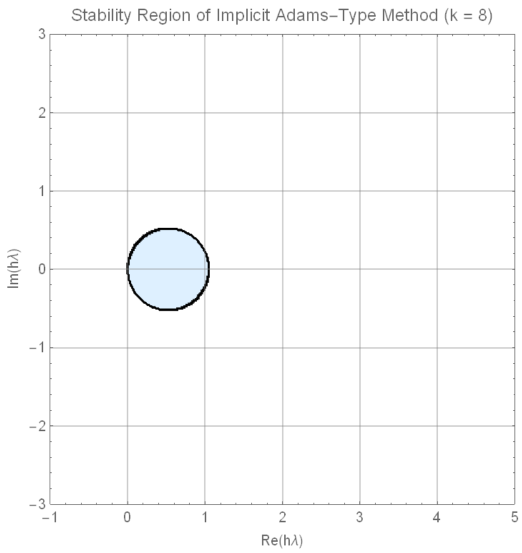

Figure 3.18: Stability plot of SDATF2 for k=8

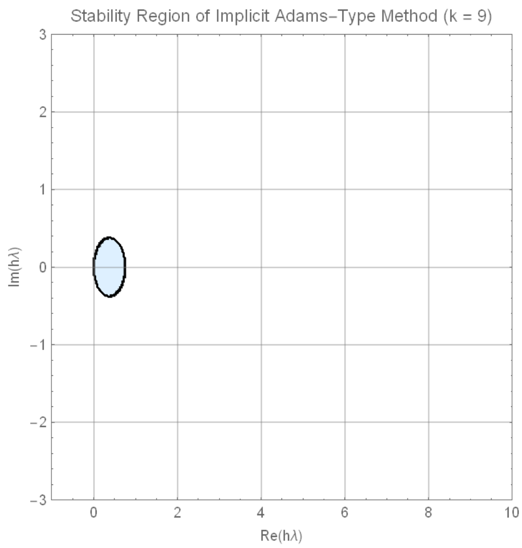

Figure 3.19: Stability plot of SDATF2 for k=9

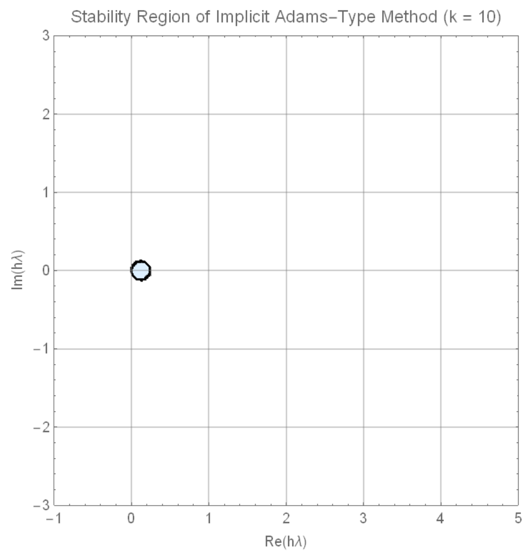

Figure 3.20: Stability plot of SDATF2 for k=10

The boundary locus plots in fig. (3.12)-(3.21), shows that $SDATF_2$ for step size $k = 2$ to $k = 10$, the stability region includes the entire left of the complex plane, meaning that $SDATF_2$ is A-stable. The bounded region is the region of instability. In fig (3.12), the bounded region is between $[0-7]$, in fig. (3.13), the bounded region is between $[0-3.2]$, in fig. (3.14), the bounded region is between $[0-2.2]$, in fig (3.15), the bounded region is between $[0-1.5]$, in fig. (3.16), the bounded region is between $[0 - 1.3]$. As the step size increases, the instability(bounded) region shrinks, and the stability region increases.

3.7 Conclusion

This chapter presented the development of families of new methods for the integration of stiff stochastic differential equations. The methods developed in this chapter are of strong order 1 and weak order 2. $SDATF_1$ and $SDATF_2$ are A-stable for step size for $k \leq 12$ and $k \leq 10$ respectively. $SDATF_1$ and $SDATF_2$ are not self-starting as they require another scheme to generate starting values. The technique of boundary locus was used to plot the stability regions of the methods.

CHAPTER FOUR

Numerical Experiment

4.1 Introduction

The implementation of the proposed methods on standard test SODEs are carried out in this Chapter. The implementation procedures are outlined in section 4.2. $SDATF_1$ and $SDATF_2$ are implemented, results generated are compared with those of existing methods in the literature. In some instances were exact/analytical solution exist. Result generated by proposed methods are compared to exact solution acertain in the solution tract adequately the solution path of the exact solution.

4.2 Implementation procedures

The $SDATF_1$ and $SDATF_2$ are implemented using the fixed point iteration. The k-step $SDATF_1$ and $SDATF_2$ can be applied to SODE (1.1.1) using the following steps:

Step 1: Choose a step size and final time T.

Step 2: Compute N, the number of iterative step. That is $N = \frac{T-t_0}{h}$, t_0 is the initial time.

Step 3: Obtain the time gride t_n. That is $t_n = \{nh\}_{n=0}^{N}$.

Step 4: Generate Brownian increments $\Delta W_n \sim \mathcal{N}(0, h)$.

Step 5: Obtain starting values using one-step method. That is; back value $X_0, X_1, \dots, N-k$.

Step 6: Use fixed point iteration to resolve implicitness.

Set the initial guess $X_{n+1}^{(0)} = X_n$.

$$X_{n+1}^{(m+1)} = \Phi\left(X_{n+1}^{(m)}\right),$$

where

$$\begin{aligned}\Phi(X_{n+1}) = X_{n+1} - X_n - h\sum_{j=0}^{k}\beta_j f_{n+j-k+1} - h^2\theta_1 G_{n+1} \\ + \sum_{j=0}^{k}\gamma_j g_{n+j-k+1}\Delta W_n \\ - \frac{1}{2}\sum_{j=0}^{k}\eta_j g'_{n+j-k+1} g_{n+j-k+1}[(\Delta W_n)^2 - h] = 0\end{aligned}$$

Continue until convergence, that is;

$\|X_{n+1}^{(m+1)} - X_{n+1}^{(m)}\| < \tau$, where τ is prescribed tolerance.

4.3 Standard Test problems

The following test problems are considered for the numerical experiments.

Problem 4.1. *Consider the non-linear one-dimensional stiff SODE with a multiplicative noise (cf: Brugnano et al (2000))*

$$\begin{aligned}dX = -\alpha(1 - X^2)\,dt + \beta(1 - X^2)\,dW(t), X_0 = 0, \quad 0 \le t \le 2, \\ \alpha = 1,, \quad \beta = 1, \quad \beta = 10^{-4}.\end{aligned} \tag{4.3.1}$$

The exact solution to (4.3.1) is:

$$X(t) = \frac{(1 + X_0)\exp(-2\alpha t + 2\beta W(t)) + X_0 - 1}{(1 + X_0)\exp(-2\alpha t + 2\beta W(t)) - X_0 + 1}.$$

Problem 4.2. *Consider the linear one-dimensional stiff SODE with multiplicative noise (cf: Yin* *and Gan (2015)):*

$$dX(t) = -20X(t)\,dt + 5X(t)\,dW(t), \quad 0 < t \leq T, \quad X(0) = 1, \tag{4.3.2}$$

whose exact solution is given by:

$$X(t) = \exp\left[(-20 - \frac{1}{2} \cdot 25)t + 5W(t)\right] = \exp\left[-32.5t + 5W(t)\right].$$

Problem 4.3. *Consider the non-linear SODE with additive noise:(Stochastic Nagumo Equation) (cf: Reidel and Wu (2020)) :*

$$dX(t) = \left(X(t) - X^3(t)\right)dt + dW(t), \quad t \in (0,1], \quad X(0) = -3.0 \tag{4.3.3}$$

Problem 4.4. *Consider the linear stiff SODE (cf: Abdulle and Cirilli(2008)):*

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t), \quad t \in [0,1], \quad X(0) = 1, \tag{4.3.4}$$

where $\lambda = -10^3$ *and* $\mu = 100$.

$$X(t) = X(0)\ \exp\left(\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu W(t)\right)$$

Problem 4.5. *Consider the 2-Dimensional vector-valued SODE (cf: Burrage and Tian (2001)):*

$$dX = \left(U - \frac{1}{2}V^2X\right)dt + VX \circ dW(t), \quad X(0) = X_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad t \in [0,1], \quad X \in \mathbb{R}^2, \tag{4.3.5}$$

where U and V are matrices given by:

$$U = \begin{bmatrix} -u & u \\ u & -u \end{bmatrix}, \quad V = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}.$$

The exact solution to (4.3.5) is:

$$X(t) = P\text{exp}(\rho_+(t))\,\text{exp}(\rho_-(t))P^{-1}X_0,$$

where

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = P,$$

and

$$\rho_\pm(t) = \left(-u - \frac{1}{2}v^2 \pm u\right)t + vW(t).$$

Problem 4.6. *Consider the following system of 2-dimensional coupled SODEs with multiplicative noise (cf: Pardoux and Zhang (2020)):*

$$\begin{aligned} dX_t &= -50X_t\,dt + 0.5\,dW_t, \\ dY_t &= -5Y_t\,dt + 0.1\,dW_t, \\ X_0 &= Y_0 = 1. \end{aligned} \tag{4.3.6}$$

The exact solutions at each step are given by:

$$X(t_{n+1}) = X_0e^{-\varepsilon h} + \sigma_1\varrho\Delta W_n,$$

$$Y(t_{n+1}) = Y_0e^{-\lambda h} + \sigma_2\varrho_Y\Delta W_n,$$

where:

$$\varrho = e^{-\varepsilon h} = e^{-50 \cdot 0.01} \approx 0.60653066,$$

$$\varrho_Y = e^{-\lambda h} = e^{-5 \cdot 0.01} \approx 0.95122942.$$

Problem 4.7. *Consider the Linear system of vector-valued d-dimensional SODE (cf: Reshniak et al 2014):*

$$dX(t) = AX(t)\,dt + \sum_{k=1}^{m} B^k X(t)\,dW^k(t), \quad X(0) = X_0 \in \mathbb{R}^d, \tag{4.3.7}$$

the matrix coefficients A and B are defined as follows:

$$A_{i,j} = \begin{cases} \frac{1}{20}, & \text{if } i \neq j, \\ -\frac{3}{2}, & \text{if } i = j, \end{cases} \quad \text{and} \quad B^k_{i,j} = \begin{cases} \frac{1}{100}, & \text{if } i \neq j, \\ \frac{1}{5}, & \text{if } i = j. \end{cases}$$

The exact solution is given by:

$$X(t) = X_0 \exp\left((A - \frac{1}{2}\sum_{k=1}^{m}(B^k)^2)t + \sum_{k=1}^{m} B^k W^k \right)$$

The numerical solution is computed for the case $d = m = 5$.

4.4 Methods in Comparison

The methods used in comparison are as follows:

(i) Stiffly Accurate Runge–Kutta Methods (SARKM) for stiff stochastic differential equations, (Burrage and Tian(2001)). The method is implicit stochastic Runge–Kutta schemes adapted from the deterministic concept of stiff accuracy (the final stage equals the method's step value) to improve numerical stability when integrating stiff stochastic systems. The method is convergent with order 1.

(ii) Chebyshev Methods for Stiff Stochastic Differential Equations (CMSSDSs), (Abdulle and Cirilli (2008)). The method is a class of explicit stabilized schemes designed to efficiently integrate stiff stochastic systems without resorting to fully implicit methods.It has a convergent order 1.

(iii) Semi-Implicit Taylor Schemes (SITS) for Stiff Rough Differential Equations. (Riedel and Wu (2020)). The method is a semi-implicit stochastic Taylor-type method developed for solving stiff Rough Differential Equations (RDEs). The method is convergent with order 1 and A-stable in the mean square sense.

(iv) An improved Milstein method (IMM) for stiff stochastic differential equations, (Yin and Gan (2015)). The method is a semi-implicit stochastic integration scheme designed to efficiently and stably solve stiff stochastic differential equations (SODEs). It is a modification of the classical Milstein method, aiming to enhance mean-square stability and accuracy when the drift term exhibits stiffness. It is of strong order 1.

(v) Adams-type Methods (ATM) for the Numerical solution of Stochastic Differential Equations, (Brugnano et al 2000). the methods are a class of linear multistep methods extended to the stochastic framework for solving Itô-type stochastic differential equations (SODEs). It is convergent with order 1.

(vi) High-order combined Multi-Step Scheme (HOCMSS) for solving forward Backward Stochastic Differential Equations, (Teng and Zhao (2021)). The method is an advanced numerical technique designed for efficiently solving forward–backward stochastic differential equations (FBSDEs). It is of strong order 1.

(vii) Split step Milstein Methods (SSMM) for multi-channel stiff stochastic differential systems, (Reshniaka et al (2014)). The method is a specialized numerical approach for solving multi-channel stiff stochastic differential systems. It is convergent with order 1.

(viii) Stochastic Taylor Expansion (STE), (Pardoux and Zhang (2020)). The method is classified as a stochastic single-step method and more specifically as a Taylor-type (Itô–Taylor) numerical method for stochastic differential equations (SODEs). It belongs to the strong approximation family of methods because it directly approximates the sample paths of the SODE rather than just the distribution. It is of strong order 1.5.

(ix) Split step Adams-Moulton-Milstein (SSAMM) method, (Reshniak et al (2014)). The method is a semi-implicit, multi-step stochastic scheme that combines the stability advantages of the implicit Adams–Moulton approach with the accuracy of the Milstein correction. The method achieves strong order 1.0 and mean-square A-stability.

(x) Modified split step Adams Moulton Milstein (MSSAMM) method, (Reshniak et al (2014)). The method is a semi-implicit, multi-step stochastic integration scheme designed for stiff SODEs. It combines the implicit Adams–Moulton formula with the Milstein correction, and introduces a stability modification term to enhance mean-square A-stability. The method achieves strong order 1.0.

METRICS

The following metrics of measurements are used in this study:

i Absolute Error (AE)

ii Pathwise Error (PE)

iii Global Mean Square Error (GMSE)

iv Strong Error (SE)

v Mean Square Error (MSE)

vi Relative Mean Square Error (RMSE)

4.5 Results

Table 4.1: Global mean-square Error (GMSE) for test problem 1, for k=2,$\alpha = 1, \beta = 1$

$\frac{1}{h}$	ATM ($p = 1$)	$SDATF_1$ ($p = 1$)	$SDATF_2$ ($p = 1$)
0.125	1.43E-01	1.22E-02	1.20E-02
0.0625	7.74E-02	2.94E-03	2.90E-03
0.03125	3.95E-02	7.31E-04	7.20E-04
0.015625	2.14E-02	1.82E-04	1.80E-04
0.007812	1.00E-02	4.52E-05	4.50E-05
0.003906	4.68E-03	1.12E-05	1.10E-05
0.001953	2.60E-03	2.79E-06	2.80E-06
0.000977	1.38E-03	6.93E-07	6.90E-07

Table 4.2: Global mean-square Error (GMSE) for test problem 1, for k=2,$\alpha = 1, \beta = 10^{-}4$

$\frac{1}{h}$	ATM ($p = 1$)	$SDATF_1$ ($p = 1$)	$SDATF_2$ ($p = 1$)
0.125	4.18E-04	4.54E-05	4.50E-05
0.0625	4.96E-05	1.14E-05	1.10E-05
0.03125	6.03E-06	2.85E-06	2.80E-06
0.015625	7.51E-07	7.13E-07	7.00E-07
0.007812	9.81E-08	1.78E-08	1.70E-08
0.003906	1.45E-08	4.44E-09	4.30E-09

Table 4.3: Mean square Error (MSE) for the test problem 2

h	IMM Error ($p = 1$)	IMM EOC	Milstein Error ($p = 1$)	Milstein EOC	$SDATF_1$ ($p = 1$)	A EOC	$SDATF_2$ ($p = 1$)	B EOC
0.031250	8.2860E-1	–	1.49090	–	1.0262E-1	–	9.954E-2	–
0.015625	3.9750E-1	1.06	7.9910E-1	0.90	5.111E-2	1.00	4.873E-2	1.03
0.007812	1.9440E-1	1.03	4.1360E-1	0.95	2.536E-2	1.01	2.431E-2	1.00
0.003906	9.790E-2	0.99	2.0820E-1	0.99	1.266E-2	1.00	1.228E-2	0.98
0.001953	4.850E-2	1.01	1.0530E-1	0.98	6.31E-3	1.00	6.13E-3	1.00

Table 4.4: Pathwise error (PE) for the test problem 3

h	SITS (p=1)	$SDATF_1$ ($p = 1$)	$SDATF_2$ ($p = 1$)
0.007813	2.9995E-2	2.8200E-2	3.0100E-2
0.003906	1.6201E-2	1.5600E-2	1.6500E-2
0.001953	8.3910E-3	8.2000E-3	8.7000E-3
0.000977	4.0810E-3	4.0000E-3	4.3000E-3

Table 4.5: RMSE for test problem 4

h	CMSSDEs (p=1)	$SDATF_1$ (p=1)	$SDATF_2$ $(p = 1)$
0.25	1.223E-1	9.040E-2	9.805E-2
0.125	3.607E-2	2.857E-2	2.923E-2
0.0625	8.807E-3	8.117E-4	7.331E-3
0.03125	2.043E-3	1.6112E-4	1.938E-3
0.015625	4.913E-4	2.858E-4	4.656E-4

Table 4.6: Absolute Error (AE) for the test problem 5, $\nu = 0.1$

h	SARKM (p=1)	$SDATF_1$ $(p = 1)$	$SDATF_2$ $(p = 1)$
0.25	4.305E−4	1.701E−4	1.911E−4
0.125	2.267E−4	1.760E−4	1.965E−4
0.0625	1.147E−4	1.137E−4	1.159E−4

Table 4.7: Absolute Error (AE) for the test problem 5, $\nu = 5$

h	SARKM (p=1)	$SDATF_1$ $(p = 1)$	$SDATF_2$ $(p = 1)$
0.25	4.878E−6	1.08E−4	4.31E−4
0.125	3.84E−7	2.41E−7	3.01E−7
0.0625	1.26E−7	1.36E−8	1.95E−8

Table 4.8: Absolute Error (AE) for test problem 6 for X-component

Step	t_n	ΔW_n	**STE-X**	$SDATF_1$-X	$SDATF_2$-X
0	0.000	—	0.0000	0.0000	0.0000
1	0.010	0.020	1.081E-1	1.081E-1	1.081E-1
2	0.020	-0.030	5.613E-2	3.392E-2	4.091E-2
3	0.030	0.010	5.145E-2	1.315E-2	2.112E-2
4	0.040	0.015	3.863E-2	2.033E-2	3.631E-2
5	0.050	-0.025	3.542E-2	1.951E-2	2.953E-2

Table 4.9: Absolute Error (AE) for test problem 6 for Y-component

Step	t_n	ΔW_n	**STE-Y**	$SDATF_1$-Y	$SDATF_2$-Y
0	0.000	—	0.0000	0.0000	0.0000
1	0.010	0.020	3.36E-2	4.16E-2	2.13E-2
2	0.020	-0.030	4.27E-2	5.07E-2	4.033E-2
3	0.030	0.010	5.45E-2	6.23E-2	3.76E-2
4	0.040	0.015	6.60E-2	7.38E-2	5.667E-2
5	0.050	-0.025	7.57E-2	8.33E-2	6.586E-2

Table 4.10: Strong error (SE), for test problem 7

h	DSSBM	MSSBM	SSAMM-	MSSAMM	$SDATF_1$	$SDATF_2$
0.5	2.207E−1	3.335E−1	1.243E−1	1.977E−1	1.876E−1	1.876E−1
0.25	1.148E−1	1.340E−1	8.368E−2	1.029E−1	8.269E−2	8.309E−2
0.125	6.241E−2	5.988E−2	4.683E−2	4.641E−2	3.952E−2	3.981E−2
0.0625	3.480E−2	3.159E−2	2.361E−2	2.425E−2	1.945E−2	1.955E−2
0.03125	1.511E−2	1.572E−2	1.339E−2	1.439E−2	9.611E−3	9.707E−3
0.015625	7.090E−3	8.037E−3	6.760E−3	6.411E−3	4.851E−3	4.981E−3
0.007812	3.960E−3	4.354E−3	3.637E−3	3.294E−3	2.460E−3	2.529E−3
0.003906	2.112E−3	1.901E−3	1.575E−3	1.774E−3	1.319E−3	1.324E−3

4.6 Results and Discussion

$ATSDF_1$, $ATSDF_2$, and ATM both of strong order 1 were applied over the interval $t \in [0, T]$ on test problem 1. The Mean square global error, Brugnano *et al* (2000) is presented in table 4.1 and table 4.2 for $k = 2$, $\alpha = \beta = 1$, $\alpha = 1$ and $\beta = 10^{-4}$ respectively and the results showed that $ATSDF_2$ generate the best result in terms of accuracy compare to other methods. The proposed method demonstrate superior performance in accuracy over other methods in comparison. The solution plot for $SDATF_1$ and $SDATF_1$ together with the exact solution is shown in Fig 4.1. This plot shows satisfactory approximation as the proposed methods mimic the exact solution in the trajectory.

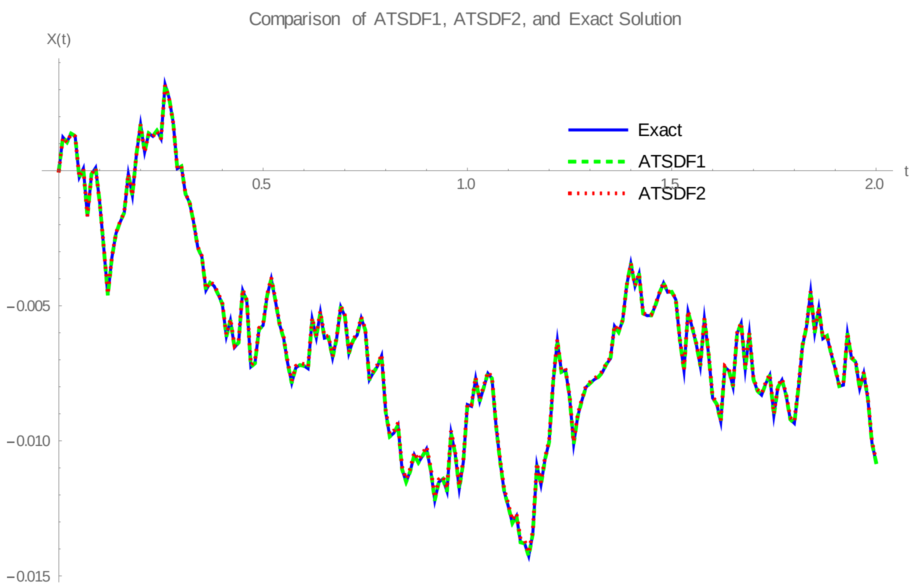

Figure 4.1: Solution to test problem 1.

For problem 2,$SDATF_1$, $SDATF_2$, IMM and Milstein methods of order 1 were applied on the interval $0 < t \leq T$ to the test problem 2. The mean square error, Yin and Gan (2015), is presented and results together with their Experimental Order of Convergence in table 4.3 shows that $ATSDF_2$ generate the best results in terms of accuracy . The numerical solution and the exact solution are presented in fig 4.2, this shows that the methods are effective.

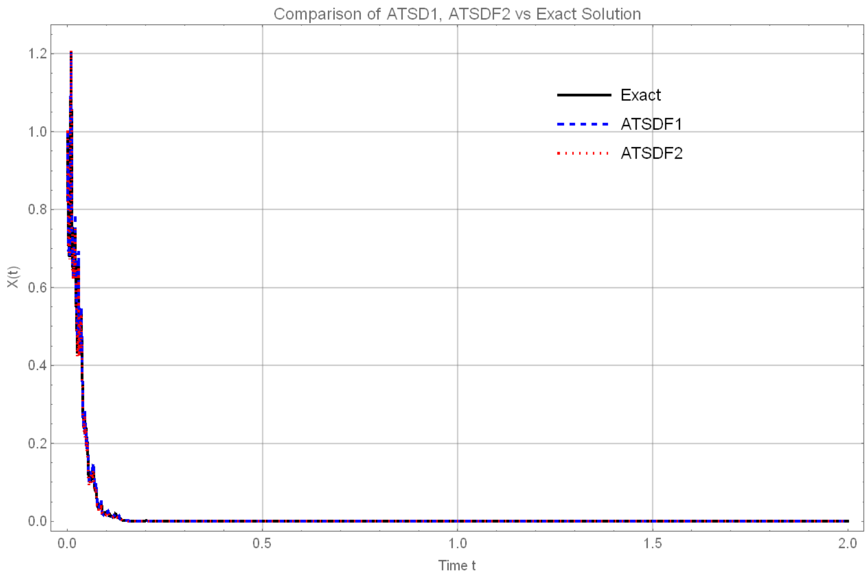

Figure 4.2: Solution to test problem 2.

$SDATF_1$, $SDATF_2$, and SITS were applied on the interval $t \in [0,1]$ to test problem 3. The pathwise error, Riedel and Wu (2020) when compared to the reference solution obtained via a finer step size of $h_{ref} = 2^{-14}$ is presented in table 4.4. The results shows that $ATSDF_1$, produces the best results than other methods. The approximate solution and exact solution are presented in Fig 4.3.

Figure 4.3: Solution to test problem 3

For test problem 4, $SDATF_1$, $SDATF_2$, and CMSSDEs of order 1 were applied and Relative Mean Square Error computed on the interval $[0, 1]$. The results were compared to Abdulle and Cirilli (2008) as seen in table 4.5, which shows that $ATSDF_2$, generate the best results. The solution plot of $SDATF_1$, $SDATF_2$, with exact is shown in Fig 4.4.

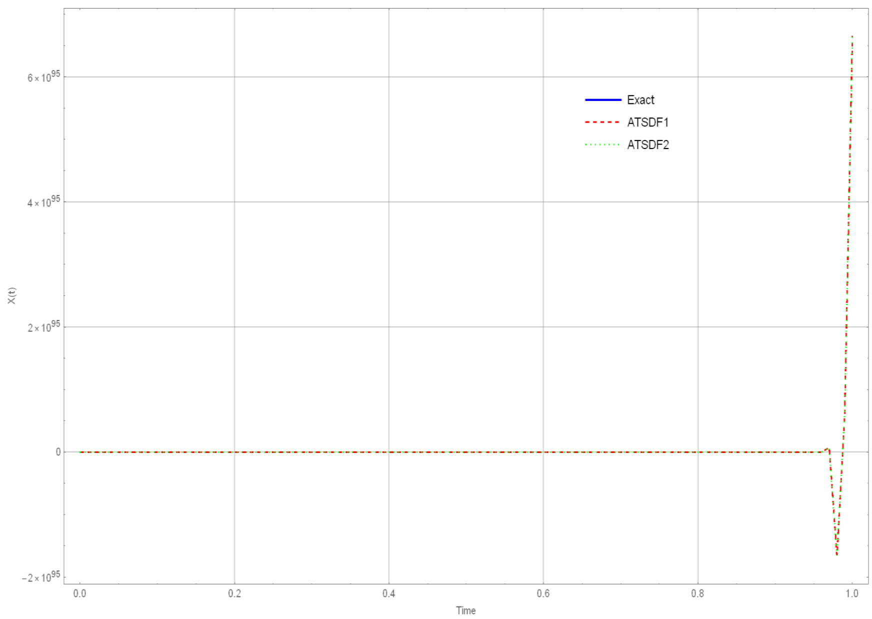

Figure 4.4: Solution to test problem 4

For Test Problem 5, the $SDATF_1$, $SDATF_2$, and SARKM of order 1 were applied on the interval $t \in [0,1]$ for $\nu = 0.1, \nu = 5$ to a two-dimensional linear SDE of a Stratonovich form. The mean of the Absolute error, Burrage and Tian (2001) is presented in table 4.6 and 4.7, and the results shows that $SDATF_1$, performed marginally better than $SDATF_2$ and SARKM techniques. From the solution plot of the approximate and exact solution in fig 4.5, it is observed that the proposed methods mimic the exact solution trajectory.

Figure 4.5: Solution to test problem 5

For Test Problem 6, the $SDATF_1$, $SDATF_2$, of order 1 were applied to a system of stiff SDE. The results of the methods were compared to STE of Pardoux and Zhang (2014) of order 1.5, using h = 0.01. The $ATSDF_1$, perform better in X-component while $ATSDF_2$ perform better in Y-component. The strong error are shown in Table 4.8 and 4.9. The solution plots in shown in Fig 4.6 and 4.7.

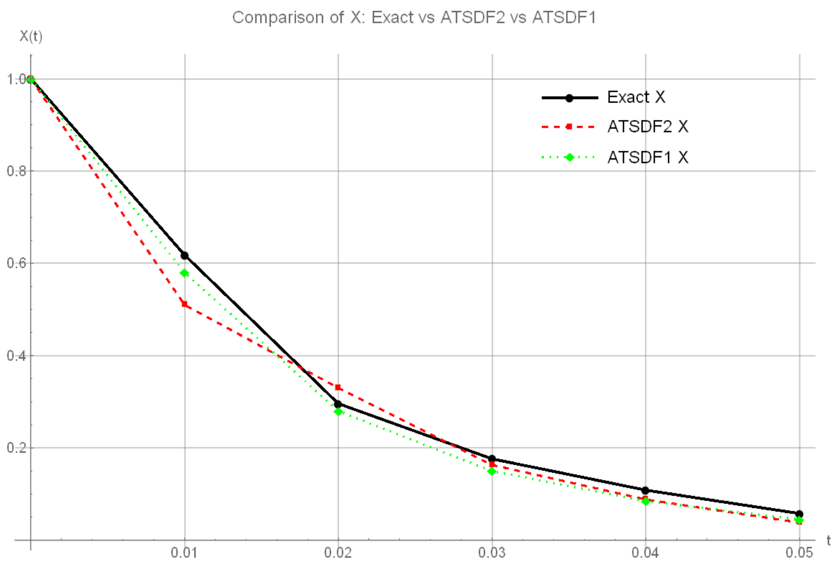

Figure 4.6: Solution to test problem 6

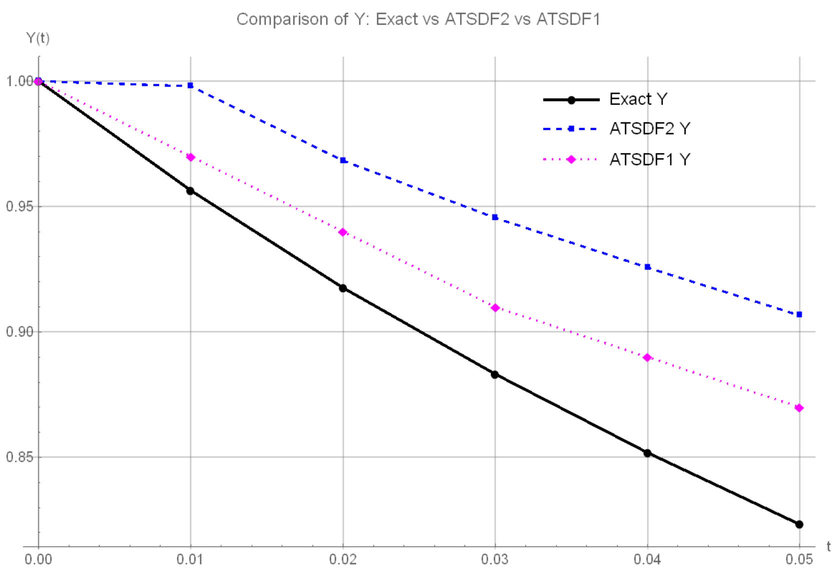

Figure 4.7: Solution to test problem 6

Test Problem 7 was solved using $ATSDF_1$, $ATSDF_2$, of order 1, and computed $d = m = 5$. The strong error were compared to the DSSBM, MSSBM, SSAMM and MSSAMM of Reshniak *et al* (2014) and the results are displayed in Table 4.10. The $SDATF_1$, returned better accuracy than the other methods in comparison. The solution plot is shown in Fig 4.8.

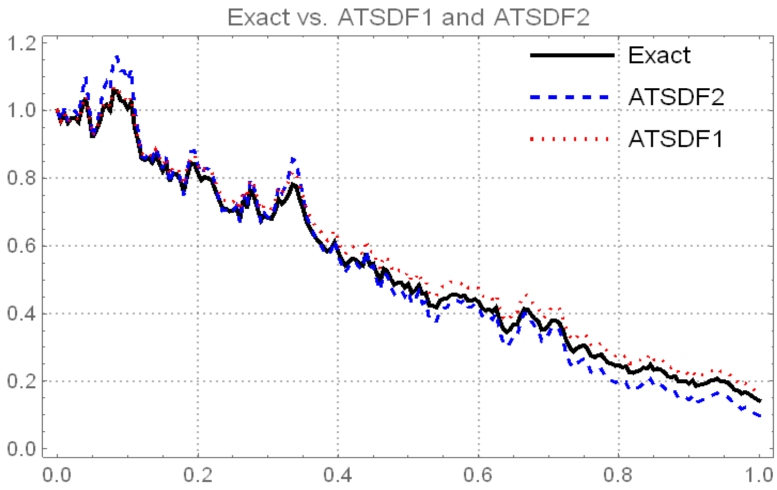

Figure 4.8: Solution to test problem 7

4.7 Conclusion

Numerical experiments were conducted on a set of standard benchmark SODEs commonly used in the literature to assess the performance of numerical schemes. The newly developed methods in this study were applied to standard test problems to evaluate their accuracy, and efficiency.

Due to the presence of implicit terms in the proposed methods, the resulting nonlinear equations at each time step were resolved using an iterative fixed-point approach, ensuring convergence to the numerical solution within a prescribed tolerance.

The computed numerical solutions obtained from the proposed methods were then compared with results produced by standard methods in the literature.

Furthermore, graphical illustrations were provided by plotting the numerical solutions alongside the corresponding exact or analytical solutions, allowing for a clear visualization of the accuracy and of the developed methods. These comparisons demonstrate the effectiveness and robustness of the proposed schemes in approximating the true dynamics of SODEs.

CHAPTER FIVE

Summary and Conclusion

5.1 Introduction

This study is devoted to the development of Second Derivative Adams Type Eormulae (SDATF) for the numerical integration of stiff Stochastic Ordinary Differential Equations (SODEs). The motivation stemmed from the well-known challenges of stiffness in SODEs, which can cause conventional explicit schemes to become unstable or inefficient. To address these issues, families of Adams-type methods incorporating Second Derivative terms were proposed and analyzed.

5.2 Summary

In Chapter one, background on methods for handling SODE (1.1.1) was introduced. Numerical Methods for integrating stiff SODE (1.1.1) were highlighted. Some basic terms were defined. Some real life example were highlighted, with motivation to the study stated, and problem statement highlighting the gap this study is designed to fill, aim and objectives of the study were listed. In Chapter two, a chronological review was taken at the developmental stages and evolution of methods designed for the integration of stiff SODEs with intension of identifying gaps in the literature. In pursuit of the stated objectives in Chapter one, two families of methods, denoted as $SDATF_1$ and $SDATF_2$, were derived in Chapter Three and subjected to rigorous theoretical analysis. Their stability properties and order of convergence were investigated, with particular emphasis on mean-square stability, which is essential in the context of stochastic systems. Chapter four was on the implementation of the proposed methods. Standard test problems were applied to methods proposed in Chapter Three, and the results were compared with that of the methods in the literature. Numerical solutions were also compared with the exact exact solutions. The study confirmed the efficiency and improved performance of the proposed schemes when compared with existing methods in the literature. Therefore, these methods provide reliable tools for solving stiff stochastic differential equations commonly arising in real-world applications.

5.3 Findings

The findings from this study are as follows: The method $\mathbf{SDATF}_1$ is A-stable for step numbers up to $k \leq 12$, while $\mathbf{SDATF}_2$ is A-stable for step numbers up to $k \leq 10$. For both methods, the instability regions shrink as the step number k increases, indicating that the stability regions enlarge for higher-order schemes. In terms of convergence, $\mathbf{SDATF}_1$ and $\mathbf{SDATF}_2$ are strongly stable of order 1 and weakly stable of order 2. Additionally, both methods exhibit mean-square stability, ensuring that the numerical solutions remain reliable over long integration intervals. It is important to note, however, that neither $\mathbf{SDATF}_1$ nor $\mathbf{SDATF}_2$ is self-starting, and therefore suitable starting procedures are required to generate the initial values of the multi-step sequence.

5.4 Contribution to knowledge

The study has contributed to knowledge in the following ways:

1. second derivative formulae for the treatment of initial value problems in Stochastic Differential Equations introduced.
2. second derivative formulae that are zero stable apriori are developed.

3. A-stable second derivative formulae are derived.

5.5 Area for future Research

Based on the results obtained in this study, the following areas are recommended for future research:

Development of self-starting variants of SDATF methods to eliminate the need for external starting schemes.

Extension of the methods to stochastic differential equations with jumps or driven by Lévy processes.

Investigation of adaptive time-stepping strategies combined with SDATF methods to improve computational efficiency.

Application of the developed methods to real-world stiff stochastic models in finance, biology, and physics to validate their practical performance.

5.6 Conclusion

The study has successfully developed and analyzed **two families of second derivative Adams-type formulae ($\mathbf{SDATF_1}$ and $\mathbf{SDATF_2}$)** for the numerical integration of stiff stochastic differential equations (SODEs). The theoretical analysis and numerical experiments collectively demonstrate that the proposed schemes possess desirable stability and convergence properties, which are essential for the accurate and efficient simulation of stiff stochastic systems.

In particular, the results reveal that both methods are A-stable for practical step sizes, ensuring that numerical solutions remain bounded even for highly stiff problems. Furthermore, the schemes exhibit *strong and weak convergence* of appropriate orders, indicating their ability to approximate individual sample paths as well as statistical moments of the exact solution with high fidelity. The *mean-square stability analysis* confirms that the proposed methods maintain long-term numerical stability in the stochastic sense, a crucial requirement for integrating stiff SODEs over extended time intervals.

Comparative numerical experiments against standard methods in the literature show that the proposed methods achieve higher accuracy with fewer computational steps, thereby offering a favourable balance between stability and efficiency. Moreover, the inclusion of second derivative terms enhances the damping of stiffness-induced oscillations, improving both the numerical robustness and precision of the computed solutions.

Overall, these properties establish $\mathbf{SDATF}_1$ and $\mathbf{SDATF}_2$ as reliable and computationally efficient numerical integrators for solving stiff stochastic models commonly encountered in fields such as chemical kinetics, population dynamics, control theory, and financial mathematics. Their combination of strong mean-square stability, convergence, and efficiency makes them promising tools for practical applications requiring high-accuracy stochastic simulations.

REFERENCES

Abdulle, A., and Cirilli, S. (2008). Chebyshev methods for stiff stochastic differential equations. Mathematics of Computation, 78(267), 581–616. https://doi.org/10.1090/S0025-5718-08-02175-5

Allen, E. (2007). Modeling With Ito Stochastic Differential Equations. Dordrecht: Springer. https://doi.org/10.1007/978-1-4020-5953-7

Arif, M. S., Abodayeh, K., and Nawaz, Y. (2025). A hybrid exponential Runge–Kutta scheme for stochastic SIQR disease modeling. *European Journal of Pure and Applied Mathematics, 18*(3), Article 6176. https://doi.org/10.29020/nybg.ejpam.v18i3.6176

Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities. Journal of Political Economy, 81(3), 637–654. https://doi.org/10.1086/260062

Brugnano, L., Burrage, K., and Burrage, P. M. (2000). Adams-type methods for the numerical solution of stochastic differential equations. Applied Numerical Mathematics, 33(1–4), 145–158. https://doi.org/10.1016/S0168-9274(99)00041-1

Burrage, K., and Burrage, P. M. (1996). Stochastic linear multistep methods. SIAM Journal on Numerical Analysis, 33(3), 1087–1104.

Burrage, K., and Tian, T. (2001). Stiffly accurate Runge–Kutta methods for stiff stochastic differential equations. BIT Numerical Mathematics, 41(5), 940–966. https://doi.org/10.1023/A:10219854013

Burrage, K., Burrage, P. M., and Tian, T. (2003). Numerical methods for strong solutions of stochastic differential equations: An overview. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 460(2041), 373–402. https://doi.org/10.1098/rspa.2003.1238

Burrage, K., Burrage, P. M., and Tian, T. (2004). Numerical methods for strong solutions of stochastic differential equations: An overview. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 460(2041), 373–402. https://doi.org/10.1098/rspa.2003.1247

Burrage, P. M. (1999). Runge–Kutta methods for stochastic differential equations (Ph.D. Thesis). Department of Mathematics, Queensland University, Brisbane, Queensland, Australia.

Bussell, D., and García-Trillos, C. A. (2023). Deep multi-step mixed algorithms for high-dimensional nonlinear PDEs and associated BSDEs. arXiv:2308.14487.

D'Ambrosio, R., and Buckwar, E. (2021). Exponential mean-square stability properties of stochastic linear multistep methods. BIT Numerical Mathematics, 61, 1001–1031. https://doi.org/10.1007/s 021-00867-4

Dai, L., and Wang, X. (2023). Adaptive Euler–Maruyama methods with step-size control for stochastic differential equations. IMA Journal of Numerical Analysis. https://doi.org/10.1093/imanum/draf

Debrabant, K., and Rößler, A. (2009). Families of efficient second order Runge–Kutta methods for the weak approximation of Itô stochastic differential equations. *Applied Numerical Mathematics, 59*(3-4), 582–594. https://doi.org/10.1016/j.apnum.2008.03.012

Debrabant, K., Kværnø, A., and Mattsson, N. C. (2022). Lawson schemes for highly oscillatory stochastic differential equations and conservation of invariants. *BIT Numerical Mathematics, 62*, 1121–1147. https://doi.org/10.1007/s10543-021-00906-8

Denk, R., and Schäffler, H. (1997). Stable and convergent multistep methods for stochastic ordinary differential equations. Journal of Computational and Applied Mathematics, 78(1), 1–22.

Di Giovacchino, S., and D'Ambrosio, R. (2024). Strong backward error analysis of symplectic integrators for stochastic Hamiltonian systems. *Applied Mathematics and Computation, 467*, 128488. https://doi.org/10.1016/j.amc.2023.128488

Enright, W. H. (1974). Second-derivative methods for stiff ODEs. SIAM Journal on Numerical Analysis, 11(2), 321–331. https://doi.org/10.1137/0711029

Fang, S., and Zhao, W. (2023). ODE-based multistep schemes for backward stochastic differential equations. Numerical Mathematics: Theory, Methods and Applications, 16(4), 1053–1086. https://doi.org/10.4208/nmtma.OA-2023-0060

Fu, Y., Zhao, W., and Zhou, T. (2018). Multistep schemes for forward–backward stochastic differential equations with jumps. Applied Numerical Mathematics, 128, 123–140. https://doi.org/10.1016/j.a
num.2018.02.015

Gardiner, C. W. (2009). Stochastic Methods: A Handbook for the Natural and Social Sciences (4th ed.). Springer.

Gaines, J. G., and Lyons, T. (1994). Variable step size control in the numerical solution of stochastic differential equations. SIAM Journal on Applied Mathematics, 54(6), 1694–1711.

Goldwyn, J. H., and Shea-Brown, E. (2011). The what and where of adding channel noise to the Hodgkin–Huxley equations. PLoS Computational Biology, 7(11), e1002247. https://doi.org/10.1371/journal.pcbi.1002247

Haghighi, A. (2023). A modified split-step truncated Euler–Maruyama method for SDEs with non-globally Lipschitz continuous coefficients. Computational Methods for Differential Equations, 11(3), 522–534.

Hairer, E., Nørsett, S. P., and Wanner, G. (1993). Solving ordinary differential equations I: Nonstiff problems (2nd ed.). Springer–Verlag.

Hairer, E., and Wanner, G. (2010). Solving ordinary differential equations II: Stiff and differential-algebraic problems (2nd rev. ed.). *Springer*.

Hallern, C., Missfeldt, R., and Rößler, A. (2024). An exponential stochastic Runge–Kutta type method of order up to 1.5 for SPDEs of Nemytskii-type. *IMA Journal of Numerical Analysis*, 45(4), 2361–2403. https://doi.org/10.1093/imanum/drae064

Han, Y. (2019). Stochastic linear quadratic optimal control problem for fractional Brownian motion. Optimal Control Applications and Methods. https://doi.org/10.1002/oca.2523

He, J., Gao, S., Zhan, W., and Guo, Q. (2024). An explicit Euler–Maruyama method for McKean–Vlasov SDEs driven by fractional Brownian motion. *Applied Mathematics Letters*, 152, 108310. https://doi.org/10.1016/j.aml.2024.108310

Higham, D. J. (2000). Mean-square and asymptotic stability of the stochastic θ-method. *SIAM Journal on Numerical Analysis*, 38(3), 753–769. https://doi.org/10.1137/S003614299834736X

Higham, D. J. (2001). An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review*, 43(3), 525–546. https://doi.org/10.1137/S0036144500378302

Huang, T. (2023). Modified Milstein schemes for fractional diffusions via rough path theory. *Stochastic Processes and Their Applications, 143, 152–173*.

Hutzenthaler, M., Jentzen, A., and Kloeden, P. E. (2011). Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proceedings of the Royal Society A*, 467(2130), 1563–1576. https://doi.org/10.1098/rspa.2010.0333

Itô, K. (1951). On stochastic differential equations. *Memoirs of the American Mathematical Society*, 4, 1–51.

Kloeden, P. E., and Platen, E. (1992). Numerical solution of stochastic differential equations. *Springer–Verlag*.

Kloeden, P. E., and Platen, E. (1995). Numerical methods for stochastic differential equations. In Nonlinear Dynamics and Stochastic Mechanics (pp. 437–461). CRC Press.

Kruse, R. (2019). Drift-randomized Milstein method for stochastic differential equations. *SIAM Journal on Numerical Analysis*, 57(4), 1984–2015.

Kushner, H. J., and Dupuis, P. G. (2001). Numerical methods for stochastic control problems in continuous time. *Springer–Verlag*.

Lateef, A., Zhao, W., and Tian, T. (2025). Antithetic multilevel Monte Carlo Milstein schemes for SPDEs with non-commutative noise. *Numerical Algorithms*, 89(2), 301–328.

Li, X., Wang, Z., Ma, Q., and Ding, X. (2023). Conservative continuous stage stochastic Runge–Kutta methods for SDEs. *Journal of Computational and Applied Mathematics*, 422, 114877. https://doi.org/10.1016/j.cam.2022.114877

Liu, Y., Huang, S., and Zhang, L. (2023). Milstein-type schemes for nonlinear time-changed stochastic differential equations with Hölder continuous coefficients. *Journal of Scientific Computing*, 95(3), 45.

Maruyama, G. (1955). Continuous Markov processes and stochastic equations. *Rendiconti del Circolo Matematico di Palermo*, 4(1), 48–90.

Maruyama, G., and Yoshida, N. (2019). Recent advances in numerical methods for stochastic differential equations. *Stochastic Analysis and Applications*, 37(6), 1041–1072. https://doi.org/10.1080/073

Mao, X. (1997). Stochastic Differential Equations and Applications. *Horwood Publishing*.

Mao, X. (2015). The truncated Euler–Maruyama method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 290, 370–384. https://doi.org/10.1016/j.cam.2015.06.

Mao, X., and Szpruch, L. (2013). Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Journal of Computational and Applied Mathematics*, 238, 14–28. https://doi.org/10.1016/j.cam.2012.08.01

Mehdizadeh Khalsaraei, A., Esmaeilbeig, R., and Ghaderi, R. (2012). Families of A-stable multistep methods based on second derivatives. *International Journal of Computer Mathematics*, 89(10), 1305–1322.

Milstein, G. N. (1974). Approximate integration of stochastic differential equations. *Theory of Probability and Its Applications*, 19(3), 557–562. https://doi.org/10.1137/1119062

Milstein, G. N. (1995). Numerical integration of stochastic differential equations. *Springer*.

Milstein, G. N., Platen, E., and Schurz, H. (1998). Balanced implicit methods for stiff stochastic systems. *SIAM Journal on Numerical Analysis*, 35(3), 1010–1019. https://doi.org/10.1137/S003614299731

Milstein, G. N., and Tretyakov, M. V. (2004). Stochastic numerics for mathematical physics. *Springer*.

Mrungowius, M. (2021). Efficient simulation of iterated integrals in stochastic numerical methods. *Journal of Computational Mathematics*, 39(4), 850–872.

Muka, K. O., and Ikhile, M. N. O. (2015). A diagraph theoretic parallelism in block methods. *Afrika Mathematika*, 26(7–8), 1651–1667.

Nwachukwu, G. C., and Mokwunyei, N. E. (2018). Generalized Adams-Type Second Derivative Methods for Stiff Systems of ODEs. *IAENG International Journal of Applied Mathematics*, 48(4), 455–465.

Øksendal, B. (2003). Stochastic Differential Equations: An Introduction with Applications (6th ed.). *Springer*.

Pardoux, É., and Zhang, S. (2020). Stochastic Taylor expansion. Stochastic Processes and Their

Applications, 130(12), 7364–7387. https://doi.org/10.1016/j.spa.2020.05.011

Platen, E., and Bruti–Liberati, N. (2010). Numerical solution of stochastic differential equations with jumps in finance. Springer–Verlag. https://doi.org/10.1007/978-3-642-13612-0

Platen, E., and Wagner, W. (1982). On Taylor formula for a class of Itô processes. *Probability and Mathematical Statistics*, 3, 37–51.

Reshniak, V., Tokman, M., and Van Vleck, E. (2014). Split-step Milstein methods for multichannel stiff stochastic differential systems. *Journal of Computational and Applied Mathematics*, 262, 12–24. https://doi.org/10.1016/j.cam.2013.09.052

Riedel, S., and Wu, H. (2020). Semi-implicit Taylor schemes for stiff rough differential equations. *IMA Journal of Numerical Analysis*, 40(3), 1741–1772. https://doi.org/10.1093/imanum/drz022

Rößler, A. (2006). Runge–Kutta methods for Itô stochastic differential equations with scalar noise. *BIT Numerical Mathematics*, 46(1), 97–110. https://doi.org/10.1007/s10543-005-0039-7

Saito, Y. (1996). Stochastic adaptive time-stepping methods for stochastic differential equations. *BIT Numerical Mathematics*, 36(3), 608–619. https://doi.org/10.1007/BF01731917

Sauer, T. (2012). Numerical analysis. Pearson.

Schurz, H. (1999). Stability, stationarity and boundedness of some implicit numerical methods for stiff stochastic systems. *Stochastic Analysis and Applications*, 17(5), 955–975. https://doi.org/10.1080/07362999908809619

Shampine, L. F., and Reichelt, M. W. (1997). The MATLAB ODE suite. *SIAM Journal on Scientific Computing*, 18(1), 1–22. https://doi.org/10.1137/S1064827594276424

Skwame, Y., Sabo, J., Kyagya, T. Y., and Bakari, I. A. (2018). A class of two-step second derivative Adams-Moulton method with two off-step points for solving second order stiff ordinary differential equations. *International Journal of Scientific and Management Research, 1*(1), 12–20. https://doi.org/10.9734/JSRR/2018/39444

Stamatiou, K. (2017). Asymptotic mean-square stability for two-step methods. *Journal of Computational and Applied Mathematics*, 314, 123–140.

Talay, D., and Tubaro, L. (1990). Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Analysis and Applications*, 8(4), 483–509. https://doi.org/10.1080/07362999008809220

Tang, X., and Xiong, J. (2018). Stability analysis of general multistep methods for Markovian backward stochastic differential equations. *Applied Numerical Mathematics*, 129, 24–44. https://doi.org/10.1016/j.apnum.2018.02.021

Teng, L., and Zhao, W. (2021). High-order combined multi-step scheme for solving forward–backward stochastic differential equations. *Journal of Scientific Computing, 87*, 81. https://doi.org/10.1007/s109 021-01505-z

Ukuedojor, E., Adinya, I., and Ogbebor, O. (2023). Euler–Maruyama method for solving first-order uncertain stochastic differential equations. Journal of Mathematical Modelling, 41(2), 101–110.

Wiktorsson, M. (2001). Joint characteristic function and simultaneous simulation of iterated Itô integrals for multiple independent Brownian motions. *The Annals of Applied Probability*, 11(2), 470–487.

Wu, Q., and Yan, J. (2025). Milstein schemes for stochastic semilinear subdiffusion equations. *SIAM Journal on Numerical Analysis*, 63(1), 112–136.

Yin, G., and Gan, S. (2015). An improved Milstein method for stiff stochastic differential equations. *Applied Numerical Mathematics*, 95, 99–116. https://doi.org/10.1016/j.apnum.2014.06.010

Zhang, H., Li, X., and Wang, Z. (2023). Adaptive time-stepping Milstein methods for piecewise continuous SODEs. *Applied Numerical Mathematics*, 176, 110–130.

Zhao, H. Y., and Wang, J. L. (2012). A modified Euler–Maruyama method for solving SDEs with jumps. *Journal of Computational and Applied Mathematics*, 236(8), 2147–2156. https://doi.org/10.10

Zhao, W., He, J., and Zhou, T. (2014). Multistep methods for coupled forward–backward stochastic differential equations. *SIAM Journal on Scientific Computing*, 36(4), A1731–A1751.

Zhao, W., Zhou, T., and Kong, T. (2017). High order numerical schemes for second-order forward–backward stochastic differential equations with applications to stochastic optimal control. *Communications in Computational Physics, 21*(3), 808–834. https://doi.org/10.4208/cicp.OA-2016-0056