A STUDY ON METRIC SPACE AND INNER PRODUCT SPACE

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UNDERTAKING

This project was carried out by OBOH SHALOM ALOAYE. All work used have been duly cited and acknowledged.

OBOH SHALOM ALOAYE ______________________

Name of student Signature and Date

CERTIFICATION

This is to certify that this project was carried out and written by OBOH SHALOM ALOAYE with Mat No. PSC1607648 of the Department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, Edo State, Nigeria under the Supervision of MR. H.O OMOKARO.

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MR. H.O. OMOKARO Date

Project supervisor

_____________________ _________________

Prof. A. A. Osagiede Date

Head of Department

DEDICATION

I dedicate this project to God Almighty for his infinite mercies, grace and protection, good health, wisdom, knowledge and understanding all of which he has graciously granted me during my years of study in the University of Benin, Benin City, Edo State, Nigeria.

ACKNOWLEDGEMENTS

Firstly, I want to thank God almighty for his infinite mercies, for the knowledge, courage and good health granted me during this research work.

I appreciate my parents, Mr. and Mrs. Oboh for all the help financially, spiritually and emotionally given to me. I also appreciate my sibling Markanthony Oboh for the love, care and support.

Apart from the people above, I also want to acknowledge my project supervisor Mr. H.O Omokaro for his support, direction and help in making this project a success. I pray that God almighty will repay him greatly in Jesus name. Amen.

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Abstract

Abstract

This project work will introduced the reader to the concept of metrics (a class of functions which is regarded as generalization of the notion of distance) and metric spaces with examples. It emphasises the notion of vector space which generalizes the concept of addition and scalar multiplication. The notion of inner product allows us to generalize the notion of the dot product of vectors. It also allows us to talk about angle between vectors, and their norms. We can discuss the notion of distance between vectors. Also the combination of inner product with a vector gives a scalar.

An inner product space is a special type of vector space that has mechanism for computing a version of dot product that can be defined in real or complex vector space, as long as it satisfies some conditions.

The properties arising from a metric in an inner product space is important example of a metric space

Chapter one

1.1 BACKGROUND TO STUDY

The proprieties of a metric arising in an inner product space is showing the relationship between a metric space and an inner product space. Starting with a topological space, since a metric space is an example of a topological space and also defining a normed space since an inner product space induces in it.

1. Introduction to Vector Spaces

We first of all define a field since a vector space is defined on a field.

1.1.1 Definition of a Field

Let F be a non-empty set on which are defined two operations, called addition and multiplication:

$$a, b \in F \Rightarrow a + b \in F, ab \in F.$$

We say that F is a field if and only if these operations satisfy the following properties:

1. $a + b \in F$ for all $a, b \in F$ (closure property w.r.t addition)
2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in F$ (associative property of addition);
3. There exists an element 0 of F such that $a + 0 = a$ for all $a \in F$ (existence of an additive identity);
4. For each $a \in F$, there exists an element $-a \in F$ such that $a + (-a) = 0$ (existence of additive inverses);
5. $a + b = b + a$ for all $a, b \in F$ (commutative property of addition);
6. $a.b = b.a$ for all $a, b \in F$ (commutative property w.r.t multiplication);
7. $(a.b).c = a.(b.c)$ for all $a, b, c \in F$ (associative property w.r.t multiplication);
8. $a.(b + c) = a.b + a.c$ for all $a, b, c \in F$ (left distributive property w.r.t multiplication w.r.t addition).
9. $(a + b).c = a.c + b.c$ for all $a, b, c \in F$ (right distributive property w.r.t multiplication w.r.t addition).
10. $a.b = b.a$ for all $a, b \in F$
11. For every $a \in F$ for which $a \neq 0$ there exist $b \in F$ such that $ab = ba = 1$

1.1.2 Examples of a Field

1. The set $\mathbb{R}$ of real number is a field

2. The set $\mathbb{C}$ of complex number is a field

3. The set Q of rational number is a field

1.1.3 Definition of vector space

Let F be a field and let V be a nonempty set. Suppose two operations are defined with respect to these sets, addition and scalar multiplication:

$$u, v \in V \Rightarrow \quad u + v \in V,$$

$$\alpha \in F, v \in V \Rightarrow \quad \alpha v \in V.$$

We say that V is a vector space over F if and only if the following properties are satisfied:

1. $u + v \in V$ for all $u, v \in V$

2. $u + v = v + u$ for all $u, v \in V$ (commutative property of addition);

3. $u + (v + w)=(u + v) + w$ for all $u, v, w \in V$ (associative property of addition);

4. There exists an element 0 of V such that $u + 0 = u$ for all $u \in V$ (existence of an additive identity);

5. For each $u \in V$, there exists an element $-u$ of V such that $u + (-u)=0$ (existence of additive inverses);

6. $\alpha(\beta u)=(\alpha\beta)u$ for all $\alpha, \beta \in F, u \in V$ (associative property of scalar multiplication);

7. $\alpha(u + v) = \alpha u + \alpha v$ for all $\alpha \in F, u, v \in V$ (distributive property);

8. $(\alpha + \beta)u = \alpha u + \beta u$ for all $\alpha, \beta \in F, u \in V$ (distributive property);

9. 1 · u = u for all u ∈ V

Examples of vector space are

1. The set $\mathbb{R}^2$ is a vector space $\mathbb{R}^2 = \{ (x, y): x, y \in \mathbb{R} \}$

PROOF

Let u, v ϵ $\mathbb{R}^2$, α,s ϵ $\mathbb{R}^2$, such that u = (x, y), v= (x′, y′) ϵ $\mathbb{R}$. Show that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$

Consider u +v = (x, y) + (x′, y′) = (x+ x′, y+ y′) ϵ $\mathbb{R}^2$. That is u + v is additive. Therefore $\mathbb{R}^2$ is closed under addition.

To show associativity w.r.t addition in $\mathbb{R}^2$. Let w ϵ $\mathbb{R}^2$ such that w = (x″, y″). Consider u +{v + w }= (x, y) + {(x′, y′) + (x″, y″)} = (x, y) + { (x+ x′) , (y+ y′)} = { x + x′ + x″, y + y′ + y″} = {x + x′ , y + y′} + (x″ , y″) = { (x, y) + (x′, y′) } + (x″, y″) = {u + v} + w. That is u + {v + w} = {u +v} + w. Therefore, $\mathbb{R}$ is associative w.r.t addition.

Let 0 ϵ $\mathbb{R}^2$ such that 0 = (0,0) ϵ $\mathbb{R}$. Consider u + 0 = (x, y) + (0, 0) = (x, y) = u. Similarly, 0 + u = (0, 0) + (x, y) = (x, y) = u.

For all u ϵ $\mathbb{R}$. Therefore, 0 is the additive identity element of $\mathbb{R}^2$.

Let u ϵ $\mathbb{R}$, put d = -(x,y). Now consider, u + d = (x, y) + (-(x, y)) = (x, y) – (x, y) = ((x- x , y – y)) = (0. 0) = 0. Similarly, d + u = (-(x, y)) + (x, y) = (-x + x , -y + y) = (0, 0). Therefore, d is the identity inverse of x in $\mathbb{R}$.

Consider u + v = (x, y) + (x′, y′) = (x + x′, y + y′) = (x′, y′) + (x, y) = v + u. Therefore , $\mathbb{R}^2$ is a commutative w.r.t addition. i.e. $\mathbb{R}$ is an additive abelian group.

Now, let α, s $\epsilon\mathbb{R}$, consider α (u + v) = α((x, y) + (x′, y′)) = α ((x + x′, y + y)) , (α (x + x′), α (y + y′)) = (αx + αx′, αy + αy′) = (αx , αy) + (αx′, αy′) = α(x, y) + α (x′ ,y′) = αu + αw

Therefore α(v + w) = αv +αw

Let α, β ϵ f and v = (x, y) ϵ $\mathbb{R}^2$ such that v = (x, y)

Consider (α + β) v = (α + β) (x, y) = ((α + β) x, (α + β)y) = (αx + βx, αy + βy) = (αx, αy) + (βx, βy) = α(x,y) + b(x,y) = αv + βv. Therefore (α + β) v = αv + βv.

Consider ($\alpha\beta$)v = ($\alpha\beta$)(x,y) =α(β(x,y))= α(βv) for all α, β ϵ F and v ϵ $\mathbb{R}$

Finally, consider 1. v = 1. (x, y) = (x, y) = v for all v ϵ $\mathbb{R}$.

Therefore, the set $\mathbb{R}^2$ of 2-tuples real numbers is a vector space.

2. The set $\mathbb{C}$ of complex numbers is a vector space over the field of real numbers.

Proof

Let $\mathbb{C}$ be the set of complex number so that is $\mathbb{C}$ = {x= x_1 +ix_2 : x_1,x_2 ϵ $\mathbb{R}$}. We are to show that $\mathbb{C}$ is a vector space over $\mathbb{R}$.

So let x, y ϵ $\mathbb{C}$ where x = x_1 + ix_2, y= y_1+ iy_2 $\forall$ x_1, x_2, x_3, y_1, y_2 ϵ $\mathbb{R}$. Consider x + y = (x_1 + y_1) + i (x_2 + y_2)= c_1 + c_2 Where c_1 = x_1 + y_1, c_2 = x_2 + y_2 ϵ $\mathbb{R}$.

Clearly, x+ y ϵ $\mathbb{R}$. Thus, $\mathbb{C}$ is closed w.r.t addition.

To show associativity in $\mathbb{C}$, where z = z_1 +s iz_2, z_1, z_2 $\epsilon\mathbb{R}$

Consider (x + y) + z = ((x_1 + ix_2) + (y_1 + iy_2)) + (z_1 + iz_2) = ((x_1 + y_1) + i(x_2+ y_2)) + (z_1 + iz_2) = ((x_1 + y_1) + z_1) + i((x2 + y2) + z2) = (x_1 +(y_1 + z_1)) + i (x_2 +(y_2 +z_2)) =

$(x_1 + ix_2) + \{ (y_1 + z_1) + i\,(y_2 + z_2) \} = (x_1 + ix_2) + \{(y_1 + iy_2) + (z_1 + iz_2)\} = x + (y + z)$

i.e. $(x + y) + z = x + (y + z) \;\; \forall\, x, y, z \in \mathbb{R}$

Since $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $\mathbb{R}$ is a field.

Therefore $\mathbb{C}$ is associative w.r.t addition.

Let $0 = 0 + i0$,

Consider $x + 0 = (x_1 + ix_2) + (0 + i0) = (x_1 + 0) + i(x_2 + 0) = x_1 + ix_2 = x$

Similarly, $0 + x = 0 + i0 + x_1 + ix_2 = (0 + x_1) + i(0 + x_2) = x_1 + ix_2 = x$

i.e. $x + 0 = 0 + x = x \;\forall\, x \in \mathbb{R}$.

Also, let $x' \in \mathbb{C}$ such that $x' = -x + i(-x_2)$, $-x_1, -x_2 \in \mathbb{R}$.

Consider $x + x' = (x_1 + (-x_1) + i(x_2 + (- x_2)) = 0 + i0 = 0$

Similarly, $x + x' = (-x_1 + x_1) + i\,(-x_2 + x_2) = 0 + i0 = 0$.

Thus x' is the additive inverse of x in $\mathbb{C}$.

Now, let $\alpha \in F$, $y, x \in \mathbb{R}$,

Consider

$\alpha\,(x + y) = \alpha((x_1 + i\,x_2) + (y_1 + iy_2)) = \alpha\,((x_1 + y_1) + i(x_2 + y_2)) = \alpha\,(x_1 + y_1) + i\alpha\,(x_2 + y_2) = (\alpha x_1 + \alpha y_1) + i(\alpha x_2 + \alpha y_2) = \alpha(x_1 + ix_2) + \alpha(y_1 + iy_2) = \alpha x + \alpha y$

i.e. $\alpha(x + y) = \alpha x + \alpha y \;\forall\, \alpha \in \mathbb{R}$, $x, y \in C$

Let $\alpha, \beta \in F$, $x \in \mathbb{R}$.

Consider

$(\alpha + \beta)\ x = \alpha + \beta\ (x_1 + ix_2) = (\alpha + \beta)x_1 + (\alpha + \beta)ix_2 = \alpha x_1 + \beta x_1 + i\alpha x_2 + i\beta x_2 = \alpha x_1 + i\alpha x_2 + \beta x_1 + i\beta x_2 = \alpha(x_1 + ix_2) + \beta(x_1 + ix_2) = \alpha x + \beta x$

i.e. $(\alpha + \beta)\ x = \alpha x + \beta x\ \forall\ \alpha,\beta \in \mathbb{R}$, $x \in C$

Also, $(\alpha\beta)x = \alpha\beta(x_1 + ix_2) = (\alpha\beta)x_1 + (\alpha\beta)x_2 = \alpha(\beta x_1) + i\alpha(\beta x_2) = \alpha(\beta x_1 + \beta x_2) = \alpha(\beta(x_1 + ix_2)) = \alpha(\beta(x)) = \alpha(\beta x)$

Thus, $(\alpha\beta)x = \alpha(\beta x)\ \forall\alpha, \beta \in \mathbb{R}$, $x \in C$

Finally, $1.x = 1.(x_1 + ix_2) = 1.x_1 + i\,(1. X_2) = x_1 + ix_2 = x$

Therefore, the set $\mathbb{C}$ of complex number is a vector space over the field of real numbers.

1.1.2 LINEAR TRANSFORMATION OF VECTOR SPACES

A function $\mathscr{F}: V_1 \rightarrow V_2$ from a vector space V_1 to a vector space V_2 is called a linear transformation if

1. $\mathscr{F}(x + y) = \mathscr{F}(x) + \mathscr{F}(y)$ (Additive property)

2. $\mathscr{F}(\alpha x) = \alpha\, \mathscr{F}(x)$ (Homogeneous property)

For all x, y $\in$ V and $\alpha \in$ F. where F is the field of V_1 and V_2.

A linear transformation is also called a **linear mapping** or an **$\mathscr{F}$- homomorphism**

1.1.3 SUBSPACE OF A VECTOR SPACE

A Subset W of a vector space over a field F is called a **Subspace** of V if W is itself a vector space over F under addition and scalar multiplication defined on v.

1.1.4 LINEAR COMBINATION OF VECTORS

Let V be a vector space over a field F, let $u_1, u_2,...,u_k$ be vectors in V , and let $\alpha_1, \alpha_2,...,\alpha_k$ be scalars in F. Then $\alpha_1 u1 + \alpha_2 u_2 + \cdots + \alpha_k u_k$ is called a **linear combination** of the vectors $u_1, u_2,...,u_k$. The scalars $\alpha_1,...,\alpha_k$ are called the **weights** in the linear combination.

1.1.5 LINEAR SPAN OF A VECTOR

The set of all linear combinations of a list of vectors $v_1,\ldots,v_m$ in V is a subspace of v generated span of $v_1,\ldots, v_m$, denoted span $(v_1,\ldots, v_m)$. In other words,

Span $(v_1,\ldots, v_m) = \{a_1v_1 +\ldots+ a_mv_m : a_1,\ldots, a_m \in F$.

The span of the empty list ( ) is defined to be (0).

1.1.6 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

Definition: linear dependence

- A list of vectors in V is called linearly dependent if it is not linearly independent.
- In other words, a list $v_1,\ldots,v_m$ of vectors in V is linearly dependent if there exist $a_1,\ldots, a_m \in F$, not all 0, such that $a_1v_1 + \ldots + a_mv_m = 0$.

Definition: linear independence

Let V be a vector space and $v_1,v_2\ldots v_k \in V$. Then $\{ v_1,v_2\ldots v_k\}$ is linearly independent if and only if the vector equation $a_1v_1 +\ldots+ a_mv_m =0$ has the unique solution $a_1=\ldots.. = a_k$

1.1.7 BASIS OF A VECTOR SPACE

Let V be a vector space over a field F, and let u1, u2,...,un be vectors in V . We say that {u1, u2,...,un} is a basis for V if and only if {u1, u2,...,un} spans V and is linearly independent.

A subspace of a vector space is a vector space in its own right, so we often speak of a basis for a subspace.

1.1.8 DIMENSION OF A VECTOR SPACE

The number of element in a basis of V is called the dimension of V.

Let Vdenote a vector space. Suppose a basis of V has n vectors, n is called the dimension of v, we write dim(v) =n.

A vector space V is said to be finite-dimensional if there exist a finite subset of V which is a basis of V otherwise, V is said to be infinite-dimensional.

Example

1. R^2, the set of all ordered pairs(x, y) where x and y are in R. The standard basis of R^2 is {(1, 0),(0,1)}. The basis has two elements, therefore, dim (R^2) = 2.

2. R^3, the set of all ordered triples(x, y, z) where x, y, z are in R. The standard basis of R^3 is {(1,0,0),(0,1,0),(0,0,1)}. The basis has three elements, therefore, dim (R^3) =3

3. N”, the set of all ordered pairs$(x ,x_2,\ldots.x_n)$ where x, y and z are in R. The standard basis of r3 is {(1,0,0),(0,1,0),(0,0,1)}. The basis has three elements, therefore, dim (R^3) =3

4. P^2, the set of polynomials of degree less than or equal to 2 a basis for P^2 is {(l, x, x^2)}. The basis has three elements, therefore, dim (P^3) = 3

1.2 AIM AND OBJECTIVE

To show

I. The concept of metric space

II. The concept of inner product space

III. Every inner product space is metric space

1.3 RESEARCH METHODOLOGY

For the purpose of this study, both the primary and secondary sources of data collection are used. Textbooks, internet sources, PDFs and notes relating to the subjects matter were also used.

1.4 DEFINITION OF TERMS

INNER PRODUCT

A function $< , > : v \times v \rightarrow \mathbb{C}$ defined on a vector space v is called an inner product on v if it satisfies the following condition:

1. For every vector u, <u, u> is a non-negative real number and <u, u> = 0 iff u = 0. This means that <,> is positive definite.

2. For all vectors u and v, $<u,v> = \overline{< v, u >}$.. we say that $<,>$ is conjugate symmetric

3. For all vectors u and v and scalar α, $< \alpha u, v > = \alpha < u, v >$. We say that $<, >$ is homogeneous in the first argument

4. For all vectors u, v and w, $<u+v,w> = <u, w> + <v, w>$. we say that $< , >$ is additive in the first argument

NORM

Let $(v, \ < , >)$ be an inner product space. The norm, length or magnitude of the vector $||u||$, is defined by $\sqrt{(u,u)}$. The norm is always defined since $<u,u> = 0$ and therefore we can always take a square root.

1.5 ORGANISATION OF STUDY

This research work is broken down into five different chapters. Chapter one is basically introductory and gives insight to the general coverage of the work through some sub-topics such as aim and objective of study, significance of study etc. Chapter 2 captures topological space. Chapter 3 is about inner product spaces, some example and inequalities. Chapter 4 is about the properties of a metric arising in inner product space. The definition of a metric space, examples and the normed space. Chapter 5 is about the summary and conclusion.

CHAPTER TWO

Literature Review

2.1Topological space

Definition 2.1.1 Let X be an non-empty set. A set T of subsets of X is said to be topology on X if

1. X and the empty set Ø belong to T

2. The union of any (finite or infinite) number of sets in T belongs to T

3. The intersection of any two set in T belongs to T.

The pair (X, T) is called a topological space.

Definition 2.1.2: In the topological space (X,T) the elements of T are called open sets. If A is in T, the elements of A are called points.

Definition 2.1.4: If A is a subset of X and p is in X, the statement that **p is a limit point of A** means that every open set which contains p contains a point of A different from p.

Definition 2.1.5: If p is a point and U is a subset of X, the statement that U is a neighborhood of p means that there exists an open set V such that p is in V and V fis a subset of U.

Definition 2.1.6: If M is a subset of X, X-M is the set of all points which are in X, but not in M.

Definition 2.1.7: The statement that the subset M is closed means X-M is open

2.3 Separation axiom

The separation axioms are about the set of topological means to distinguish disjoint sets and disjoint points.

Separation Properties for Topological Spaces

T_O If p and q are distinct two points; there is an open set which contains one and not the other.

T_I If p and q are distinct two points; there is an open set which contains p and not q.

T_2 If p and q are distinct two points; there exist two disjoint open sets, one containing p and the other q. A **T_2** is also called a **Hausdorff space**

T_3 If p is a distinct point and k is a closed set not containing p, then there exist two disjoint open sets, one containing k and the other p. A **T_3** is also called **Regular space**

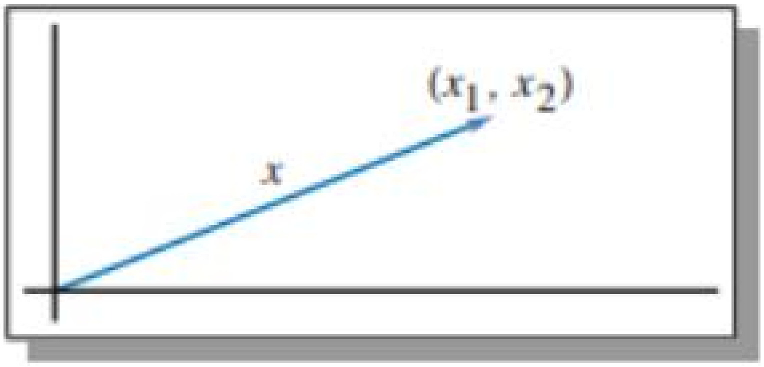
$T_{3.5}$ A topological space **X** is said to be completely regular space if every closed set **A** in **X** and a point x ϵ X, x

T_4 If k and m are two disjoint closed sets, and then there exist two disjoint open sets, one containing k, the other m. (Normal)

T_5 If k and m are mutually separated sets, and then there exist two disjoint open sets, one containing k, the other m. (Completely Normal)

Chapter 3

Inner Products space



The length of this vector is $\sqrt{X_1^2 + X_2^2}$

3.1 To understand the concept of inner product, think of vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ as arrows with initial point at the origin. The length of a vector x in R^2 or R^3 is called the **norm** of x, denoted by ||x||. Thus for x= $(x_1, x_2) \in R^2$, we have ||x|| = $\sqrt{x_1^2 + x_2^2}$

Similarly, if x =$(x_1, x_2, x_3) \in R^3$, then ||x||=$\sqrt{x_1^2 + x_2^2 + x_3^2}$.

Even though we cannot draw pictures in higher dimensions, the generalization to $\mathbf{R}^n$ is obvious: we define the norm of x = $(x_{1,\dots,}x_n) \in \mathbf{R}^n$ by

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$$

The norm is not linear on $\mathbf{R^n}$. To inject linearity into the discussion, we introduce the dot product.

3.2 Definition dot product

For x, y ϵ R^n, the dot product of x and y, denoted x · y, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.

Note that the dot product of two vectors in R^n is a number, not a vector. Obviously x · x = ||x|| for all x ϵ R^n . The dot product on R^n has the following properties:

- $x \cdot x \geq 0$ for all x ϵ $\mathbb{R}^n$;
- $x \cdot x = 0$ if and only if $x = 0$;
- for y ϵ $\mathbb{R}^n$ fixed, the map from $\mathbb{R}^n$ to $\mathbb{R}$ that sends x ϵ $\mathbb{R}^n$ to x · y is linear;
- $x \cdot y = y \cdot x$ for all x; y ϵ $\mathbb{R}^n$

3.3 Definition inner product

Let V be a vector space over F. An inner product on v is a function when assign to each element in V × V a scalar in F denote by $< u, v >$ satisfying the following conditions:

1. $< v, v > \geq 0$ for all v ϵ V (positivity)

2. $< v, v > = 0$ if and only if v = 0 (definiteness)

3. $< u, v > \; = \overline{< v, u >}$ for all u, v ϵ V (conjugate symmetry)

4. $< u + v, w > \; = \; < u, w > \; + \; < v, w >$ for all u ,v ,w ϵ V (additivity in first slot)

5. $< \alpha u, v > \; = \; \alpha < u, v >$ for all u, v ϵ V and α ϵ F (homogeneity in first slot)

A vector space V on which an inner product is defined on is called an inner product space

3.4 Basic properties of an inner product

(a) For each fixed u ϵ V, the function that takes v to $<v , u>$ is a linear map from V to **F**.

(b) $<0,u> = 0$ for every u ϵ V.

(c) $<u, 0> = 0$ for every u ϵ V.

(d) $<u, v + w> = < u, v > + < u, w >$for all u, v, w ϵ V.

(e) $< u, \lambda v > \; = \overline{\lambda} < u, v >$ for all λ ϵ F and u, v ϵ V.

Proof

(a) Part (a) follows from the conditions of additivity in the first slot and homogeneity in the first slot in the definition of an inner product.

(b) Part (b) follows from part (a) and the result that every linear map takes 0 to

(c) Part (c) follows from part (a) and the conjugate symmetry property in the definition of an inner product.

(d) Suppose u, v, w ϵ V. Then

$$< u, v + w > = \overline{< v + w, u >}$$

$$= \overline{< v, u > + < w, u >}$$

$$= \overline{< v, u >} + \overline{< w, u >}$$

$$= < u, v > + < u, w >$$

(e) Suppose λ ϵ **F** and u, v ϵ V. Then

$$< u, \lambda u > = \overline{< \lambda v, u >}$$

$$= \overline{\lambda < u, u >}$$

$$= \overline{\lambda}\, \overline{< v, u >}$$

$$= \overline{\lambda} < u, v >,$$

as desired.

3.5 Example of an inner products

(a) Let V be the set of all continuous real valued functions defined on the closed interval [0, 1]. Then V is an inner product space with inner product defined by

$< f, g > \ = \int_0^1 f(t)g(t)dt$

Solution

Given $< f, g > = \int_0^1 f(t)g(t)dt$

Let f, g, h ϵ V and α ϵ ℝ

(i) $< f, f > = \int_0^1 f(t)f(t)dt$

$= \int_0^1 [f(t)]^2 dt \geq 0$

(ii) $< f, f > = 0$ this implies $\int_0^1 [f(t)]^2 dt = 0$

This implies f (t) = 0 for all t ϵ [0,1]

Therefore f = 0

(iii) $< f + g, h > = \int_0^1 (f + g)(t)h(t)dt$

$= \int_0^1 [f(t) + g(t)]\, h(t)dt$

$= \int_0^1 [f(t)h(t) + g(t)h(t)]\, dt$

$= \int_0^1 f(t)h(t)dt + \int_0^1 g(t)h(t)dt$

$= < f, h > + < g, h >$

(iv) $< \alpha f, g > = \int_0^1 (\alpha f)(t)g(t)dt$

$= \int_0^1 \alpha f(t)g(t)dt$

$= \alpha \int_0^1 f(t)g(t)dt$

$= \alpha < f, g >$

(v) $\overline{< g, f >} = \overline{\int_0^1 g(t)f(t)dt}$

$= \int_0^1 \overline{g(t)}\,\overline{f(t)}\,dt$

$= \int_0^1 g(t)f(t)dt$ (g(t) & f(t) are real)

$= \int_0^1 f(t)g(t)dt$

$= < f, g >$

2. Show that the following function defines an inner product on $\mathbb{R}^2$, where u = (u_1, u_2) and v = (v_1, v_2). $< u, v > = u_1v_1 + 2u_2v_2$.

Solution

$< u, v > = u_1v_1 + 2u_2v_2$.

1. $< u, v > = u_1v_1 + 2u_2v_2$.

$= v_1u_1 + 2v_2u_2$

$= < v, u >$

2. Let W = (w_1, w_2)

$< u, v + w > = < (u_1, u_2), (v_1 + w_1, v_2 + w_2) >$

$= u_1(v_1 + w_1) + 2u_2(v_2 + w_2)$

$= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2)$

$= < u, v > + < u, w >$

3. Let c $\epsilon \mathbb{R}$, then

$c < u, v > = c(u_1v_1 + 2u_2v_2)$

$= (cu_1)v_1 + 2(cu_2)v_2$

$= < cu, v >$

4. $< v, v > = v_1^2 + v_2^2 \geq 0$

Moreover, $< v, v > \; = 0$

This implies $v_1^2 + 2v_2^2 = 0$

This implies $v_1 = v_2 = 0$, since $v_1^2 \geq 0$ and $v_2^2 \geq 0$

This implies v = (0,0)

3. For a < b define C[a, b] to be the vector space of all continuous real functions on [a, b]. For f, g ∈ C[a, b] define $<f, g> = \int_b^a f(x)g(x)\, dx$

4. The usual inner product on R^n.

3.6 Norms

Our motivation for defining inner products came initially from the norms of vectors on R^2 and R^3. Now we see that each inner product determines a norm.

Definition: norm, ||v||

For v ϵ V, the norm of v, denoted ||v||, is defined by ||v||, is defined by

$$||v|| = \sqrt{< v, v >}$$

3.7 Inequalities

There are two absolutely basic inequalities that are valid for any inner product space. It is named after two of the founders of modern analysis, the nineteenthcentury mathematicians Augustin Cauchy, of France, and Herman Schwarz, of Germany, who established it in the case of the L2 inner product on function space. The more familiar triangle inequality, that the length of any side of a triangle is bounded by the sum of the lengths of the other two sides, is, in fact, an immediate consequence of the Cauchy–Schwarz inequality, and hence also valid for any norm based on an inner product.

Theorem 3.7.1

The Cauchy-Schwartz Inequality

If x and y are any two vectors in an inner product space, then $|\langle x , y\rangle| \leq \| x\| \|y\|$

Proof

If y = o, then $\langle x , y\rangle = \langle x , 0\rangle = \overline{\langle 0, x\rangle} = 0\overline{\langle x, x\rangle} = 0$

And the proof follows.

If $y \neq 0$, then for any scalar λ, we have

$0 \leq \|x - \lambda y \| = \langle x - \lambda y, x - \lambda y \rangle$

$= \langle x, x\rangle - \bar{\lambda}\langle x,y\rangle - \lambda \langle y,x\rangle + \lambda\bar{\lambda} \langle y ,y\rangle$

$= \|x\| - \bar{\lambda} \langle x,y\rangle - \lambda[\langle y,x\rangle - \bar{\lambda} \|y\|^2]$

If we choose λ ∋

$\bar{\lambda} = \langle y,x\rangle / \|y\|^2$

Then we have

$0 \leq ||x||^2 - \frac{|<x,y>|^2}{||y||}$

This implies $0 \leq ||x||^2\ ||y||^2 - |<x\ ,\ y>|^2$

Therefore, $|<x,y>| \leq ||\ x\ ||\ \ ||y||$

Theorem 3.7. 2 Triangle Inequality

Suppose u, v ϵ V. Then

$$||u + v|| \leq ||u||+||v||.$$

This inequality is an equality if and only if one of u, v is a nonnegative multiple of the other.

Proof We have

$$||u + v||^2 = < u + v, u + v >$$

$$= < u,u > + < v,v > + < u,v > + < v,u >$$

$$= < u,u > + < v,v > + < u,v > + \overline{< u,v >}$$

$$= ||u||^2 + ||v||^2 + 2Re < u,v >$$

$$\leq ||u||^2 + ||v||^2 + 2| < u,v > |$$

$$\leq ||u||^2 + ||v||^2 + 2\ \big||u|\big|\ ||v|| \qquad 3.7.3$$

$$= (||u|| + ||v||)^2$$

Where 3.7.3 follows from the Cauchy- Schwarz inequality(3.7.1).

Chapter 4

METRIC SPACE

4.1 Metrics

A metric on a set X is a function that assigns a distance to each pair of elements of X. This distance function cannot be completely arbitrary—it must have certain properties similar to those of the physical distance between points in R^d . For example, the distance between any two points x, y in R d is nonnegative and finite, and it is zero only when the two points are identical. Further, the distance between x and y is the same as the distance between y and x, and if we have three points x, y, z, then the length of any one side of the triangle that they determine is less than or equal to the sum of the lengths of the other two sides (this is called the Triangle Inequality).

A metric is a function, defined on pairs of elements of a set, that has similar properties. Here is the precise definition

4.2 DEFININTION OF METRIC SPACE

(Metric Space). Let X be a nonempty set and R be the set of real number. A metric on X is a function $d: X \times X \to R$ such that for all $x, y, z \in X$ we have:

(a) Non-negativity: $d(x, y) \geq 0$

(b) Uniqueness: $d(x, y) = 0$ if and only if $x = y$,

(c) Symmetry: $d(x, y) = d(y, x)$, and

(d) The Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

If these conditions are satisfied, then X is a called a metric space. The number

$d(x, y)$ is called the distance from x to y.

Theorem 4.3. (Rearrangement of the Triangle Inequality)

Suppose X is a metric space and $a, b, c \in X$. Then $|d(a, b) - d(b, c)| \leq d(a, c)$.

Proof

The triangle inequality for d yields first $d(a, b) \leq d(a, c) + d(c, b)$ and second $d(c, b) \leq d(c, a) + d(a, b)$. Using symmetry, rearrangement of the first of these two inequalities gives $d(a, b) - d(b, c) \leq d(a, c)$ and rearrangement of the second gives $d(b, c) - d(a, b) \leq d(a, c)$. The two together prove the theorem

4.4 EXAMPLES OF METRIC SPACE

Usual metric on $\mathbb{R}$

Let $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a metric on $\mathbb{R}$ given by $d(x_1, x_2) = |\ x_1 - x_2\ |$. Then "d" is called a usual metric on $\mathbb{R}$ and $(\mathbb{R},d)$ is called the usual metric space.

Example

Let x = r be the set of all real number and let $d: R \times R \rightarrow R$ be defined by $d(x_1, x_2) = |\ x_1 - x_2|$ denotes the absolute value of the number $x_1 - x_2$. Show that (R, d) is a metric space.

Proof:

Here the function $d: R \times R \rightarrow R$ is defined as $d(x_1, x_2) = | x_1 - x_2 |$

Property (1)

Let $x_1, x_2 \in \mathbb{R}$, since $| x_1 - x_2 | \geq 0 \ \forall \ x_1, x_2 \in \mathbb{R}$

Therefore $d(x_1, x_2) \geq 0$

Property (2)

Let $d(x_1, x_2) = 0$ this implies $| x_1 - x_2 | = 0$

This implies $x_1 - x_2 = 0$

Therefore $x_1 = x_2$

Also, let $x_1 = x_2$ this implies $x_1 - x_2 = 0$

This implies $| x_1 - x_2 | = 0$

This implies $d(x_1 - x_2)| = 0$

Thus $d(x_1, x_2) = 0$ i.e $x_1 = x_2$

Property (3)

Since $d(x_1, x_2) = | x_1 - x_2 |$

$= | -x_2 + x_1 | = | - (x_2 - x_1) |$

$= | x_2 - x_1 |$

$= d(x_2, x_1)$

Therefore $d(x_1, x_2) = d(x_2, x_1)$

Property (4)

Let $x_1, x_2, x_3 \in \mathbb{R}$, then

$d(x_1, x_2) = | x_1 - x_2 |$, $d(x_1, x_3) = | x_1 - x_3 |$ and $d\,(x_2 - x_3) = | x_2 - x_3 |$

Now $d(x_1, x_3) = | x_1 - x_3 |$

$= | x_1 - x_2 + x_2 - x_3 |$

$\leq | x_1 - x_2 | + | x_2 - x_3 |$

$= d(x_1, x_2) + d(x_2, x_3)$

Therefore $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$

Thus (R,d) is a metric space

Usual metric on $\mathbb{R}^2$

Let $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be metric on $\mathbb{R}^2$ given by $d[(x_1,y_1),(x_2,y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Then "d" is called a usual metric on $\mathbb{R}^2$ and ($\mathbb{R}^2$, d) is called the usual metric space.

Usual metric on $\mathbb{R}^3$

Let $d : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a metric on $\mathbb{R}^3$ given by $d\,[(x_1,y_1,z_1), (x_2,y_2,z_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$. Then "d" is called a usual metric on $\mathbb{R}^3$ and ($\mathbb{R}^3$, d) is the usual metric space.

Euclidean metric $\mathbb{R}^n$

Let d: $R^n \times \mathbb{R}^n \to \mathbb{R}$ be a metric given by d (x,y) = $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ or d(x,y) = max $|x_i - y_i|$

Then "d" is called the Euclidean metric on $\mathbb{R}^n$.

Example 2: (The Euclidean metric on $\mathbb{C}$ i.e extension of Euclidean metric on $\mathbb{R}$).

Let $\mathbb{C}$ be the set of all complex numbers and $d: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be a function defined as

$$d(z, z^\iota) = |z - z^\iota| \; for \; all \; z, z^\iota \in \mathbb{C}.$$

Then d is a metric on $\mathbb{C}$, called the usual metric or Euclidean metric on $\mathbb{C}$. Of course, d is an extension to $\mathbb{C} \times \mathbb{C}$ of the Euclidean metric u on $\mathbb{R}$ i.e $u = d/_{\mathbb{R}}$.

Example 3: (The Euclidean Plane $\mathbb{R}^2$)

Let $X = \mathbb{R}^2$ be the set of all ordered pairs of real numbers and $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined as $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} for \; all \; x = (x_1, x_2), y = (y_{1,} y_2) \in \mathbb{R}^3$.

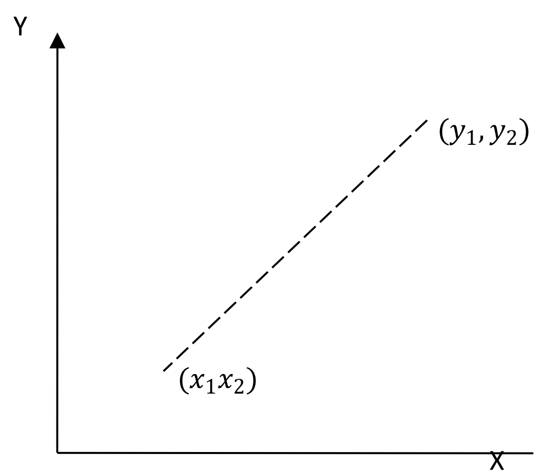


Figure 1

We shall show that d is a metric on $\mathbb{R}^2$.

By definition, $d(x, y) \geq 0 \; for \; all \; x, y \in \mathbb{R}^2$.

For any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$,

$d(x, y) \Leftrightarrow \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0$

$$\Leftrightarrow x_1 - y_1 = 0 \; and \; x_2 - y_2 = 0$$
$$\Leftrightarrow x_1 = y_1 \; and \; x_2 = y_2$$
$$\Leftrightarrow x = y$$

For all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$,

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} = d(y, x).$$

Now, suppose $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$ be points. Then

$$d(x, y) = \sqrt{[(x_1 - y_1)^2 + (x_2 - y_2)^2]}$$
$$= \sqrt{[(x_1 - z_1) + (z_1 - y_1)]^2 + [(x_2 - z_2) + (z_2 - y_2)]^2}$$
$$= \sqrt{(a + b)^2 + (c + d)^2},$$

where $a = x_1 - z_1, b = z_1 - y_1, c = x_2 - z_2 \; and \; d = z_2 - y_2$. Applying Theorem 4.7, we get

$$d(x, y) \leq \sqrt{a^2 + c^2 +} \sqrt{b^2 + d^2}$$
$$\leq \sqrt{(x_1 - z_1)^2 + (z_1 - y_1)^2} + \sqrt{(x_2 - z_2)^2 + (z_2 - y_2)^2}$$
$$\leq d(x, z) + d(z, y).$$

Thus, all the four conditions (or axioms) are satisfied. It follows that d is a metric on $\mathbb{R}^2$ and the ordered pair $(\mathbb{R}^2, d)$ is a metric space. The metric d is called the Euclidean metric on $\mathbb{R}^2$, and the metric space $(\mathbb{R}^2, d)$ is called the 2-dimensional Euclidean Space$\mathbb{R}^2$.

Example 4: (Taxi Cab Metric on $\mathbb{R}^2$)

Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers and $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

$d(x, y) = |x_1 - y_1| + |x_2 - y_2| \; for \; all \; x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

We shall show that d is a metric on $\mathbb{R}^2$.

By definition,

$$d(x,y) \geq 0 \; for \; all \; x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$$

and hence (1) holds.

For (2), consider any$x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, then

$$d(x,y) = 0 \iff |x_1 - y_1| + |x_2 - y_2| = 0$$

$$\iff |x_1 - y_1| = 0 \; and \; |x_2 - y_2| = 0$$

$$\iff x_1 = y_1 \; and \; x_2 = y_2$$

$$\iff (x_1, x_2) = (y_1, y_2)$$

i.e $x = y$.

Now again for (3), consider any $x = (x_1, x_2), y = (y_1, y_2) \; in \; \mathbb{R}^2$, then

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y,x).$$

Thus, (3) is satisfied.

To show triangle inequality (4), let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$ be any points. Then

$$\begin{aligned} d(x,y) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |(x_1 - z_1) + (z_1 - y_1)| + |(x_2 - z_2) + (z_2 - y_2)| \\ &\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &\leq |x_1 - z_1| + |x_2 - z_2| + |z_1 - y_1| + |z_2 - y_2| \leq d(x,z) + d(z,y). \end{aligned}$$

Hence, all the four conditions (or axioms) of a metric is satisfied by d, therefore d is a metric on $\mathbb{R}^2$. Also, it forms a metric space denoted $(\mathbb{R}^2, d)$.

Example 5: (Maximum Metric of $\mathbb{R}^2$)

Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers and $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

$$d(x,y) = max\{|x_1 - y_1|, |x_2 - y_2|\} \; for \; all \; x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

We shall show that d is a metric on $\mathbb{R}^2$.

By definition, $d(x,y) \geq 0$ is a non-negative function and hence (1) holds.

For (2), consider $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, then

$$d(x,y) = 0 \Leftrightarrow max\{|x_1 - y_1|, |x_2 - y_2|\} = 0$$

$$\Leftrightarrow |x_1 - y_1| = 0, |x_2 - y_2| = 0$$

$$\Leftrightarrow x_1 = y_1 \ and \ x_2 = y_2$$

$$\Leftrightarrow (x_1, x_2) = (y_1, y_2)$$

i.e, $x = y$.

Now, for (3), for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$,

$$d(x,y) = max\{|x_1 - y_1|, |x_2 - y_2|\} = max\{|y_1 - x_1|, |y_2 - x_2|\} = d(y,x).$$

Thus (3) is satisfied.

To prove the triangle inequality (4), let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^3$. Then

$$|x_1 - y_1| = |(x_1 - z_1) + (z_1 - y_1)| \le |x_1 - z_1| + |z_1 - y_1|$$

$$\le max\{|x_1 - z_1|, |x_2 - z_2|\}$$

$$+ max\{|z_1 - y_1|, |z_2 - y_2|\}$$

$$= d(x,z) + d(z,y) \ i.e \ |x_1 - y_1| \le d(x,z) + d(z,y) \ldots\ldots\ldots . A.$$

Similarly,

$$|x_2 - y_2| \le d(x,z) + d(z,y) \ldots\ldots\ldots . B$$

From (A) and (B) it follows that

$$max\{|x_1 - y_1|, |x_2 - y_2|\} \le d(x,z) + d(z,y).$$

i.e $$d(x,y) \le d(x,z) + d(z,y).$$

Hence, all the four conditions(or axioms) of a metric are satisfied by d, therefore d is a metric on $\mathbb{R}^2$ and the ordered pair $(\mathbb{R}^2, d)$ is the metric space.

Example 6:

Let $X = \mathbb{R}^2, and \ let \ d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y)=\begin{cases} |x-y| & if\ xand\ y\ are\ in\ the\ same\ ray\ from\ the\ origin \\ |x|+|y| & otherwise \end{cases}$$

where $x=(x_1,x_2)\ and\ y=(y_1,y_2)\in\mathbb{R}^2$.

Show that d is a metric on $\mathbb{R}^2$. (Here $|x-y|=u(x,y)\ and\ |x|=u(x,0)$ and u is Euclidean metric on $\mathbb{R}^2$.

Proof:

Clearly, $d(x,y)\geq 0\ for\ all\ x,y\in\mathbb{R}^2$. For any $x,y\in\mathbb{R}^2$,

$$d(x,y)=\begin{cases} |x-y| & if\ xand\ y\ are\ in\ the\ same\ ray\ from\ the\ origin \\ |x|+|y| & otherwise \end{cases}$$

$$=\begin{cases} u(x,y) & if\ xand\ y\ are\ in\ the\ same\ ray\ from\ the\ origin \\ u(x,0)+u(y,0) & otherwise \end{cases}$$

$$=\begin{cases} u(y,x) & if\ xand\ y\ are\ in\ the\ same\ ray\ from\ the\ origin \\ u(y,0)+u(x,0) & otherwise \end{cases}$$

$$=d(y,x)$$

By definition of d, observe that

$$d(x,y)\geq|x-y|\ for\ all\ x,y\in\mathbb{R}^2 \ldots\ldots. A$$

Thus for any $x,y\in\mathbb{R}^2$,

$$d(x,y)=0\Longrightarrow|x-y|=0\Longrightarrow x=y.$$

Also, $x=y$ implies that $x\ and\ y$ are in same ray from the origin and therefore

$$d(x,y)=|x-y|=0.$$

Finally, to prove triangle inequality, consider and $x,y,z\in\mathbb{R}^2$.

Case 1: $x\ and\ y$ are in the same ray from the origin. Then

$$d(x,y)=|x-y|=u(x,y)\leq u(x,z)+u(z,y)=|x-z|+|z-y|$$
$$\leq d(x,z)+d(z,y)[using\ (A)].$$

Case 2: $x\ and\ y$ are in the different rays from the origin.

Subcase 1: $z\ and\ x$ are in different rays from the origin. Then

$$d(x,y) = |x| + |y| = u(x,0) + u(0,y) \leq u(x,0) + [u(0,z) + u(z,y)]$$
$$= |x| + [|z| + |y-z|] = [|x| + |z|] + |y-z| \leq d(x,z) + d(y,z)$$
$$= d(x,z) + d(z,y).$$

Subcase 2: $z \text{ and } x$ are in same ray from the origin. Then $z \text{ and } y$ are in different rays from the origin. Therefore,

$$d(x,y) = |x| + |y| = u(x,0) + u(0,y) \leq u(x,z) + u(z,0) + u(0,y)$$
$$= |x-z| + |z| + |y| = d(x,z) + d(z,y).$$

Thus in all the cases triangle inequality is satisfied and hence d is a metric on $\mathbb{R}^2$.

Example 7: (Metric on the set of Interval of $\mathbb{R}$).

Let ρ denote the collection of closed intervals of $\mathbb{R}$ of the type [a, b]. i.e,

$$\rho = \{[a,b]: a,b \in \mathbb{R}, a \leq b\}.$$

The function $d: \rho \times \rho \to \mathbb{R}$ given by

$d(I,J) = \max\{|c-a|, |d-b|\}$ $for\ all\ I = [a,b], J = [c,d]$ in ρ is a metric on ρ.

On the lines similar to example 5 above, it is straightforward to check that condition (1), (2), (3) of the definition of a metric holds.

Now, we need to check triangle inequality (4), let $I = [a,b], J = [c,d], K = [e,f] \in \rho$ be any closed intervals of $\mathbb{R}$. Then

$$d(I,J) = \max\{|c-a|, |d-b|\} = \max\{|(c-e) + (e-a)|, |(d-f) + (f-b)|\}$$
$$\leq \max\{|c-e| + |e-a|, |d-f| + |f-b| \ldots\ldots. A$$

Now,

$$|c-e| \leq max\{|c-e|, |d-f|\} = d(K,J) \ldots\ldots. (*)$$

$$|e-a| \leq max\{|e-a|, |f-b|\} = d(I,K) \ldots\ldots (**)$$

Adding (*) and (**), we get

$$|c-e| + |e-a| \leq d(I,K) + d(K,J) \ldots\ldots\ldots\ldots B$$

Similarly,

$$|d-f|+|f-b| \leq d(I,K)+d(K,J) \ldots\ldots\ldots C$$

From (B) and (C),

$$\max\{|c-e|+|e-a|, |d-f|+|f-b| \leq d(I,K)+d(K,J) \ldots\ldots\ldots (D)$$

Now, from (A) and (D), we conclude that

$$d(I,J) \leq d(I,K)+d(K,J).$$

Thus condition (4) holds and hence d satisfies all the four axioms (or conditions) of a metric. So, d is a metric on ℝ.

NOTE: In the above example we restricted ourselves to only closed intervals of the type $[a,b]$ and it is necessary to do so. Excluding intervals of the type $(-\infty,a),(-\infty,a],(a,\infty),[a,\infty)$ and $(-\infty,\infty)$ ensures that d takes values in ℝ. Also, if we have included all intervals of the type $(a,b),(a,b]$ *and* $[a,b)$, then the function d would have failed to satisfy condition (2) of the definition. Infact

$$d\big([a,b],[a,b)\big)=0 \; but \; [a,b] \neq [a,b).$$

Example 8: (The Space of all functions from [0,1] to [0,1]

Let F be the set of all functions from [0,1] to [0,1] and d be the real valued function on $F \times F$ given by

$$d(f,g)=\sup\{|f(x)-g(x)|: x \in [0,1]\} \; for \; all \; f,g \in F.$$

Then (F,d) is a metric space.

Proof:

To verify the fact that (F,d) is a metric space, we shall first verify that d is indeed a function i.e, d is well defined. For any $f,g \in F$,

$$f(x), g(x) \in [0,1] \Rightarrow |f(x) - g(x)| \leq 1 \; for \; all \; x \in [0,1]$$
$$\Rightarrow \{|f(x) - g(x)| : x \in [0,1]\} \; is \; bounded \; above$$
$$\Rightarrow \sup\{|f(x) - g(x)| : x \in [0,1]\} \; exists.$$

It follows that $d(f,g) < \infty \; for \; all \; f,g \in F$. hence d is well defined.

Next to show that d is a metric on F, we shall verify triangle inequality (other properties can be easily verified). Consider any $f,g,h \in F$. Then for any $x \in [0,1]$, we observe that

$$|f(x) - h(x)| \leq \sup\{|f(x) - h(x)|\}$$

And

$$|h(x) - g(x)| \leq \sup\{|h(x) - g(x)|\}.$$

Then

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$$
$$\leq \sup\{|f(x) - g(x)|\} + \sup\{|h(x) - g(x)|\} = d(f,h) + d(h,g).$$

Where $x \in [0,1]$. i.e, $|f(x) - g(x)| \leq d(f,g) + d(h,g) \; for \; each \; x \in [0,1]$.

Now taking supremum over all $x \in [0,1]$, we get

$$\sup\{|f(x) - g(x)|\} \leq d(f,h) + d(h,g)$$

i.e, $d(f,g) \leq d(f,h) + d(h,g)$.

Hence, triangle inequality follows, therefore d is a metric on F and (F,d) is a metric space.

4.5 SOMETYPES OF METRIC SPACE

4.5.1 Complete metric space

A metric space (x,d) is said to be complete if every Cauchy sequence converges in (x,d)

4.5.2 Bounded space

A metric space (x,d) is called bounded if there exist some number r ∍ d(x,y) ∀ x , y ϵ (x ,d). The smallest such possible r is called the diameter of (x,d)

4.5.3 Pre- Compact (totally bounded) space:

The metric space (x,d) is called pre-compact or totally bounded if ∀ r > 0 ∃ finitely many open balls of radius r whose union covers (x ,d).

4.5.4 Compact metric space:

A metric space (x,d) is compact if every sequence in (x,d) has a subsequence in (x,d) has a subsequence that converges to a point in (x,d).

4.5.5 Connected metric space:

A metric space (x,d) is said to be connected if the only subsets that are both open and closed are the empty set and (x,d) itself.

4.5.6 Separable spaces:

A metric space is separable if it has a countable dense subset

Inequalities:

- Cauchy –Schwarz Inequality: Let $x_i, y_i \in \mathbb{R}\ for\ i = 1,2, \dots, n,$ then the following inequality holds:

$$\sum_{i=1}^{n} |x_i, y_i| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} \sqrt{\sum_{i=1}^{n} |y_i|^2}.$$

- Minkowski's Inequality: Let $x_i, y_i \in \mathbb{R}\ for\ i = 1,2, \dots, n\ and\ p \geq 1$ be any real number. Then

$$\left(\sum_{i=1}^{n} |x_i, y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

- Minkowski's Inequality For Infinite Sums: Let $p \geq 1$ be any real number and $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ be real sequences such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty\ and\ \sum_{n=1}^{\infty} |y_n|^p < \infty.$$

Then

$$\sum_{n=1}^{\infty} |x_n + y_n|^p$$

is convergent. Moreover,

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}.$$

Theorem 4.7:

For any $w, x, y, z \in \mathbb{R}$,

$$\left[\sqrt{(w^2 + y^2)}\sqrt{(x^2 + z^2)}\right]^2 \geq [w + x]^2 + [y + z]^2.$$

Proof: Consider, $(wz - xy)^2 \geq 0$, this implies that:

$$w^2z^2 + x^2y^2 - 2wzxy \geq 0$$

$$w^2z^2 + x^2y^2 \geq 2wzxy.$$

Now, adding $w^2x^2 + y^2z^2$ to both sides, we have

$$w^2x^2 + y^2z^2 + w^2z^2 + x^2y^2 \geq w^2x^2 + y^2z^2 + 2wzxy$$

$$w^2(x^2 + z^2) + y^2(x^2 + z^2) \geq (wx + yz)^2$$

$$(w^2 + y^2)(x^2 + z^2) \geq (wx + yz)^2$$

$$\sqrt{(w^2 + y^2)(x^2 + z^2)} \geq |wx + yz|$$

$$2\sqrt{(w^2 + y^2)(x^2 + z^2)} \geq 2|wx + yz|$$

$$\Rightarrow (w^2 + y^2) + (x^2 + z^2) + 2\sqrt{(w^2 + y^2)(x^2 + z^2)} \geq (w^2 + y^2) + (x^2 + z^2) + 2|wx + yz|$$

$\Longrightarrow \left[\sqrt{(w^2 + y^2)} + \sqrt{(x^2 + z^2}\right]^2 \geq [w + x]^2 + [y + z]^2$, Since any of the real numbers w, x, y, z can be replaced by its negative if it is not positive for the condition $wx + yz \geq 0$ to hold. Hence, the inequality as required. The proof is now complete.

4.7 NORM

Definition

Let v be any vector space over the field F, where F is ℝ or ℂ. A function ||.|| : v→ R is said to be a norm on X, if it satisfies the following conditions

(i) ||u|| = 0 for all u ϵ V and ||u|| = 0 if and only if u = 0

(ii) ||αu|| = |a| ||u|| for all α ϵ F & u ϵ V

(iii) ||u + v|| ≤ ||u|| + ||v|| for all u, v ϵ V

Theorem 4.8

Let V be an inner-product space, and define $d(x, y) = \|x - y\|$. Then d is a metric. Proof: From definition above, we see easily that $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x - y = 0$, i.e., $x = y$.

Also $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$.

Finally $d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x,y) + d(y,z)$. So every inner-product space is a metric space.

CHAPTHER FIVE

SUMMARY

In which shown many concepts such as vector space, norm, orthogonally used to explain inner product space.

Also the concept such as topological space used to explain metric space.

Each chapter has an outline which I followed. Firstly, the introduction and definition of basic term; starting with vector space since an inner product is define d on it. Then topological space and the separation axiom. Finally, the concept of metric space and inner product space.

CONCLUSION

The inner product is more general than a metric - any inner product induces a metric by its norm $d(x,y)=\|x-y\|=\sqrt{\langle x-y, x-y\rangle}$, but not all metric spaces are a consequence of some inner product. An inner product is more general, since it can also give you concepts like angles, projections, and many other geometric features (even if the space is not Euclidean).

Every inner product space is definitely a metric space, which gives it plenty of interesting properties, especially when it is a complete metric space (complete inner product spaces are called Hilbert spaces, and they are some of the richest structures in mathematics).

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