LEGENDRE'S POLYNOMIAL AND SERIES

BY

Somtochukwu Blessing OFFOR

PSC1707721

DEPARTMENT OF MATHEMATICS

FACULTY OF PHYSICAL SCIENCES

UNIVERSITY OF BENIN

BENIN CITY, NIGERIA

JANUARY 2023

LEGENDRE’S POLYNOMIAL AND SERIES

BY

Somtochukwu Blessing OFFOR

PSC1707721

SUPERVISOR: PROF. E.O. OGHRE

A PROJECT SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, FACULTY OF PHYSICAL SCIENCES, UNIVERSITY OF BENIN, BENIN CITY, IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF BACHALOR OF SCIENCE (B.SC HONOURS) DEGREE IN PURE AND APPLIED MATHEMATICS

JANUARY, 2023

UNDERTAKING

This project work was carried out by **Somtochukwu Blessing OFFOR** with matriculation number **PSC1707721**. I have not copied the work of other author(s). All work used have been duly cited and acknowledged.

Somtochukwu Blessing OFFOR **Date**

CERTIFICATION

This is to certify that this project titled "**LEGENDRE'S PLOYNOMIAL AND SERIES"** was carried out by **Somtochukwu Blessing OFFOR** under the supervision of **PROF. E.O. OGHRE** in the department of Mathematics, University of Benin, Benin City.

______________________ ______________________

Prof. E. O. OGHRE **Date**

(Project Supervisor)

______________________ ______________________

Prof. R. I. Okuonghae **Date**

(Head of Department)

DEDICATION

This project is dedicated to the Almighty God, who brought me this far with His grace, and also to my lovely family for their endless support all through the way.

ACKNOWLEDEMENT

This project wouldn’t have been a success if not for the work of the Almighty God, who saw me through the period of struggle day and night, and who upheld me and gave me the physical and Spiritual strength to face the rigours in the cause of the work.

Secondly, my profound gratitude goes to my hardworking supervisor **PROF. E.O. OGHRE** who in no small measure contributed immensely with his directions, corrections, criticisms, and advice during the period of this work.

I continue to be indebted to my lovely parents; Eng and Dr. Mrs Nkechi Offor, my aunts, siblings, cousins, Mr and Mrs Nathaniel, and Mr and Mrs Flora Ekhibise, for their determination and inspiration in shouldering all financial burden in spite all odds. Thanks a million for all your care and support and May God reward you uncountable fold.

This acknowledgement would not be complete if I fail to acknowledge my outstanding course adviser; Dr.Chigozie Ibe, the dean of physical sciences Prof. Joseph Osawenkhae,the H.O.D of mathematics department; Prof. Robert .I. Okuonghae, my ever hardworking and committed lecturers; Dr. Omokaro, Ass. Prof. Orobosa, Dr. Osawaro, Dr. Alufai, and staff members of Mathematics Department University of Benin, who in one way or the other have been a source of motivation and support in my academic pursuit.

Finally, I acknowledge the various roles and contributions made by my course mates and friends; and a host of others.

TABLE OF CONTENT

ABSTRACT

Legendre polynomial is a second order ordinary differential equation as well as a type of Fourier Series written in the system of orthogonal polynomials with a vast number of mathematical properties and numerous applications. It is obtained through linear differential equation methods based on Sturm-Liouville theory.

CHAPTER ONE

INTRODUCTION

1.1 INTRODUCTION TO LEGENDRE'S DIFFERENTIAL EQUATIONS AND POLYNOMIALS.

Definition:

In the Sturm-Liouville boundary value problem, there is an important special case called the Legendre's differential equation which arises in numerous problems especially in those exhibiting spherical symmetry.

The Legendre's polynomials generate the power series that help in solving the Legendre's differential equation and polynomials which is given as.

$$(1-x^2)y''(x)-2xy'(x)+n(n+1)y(x)=0 \qquad (1.1.1)$$

This ordinary differential equation with variable coefficient is named in honor of Adrie-Marie Legendre (1752 – 1839) where n is a real number. The solutions of this equation are called the Legendre functions of degree n.

Thus the Legendre polynomials are obtained whenever n is a non-negative integer; i.e. n = 0, 1, 2…

Equation (1.1.1) may be solved by the Frobenius method. In this case, the indicial equation gives c = 0 and c = 1 and the two corresponding solutions are:

(a) $$c=0: y=a_0\left\{1-\frac{n(n+1)n^2}{2!}+\frac{n(n-2)(n+1)(n+1)n^4\cdots}{4!}\right\}$$

(b) $$c=1: y=a_1\left\{n-\frac{n(n-1)(n+2)n^3}{3!}+\frac{(n-1)(n-3)(n+2)(n+4)n^5\cdots}{5!}\right\}$$

where a_0 and a_1 are the arbitrary constants.

1.1.1 A polynomials is a finite sum of terms like $a_k x^k$, where k is a positive integer or zero. There are two sets of polynomials such that the product of any two different ones multiplied by function

W(x) called a weight function and integrate over a certain interval vanishes. Whether Legendre polynomial is even or odd function depends on its degree n.

The polynomials can also be found by solving the differential equation by determining the coefficient of a power series substituted in the equation.

Consider the polynomials $G_n(x) = \frac{d^n}{dX^n}(n^2-1)^n \cdots$ …(1.1.2)

The quantity to be differential is indeed a polynomial of degree 2n and consisting of only even powers. When differentiated n times, it becomes a polynomials of order n consisting of either an odd or all even powers of x, and n is odd or even.

1.1.2 RODRIGUE'S FORMULA

Rodrigue's was awarded a doctorate in Mathematics from the Ecole Normale in (1816) for a thesis that contains one of the two results for which he is know today namely the Rodrigue's formula for Legendre polynomials.

The story of the Rodrigue's formula for Legendre polynomials is somewhat more complicated due to the fact that Rodrigue's paper on the subject does not appear to have been noticed at the time, or if it was then, it was quickly forgotten. Thus, the Rodrigue's formula is a solution to the Legendre's differential equation. The Rodrigue's formula is given by:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \quad n = 0,1,2,\cdots, \tag{1.1.3}$$

The first six polynomials using the Rodrigue's formula are:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$$

$$P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$$

$$P_5(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right).$$

1.2 BACKGROUND OF THE STUDY

Mathematics in its varied form has influenced the trend of things in many other fields of study. Following with recent trends, one notices that mathematics is a bold subject for solving many natural occurring phenomena. To mention but a few are in the area of study life mathematics Physics, Mathematical biology, Mathematical economics, engineering and so on and so forth.

However, the possibility of these varied applications came as a result of the many ramifications of mathematics. e.g statistics, probably theory, algebra, topology, which has given birth to calculus etc. These have further led researchers into many other branches of mathematics (Differential equation) is not left out in this case.

More so, just like some other branches of mathematics (real analysis, variational calculus), Differential equation emerged from the infinitesimal theories of calculus. This was in the late seventeenth, and eighteenth centuries. The study arose when early practitioners of calculus attempt to describe some physical problem originating from mechanics and as well astronomy: Gravitational attraction, pendulum motion, elasticity, and orthogonal trajectories and curves.

The name “differential equation” was given to these equation by LEIBNITZ in the year (1646-1716) which shall be used or considering in solving the Legendre’s Differential and polynomials.

Legendre’s differential and polynomials (also known as **Legion polynomials**) introduced by Adrie-Marie in the year 1752- 1833, is just one among many solutions to various differential equations existing today. From the implication of the naming of these functions, Adrie-Marie Legendre is acclaimed to be the one who developed the Legendre differential and polynomials. His essay "Research Sar La Trajeetoine in resistant environments" won the prize and launched Legendre on his research career. Legendre then introduced what we called today the Legendre functions and used these to determine, using power series, the attraction of an ellipsoid at any exterior point. In particular he published on celestial mechanics with papers such as "research for the figure of the planets" in 1784 which contains the Legendre polynomials number theory with, for example "Undetermined Analysis Research" in 1785 and the theory of elliptic function with papers on integrations by elliptic ones in 1786. (At Aubry, on the arithmetic work of Lagrange; Gauss, Mathematics Education 1909).

1.3 OBJECTIVE OF THE STUDY

This work is aimed at achieving the following;

(i) To enumerating some properties and theories involving Legendre functions and polynomials.

(ii) To determine the finite solutions of Legendre's differential equations.

(iii) To describe the applicability of these functions in some areas of Mathematics Physics.

1.4 SCOPE OF THE STUDY

In many occasions in Physics, Legendre's polynomials in terms of angles occur where spherical symmetry is involved. The co latitude angle in spherical coordinates is the angle θ.

The longitude angle, φ, appears in a multiplying factor. Together, they make a set of function called spherical harmonics. These functions express the symmetry of the two-sphere under the action of the lie group δ0(3). As such, Legendre polynomials can be generated to express the symmetries of semi-simple lie groups and Riemannian symmetric spaces.

Since Legendre's differential equation is a second order differential equation, two sets of functions are needed to form the general solution. Legendre polynomials of the second kind $Q_n(x)$ are then introduced. The Legendre polynomials can also be found by solving the differential equation by determining the coefficient of the power series substituted in the equation. This method was often used in quantum mechanics test.

1.5 IMPORTANCE AND APPLICATON OF THE STUDY

In the course of various researches, researchers often find great treasure in Legendre polynomials function. The formulas (which though often lack theoretical interest) have been required in many physical applications.

However, over the years talks about these important functions have died down.

To this end, this study tends to bring back to limelight some of the important properties of Legendre's polynomials function. It is also expressed in precise and succinct languages the theories resulting from dealing with these functions.

Furthermore, this study applies the theories of Legendre polynomials functions for solving natural occurring phenomena. It is also note worthy that this study will be immense help those who wish to delve into studies involving Legendre polynomials. For example, for spherical coordinates, the Legendre polynomials are sufficient so long the problem is axially symmetric.

Differential equation is remarkable in predicting the world around us. They are used to describe population growth of species, the change in investment return over time, even in solving radioactive decay problems, continuous compound interest problems, orthogonal trajectories, seismic waves. They are used in specific field such as, in the field of medicine where differential equations are used for modeling cancer growth or the spread of disease. In chemistry, they are used for modeling chemical reactions and to compute radioactive half-life. In economics, they are used to find optimum investment strategies. In physics, they are used to describe the motion of waves, pendulums. In Engineering, they are used for describing the movements of electricity. Differential equations are essential tools for describing the nature of the physical universe and also an essential part of models for computer graphics and vision. Even in Bots (short for robots), partial and ordinary differential equations helps to provide shapes and interior and exterior designs for machines.

Many differential equations are formed from physical problems in the environment and other practical fields using the basic equations of science and engineering.

a) **MECHANICS:** The basic law of dynamics in the Newton law momentum i.e force, F, is proportional to the rate of change of momentum.

$$\Rightarrow F = \frac{d}{dt}(mv) \qquad (1.1.4)$$

where m is the mass, and v is velocity in time t.

b) **ENVIRONMENT:** According to Newton's law of cooling, the rate of change of temperature, T, of a body with time t, is directly proportional to the difference in temperature of the body, T, and that of the surrounding medium, T_m, i.e.

$\frac{dT}{dt} = -K(T - T_m)$, where K is a constant, i.e.

$$\frac{dT}{dt} + KT = KT_m \qquad (1.1.5)$$

c) **ELECTRIC CIRCUIT:** A simple series electric circuit, a battery or a source of electromotive force (e.m.f), E, voltage, a resistor having resistance R ohm, an inductor having an inductance, L Henries or a capacitor or condenser having a capacitance, C, Farads. The current, I, measured in Ampere, A, is the instantaneous time rate of change of charge, Q, on the condenser measured in Coulombs. i.e.

$$I = \frac{dQ}{dt} \qquad (1.1.6)$$

From basic principle pf electricity;

Potential drop across resistor $= IR$

Potential drop across inductor $= L\frac{dl}{dt}$

Potential drop across capacitor $= \frac{Q}{C}$

A circuit consisting of only a source of e.m.f, a Resistor R and an inductor L is called an RL circuit while the one consisting of a source of e.m.f, a resistor R and capacitance C is called an RC circuit. According to Kirchhoff's law for an RL circuit, the governing equation is;

$$L\frac{dl}{dt} + IR = E \qquad (1.1.7)$$

$$\therefore \frac{dI}{dt} + \frac{R}{L} = \frac{E}{L}$$

For an RC circuit, we have the equation as;

$$IR + \frac{Q}{C} = E$$

$$\therefore P\frac{d0}{dt} + \frac{Q}{C} = E$$

$$\Rightarrow \frac{d0}{dt} + \frac{1}{RC}Q = \frac{E}{R} \qquad (1.1.8)$$

Equations (1.1.4), (1.1.5), (1.1.7), and (1.1.8) are the differential equations formulated from physical problems.

CHAPTER TWO

DEFINITION OF BASIC TERMS

2.1 Differential equation:

A differential equation is an equation relating two or more variables in terms of derivatives or differentials. Therefore, in a differential equation we have the derivatives or differentials, and the independent x, and dependent variables, y, may also be present.

Examples:

$$\left(y^3+1\right)\frac{dy}{dx}-xy^2=x \tag{2.1.1}$$

$$\frac{\partial^2 s}{\partial t^2}-\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial z^2}=0 \tag{2.1.2}$$

$$xy\left(\frac{d^2y}{dx^2}\right)-y^2\sin x=0 \tag{2.1.3}$$

$$\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)+15y=\sin 3x \tag{2.1.4}$$

$$\frac{\partial r}{\partial x}=2rt \tag{2.1.5}$$

A differential equation involving only two variables, a dependent and independent variables and their derivatives as in equation (2.1.1), (2.1.3), and (2.1.4) above is called an ordinary differential equation. If however there is more than one independent variable or the derivative is partial, then the differential equation is called partial differential equation. Examples are equations (2.1.2) and (2.1.5).

2.2 An **ordinary differential equation** is an equation which involves one independent variable say x, the dependent variable y and at least one of the derivatives.

$$\left(\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^ny}{dx^n},\ldots\right) \tag{2.1.6}$$

If the dependent variable y depends on two or more independent variables say x and t (i.e $y = F(x,t)$, then the differential equation is called a **partial differential equation**.

Differential equation came into existence by Newton. It is one of the equations he initially conceptualized. Differential equations are equations involving derivatives of a function or functions.

2.3 Order and Degree of a Differential Equation:

The order of the differential equation is the order of the highest derivative present in the equation.

Examples;

1. $y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$ is an equation of the **second order.**

2. $\left(\frac{d^3y}{dx^3}\right)^5 + 6\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^7 = 0$ is an equation of the **third order.**

In a differential equation, the exponent of the highest derivative when it is cleared of fraction is called the **degree** of the differential equation. Below are examples of the degree of differential equation;

3. $\left(\frac{d^4y}{dx^4}\right)^5 + 3\left(\frac{d^3y}{dx^3}\right)^2 + \frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^8 = 0$ is an equation of **degree 5.**

4. $(y'')^3 + 10y' + 24y = 0$ is an equation of **degree 3.**

5. $\left[\frac{d^2y}{dx^2}\right]^{\frac{1}{2}} - 4 + y = 0$ is an equation of **degree 1.**

   $\Rightarrow \frac{d^2y}{dx^2} = (4-y)^2$

6. $\left(\frac{d^2y}{dx^2}\right)^3 - y^{\frac{1}{2}} = 0$ is an equation of **degree 3.**

It should also be pointed out that not all differential equations have degree. If the highest ordered derivative is in term of trigonometric ratios, hyperbolic functions, logarithm, exponential, then such equations have no degree(because these functions sin, cos, log … are expressible in infinite series). For example, the following equations;

$$\log\left(\frac{d^2y}{dx^2}\right)+\sin\left(\frac{dy}{dx}\right)+y=0 \tag{2.1.7}$$

$$\sin\left(\frac{dy}{dx}\right)+\cos x=0 \tag{2.1.8}$$

all have no degree whereas;

$$\left(\frac{d^2y}{dx^2}\right)^3+\sin\left(\frac{dy}{dx}\right)+y=0 \tag{2.1.9}$$

$$\frac{d^2y}{dx^2}+\log y=0 \tag{2.1.10}$$

have degrees 3 and 1 respectively.

The most general nth order ordinary differential equation has the form;

$$F\left(x,y,y',y'',y'''...,y^{(n)}\right) \tag{2.1.11}$$

It can also be written as;

$$y^{(n)}=F\left(x,y,y',y'',y''',...,y^{(n-1)}\right) \tag{2.1.12}$$

NOTE:

Order and degree (if defined) of a differential equation are always positive integers.

2.4 KINDS OF DIFFERENTIAL EQUATION

Differential equation in their different forms can be linear or non-linear homogeneous or non-homogeneous.

Definition: A linear differential equation of order n in the dependent variable y and the independent variable x is an equation which can be expressed as in the form;

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=f(x) \tag{2.1.13}$$

Remarks:

(i) If the differential equation cannot be written in the form (2.13), then it is said to be non-linear.

(ii) If $f(x) = 0$, then (2.13) is said to be homogeneous or otherwise, it is non-homogeneous.

2.5 ANALYTIC FUNCTIONS

A function $F(z)$ is said to be analytic at a point z_0, if $F(z)$ is differentiable not only at z_0, but at every point of some neighbourhood of z_0.

A function $F(z)$ is said to be analytic at a point z_0,if $\exists$ a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

An analytic function is also known as “holomorphic”,”regular” or “monogenic”.

Examples:

1. The function;

$F(z) = \dfrac{3z}{(z-2)^2}$ is analytic everywhere in $\not\subset$ except at $z = 2$. Where $\not\subset$ is complex number.

2. Polynomial functions are analytic everywhere in $\not\subset$, where $\not\subset$ is complex number.

2.6 SINGULAR POINTS OF LINEAR SECOND ORDER DIFFERENTIAL EQUATION

If a homogeneous second order differential equations written in the form;

$$y'' + c_1xy' + c_2xy = 0 \tag{2.1.14}$$

the behavior of the solution of the equation near a point $x = a$,is found to depend upon the behavior of the coefficients $c_1(x)$ and $c_2(x)$ near $x = a$. The point $x = a$ is said to be an **ordinary point** of the differential equation if both $c_1(x)$ and $c_2(x)$ are **regular** at $x = a$ .i.e. they both can be expanded in the power series in an interval including $x = a$.

If however, both or either $c_1(x)$ and $c_2(x)$ are/is not regular at $x = a$, then the point is said to be a **singular point** of the differential equation. In such a case, if the product $x = ac_1(x)$ and

$(x-a)^2 c_2(x)$ are both regular at $x=a$, then the point $x=a$ is called **a regular singular point** of the differential equation, otherwise the point is said to be an **irregular singular point.**

2.7 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Definition:

Let $y=f(x)$ be a function associated with x and y. If $y(x)$ satisfies a given differential equation and is differentiable within an interval $a<x<b$, y is said to be the classical solution of the differential equation within the interval.

The methods of obtaining the solution of differential equations are numerous. This is due to the fact that there exists varied forms of differential equations arising from various modeling process. Such methods include: separation of variables, integrating factors, D-operators, numerical methods, power series solution just to mention but a few. However, for the purpose of this study, particular attention will be on the power series methods.

The power series method has being described as the standard basic method for solving linear differential equation with constant or variable coefficient.

2.7.1 POWER SERIES

Definition:

A power series is an infinite series of the form:

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \tag{2.1.15}$$

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\ldots \tag{2.1.16}$$

centered at $x=a$,equation (2.1.16) becomes;

$$\begin{aligned} &f(a)=c_0+c_1(0)+c_2(0)+\ldots \\ &\Rightarrow c_0=f(a) \end{aligned} \tag{2.1.17}$$

Differentiating equation (2.1.16), we have;

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ... \quad (2.1.18)$$

at $x = a$, we have;

$$c_1 = f'(a) \quad (2.1.19)$$

Differentiating equation (2.1.18), we have;

$$f''(x) = 2c_2 + 3 \bullet 2c_3(x-a) + 4 \bullet 3(x-a)^2 + ... \quad (2.1.20)$$

at $x = a$, we obtain;

$$f''(a) = 2c_2$$

$$c_2 = \frac{f''(a)}{2!} \quad (2.1.21)$$

Differentiating equation (2.1.20), we have;

$$f'''(x) = 3 \bullet 2c_3 + 4 \bullet 3 \bullet 2c_4(x-a) + ... \quad (2.1.22)$$

at $x = a$,we obtain;

$$f'''(a) = 3 \bullet 2c_3$$

$$c_3 = \frac{f'''}{3!} \quad (2.1.23)$$

Now, if $f(x)$ can be represented as such and it is possible to differentiate n times at such point a, then;

$$\mathbf{c_n = \frac{f^{(n)}(a)}{n!}} \quad \mathbf{(}2.1.24\mathbf{)}$$

Recall from equation (2.1.16) that;

$$f(x) = c_0 + c_1x + c_2x + c_3x + ...$$

Input equations (2.1.17), (2.1.19),(2.1.21) and (2.1.23) into equation (2.1.16), we get;

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... \quad (2.1.25)$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{2.1.26}$$

Therefore, equation (2.1.26) is called the **Taylor's series** of $f(x)$ near $x = a$. However, not all functions can be expressed in this form. If $f(x)$ can be expressed in Taylor's series as $x = a$, then the function $f(x)$ is said to be analytic at that point.

Examples:

Consider the second order differential homogeneous equation;

$$y'' + y = 0 \tag{2.1.27}$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$, where $a_0 \neq 0$ (*)

Where $a_0 \neq 0$

i.e., the equation is of the form:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots \tag{**}$$

Differentiating (*), we get;

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{***}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \tag{****}$$

Substitute equation (*), and (****) into equation (2.1.27);

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{2.1.28}$$

To make both powers of $'x'$ in equation (2.28) equal, put $n = n + 2$ in the first summation and factorizing we have;

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n = 0 \tag{2.29}$$

$$\Rightarrow (n+2)(n+1)a_{n+2} + a_n = 0 \tag{2.1.30}$$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \tag{2.1.31}$$

Therefore equation (2.1.31) is the **recurrent relation.**

To obtain $a_2, a_3, a_4, \ldots$ from equation (2.1.31), let $n = 0, 1, 2, \ldots$

When;

$$n = 0, a_2 = -\frac{a_0}{2!} \tag{2.1.32}$$

$$n = 1, a_3 = -\frac{a_1}{3!} \tag{2.1.33}$$

$$n = 2, a_4 = -\frac{a_2}{4 \bullet 3} = \left(-\frac{1}{4 \bullet 3}\right) \times \left(-\frac{a_0}{2!}\right) = \frac{a_0}{4!} \tag{2.1.34}$$

$$n = 3, a_5 = \left(-\frac{a_3}{5 \bullet 4}\right) = \left(-\frac{1}{5 \bullet 4}\right) \times \left(-\frac{a_1}{3!}\right) = \frac{a_1}{5!} \tag{2.1.35}$$

and so on.

From equation (**),

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

Input equation (2.1.32), (2.1.33), (2.1.34) and (2.1.35) into equation (*), we have;

$$y = a_0 + a_1 x - \left(\frac{a_0}{2!}\right) x^2 - \left(\frac{a_1}{3!}\right) x^3 + \left(\frac{a_0}{4!}\right) x^4 + \left(\frac{a_1}{5!}\right) x^5 + \ldots \tag{2.1.36}$$

Collecting like terms and factorizing, we obtain;

$$y = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) \tag{2.1.37}$$

Therefore;

$$y = a_0 \cos x + a_1 \sin x \tag{2.1.38}$$

2.7.2 FROBENUIS METHOD

This method is used in solving second order homogeneous differential equation with variable coefficients.

Examples:

$$3xy'' + 2y' + y = 0 \tag{2.1.39}$$

Put $y = \sum_{n=0}^{\infty} a_n x^{n+v}$ (2.1.40)

Differentiating equation (2.1.40), we obtain;

$$y' = \sum_{n=0}^{\infty} (n+v) a_n x^{n+v-1} \tag{2.1.41}$$

$$y'' = \sum_{n=0}^{\infty} (n+v)(n+v-1) a_n x^{n+v-2} \text{ , where } a_0 \neq 0 \tag{2.42}$$

Input equations (2.1.40), (2.1.41), and (2.1.42) into (2.1.39), we have;

$$3x\sum_{n=0}^{\infty} (n+v)(n+v-1) a_n x^{n+v-2} + 2\sum_{n=0}^{\infty} (n+v) a_n x^{n+v-1} + \sum_{n=0}^{\infty} a_n x^{n+v} = 0 \tag{2.1.43}$$

$$\sum_{n=0}^{\infty} 3(n+v)(n+v-1) a_n x^{n+v-1} + \sum_{n=0}^{\infty} 2(n+v) a_n x^{n+v-1} + \sum_{n=0}^{\infty} a_n x^{n+v} = 0 \tag{2.1.44}$$

Combining the first and second term of equation (2.1.44), having the lowest power of x , i.e $(n+v-1) < (n+v)$,we have;

$$\sum_{n=0}^{\infty} a_n \left[3(n+v)(n+v-1) + 2(n+v)\right] x^{n+v-1} + \sum_{n=0}^{\infty} a_n x^{n+v} = 0 \tag{2.1.45}$$

$$\sum_{n=0}^{\infty} a_n \left[(n+v)(3(n+v-1)+2)\right] x^{n+v-1} + \sum_{n=0}^{\infty} a_n x^{n+v} = 0 \tag{2.1.46}$$

$$\sum_{n=0}^{\infty} a_n \left[(n+v)(3n+3v-1)\right] x^{n+v-1} + \sum_{n=0}^{\infty} a_n x^{n+v} = 0 \tag{2.1.47}$$

To get the indicial roots, take the series with the lowest power of x and put $n = 0$ in equation (2.1.47),

$$\Rightarrow a_0(v)(3v-1) = 0 \tag{2.1.48}$$

and since $a_0 \neq 0$ by definition.

$$\therefore v(3v-1) = 0 \tag{2.1.49}$$

$$\Rightarrow v_1 = 0 \text{ and } v_2 = \frac{1}{3} \tag{2.1.50}$$

Equations (2.1.49) and (2.1.50) are called **indicial equation and indicial roots** respectively.

To get the **Recurrent Relation,** we make both powers of x in equation (2.1.47) the same by putting $n = n+1$ in the first series expression of equation (2.1.47).

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+1}(n+v+1)(3n+3v+2)x^{n+v} + \sum_{n=0}^{\infty} a_n x^{n+v} = 0 \tag{2.1.51}$$

Factorizing equation (2.1.51), we have;

$$\sum_{n=0}^{\infty} \left[a_{n+1}(n+v+1)(3n+3v+2) + a_n\right]x^{n+v} = 0 \tag{2.1.52}$$

$$\Rightarrow a_{n+1}(n+v+1)(3n+3v+2) + a_n = 0 \tag{2.1.53}$$

$$\therefore a_{n+1} = -\frac{a_n}{(n+v+1)(3n+3v+2)} \tag{2.1.54}$$

Hence equation (2.1.54) is known as the **Recurrent Relation.**

Put $n = 0,1,2,\ldots$ to get $a_1, a_2, a_3, \ldots$ in equation (2.1.54).

When;

$$n = 0, a_1 = -\frac{a_0}{(v+1)(3v+2)} \tag{2.1.55}$$

$$n = 1, a_2 = -\frac{a_1}{(v+2)(3v+5)} \tag{2.1.56}$$

$$n = 2, a_3 = -\frac{a_2}{(v+3)(3v+8)} \tag{2.1.57}$$

and so on.

Input the indicial roots into equations (2.1.55), (2.1.56), and (2.1.57).

For the first root; $v_1 = 0$, we have;

$$a_1 = -\frac{a_0}{2} \tag{2.1.58}$$

$$a_2 = -\frac{a_1}{10} = \left(-\frac{1}{10}\right)\left(-\frac{a_0}{2}\right) = \frac{a_0}{20} \tag{2.1.59}$$

$$a_3 = -\frac{a_2}{24} = \left(-\frac{1}{24}\right)\left(\frac{a_0}{20}\right) = -\frac{a_0}{480} \tag{2.1.60}$$

Recall from equation (2.1.40) that;

$$y = \sum_{n=0}^{\infty} a_n x^{n+v}$$

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n x^v \ ,$$

$$\therefore y = x^v \sum_{n=0}^{\infty} a_n x^n$$

Therefore, the solution becomes;

$$y_1 = x^{v_1}\left[a_0 + a_1 x + a_2 x + ...\right] \tag{2.1.61}$$

$$y_1 = x^{v_1}\left[a_0 - \frac{a_0}{2}x + \frac{a_0}{20}x^2 - \frac{a_0}{480}x^3 + ...\right] \tag{2.1.62}$$

Since $v_1 = 0$, we have;

$$y_1 = a_0\left[1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + ...\right] \tag{2.1.63}$$

For the second root, $v_2 = \frac{1}{3}$;

$$a_1 = -\frac{a_0}{\left(\frac{4}{3}\right)(3)} = -\frac{a_0}{4} \tag{2.1.64}$$

$$a_2 = -\frac{a_1}{\left(\frac{7}{3}\right)(6)} = -\frac{a_1}{14} = \left(-\frac{1}{14}\right)\left(-\frac{a_0}{4}\right) = \left(\frac{a_0}{56}\right) \tag{2.1.65}$$

$$a_3 = -\frac{a_2}{\left(\frac{10}{3}\right)(9)} = -\frac{a_2}{30} = \left(-\frac{1}{30}\right)\left(\frac{a_0}{56}\right) = -\frac{a_0}{1680} \tag{2.1.66}$$

and so on.

Therefore, the second solution becomes;

$$y_2 = x^{v_2}\left[a_0 + a_1x + a_2x^2 + \ldots\right] \tag{2.1.67}$$

$$y_2 = x^{\frac{1}{3}}\left[a_0 - \frac{a_0}{4}x + \frac{a_0}{56}x^2 - \frac{a_0}{1680}x^3 + \ldots\right] \tag{2.1.68}$$

$$y_2 = x^{\frac{1}{3}}a_0\left[1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \ldots\right] \tag{2.1.69}$$

$$y = y_1 + y_2 \tag{2.1.70}$$

$$y = a_0\left[1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \ldots\right] + b_0x^{\frac{1}{3}}\left[1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \ldots\right] \tag{2.1.71}$$

Where $a_0 = b_0$ in equation (2.69).

$$\therefore y = a_0\left[1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \ldots\right] + b_0\left[x^{\frac{1}{3}} - \frac{x^{\frac{4}{3}}}{4} + \frac{x^{\frac{7}{3}}}{56} - \frac{x^{\frac{10}{3}}}{1680} + \ldots\right] \tag{2.1.72}$$,

where a_0 and b_0 are arbitrary constants.

CHAPTER THREE

3.1 LEGENDRE POLYNOMIAL

The second order differential equation with variable coefficient also known as Legendre Differential Equation was named after ADRIEN MARIE (September 18, 1752-January 10, 1833) a French Mathematician who became a professor in Paris in 1775 and who is best known for his work in the field of elliptic integrals and theory of numbers;

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0 \tag{3.1.1}$$

Where n is a non-negative integer.

Legendre differential equation occurs in many physical engineering problems involving spherical geometry and gravitation. Legendre and associate Legendre Polynomial are widely used in the determination of wave functions of electrons in the orbits of an atom and in the determination of potential functions in the spherical symmetric geometry.

3.2 LEGENDRE DIFFERENTIAL EQUATION

The Legendre Differential Equation is the differential equation of the form; $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$

(3.1.2)

Where n is a non-negative integer.

This equation can also be put in the form;

$$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dy}{dx}\right\} + n(n+1)y=0 \tag{3.1.3}$$

Hence the only singular points of equation (3.1.1) are x=1, x=-1 and x=∞ which are regular. Therefore the Legendre equation is a Fuchsian differential equation. The points other than singular points, e.g. $x=0$, $x=2$ behave like ordinary points of equation (3.1.1). The equation above can be integrated in series of ascending and descending powers of x. The solution of Legendre's differential equation in descending order is more important than that of the ascending powers of x.

3.3 SOLUTION OF LEGENDRE DIFFERENTIAL EQUATION IN ASCENDING POWERS

Consider the Legendre differential equation of the form;

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0 \tag{3.1.4}$$

Where n is a non-negative integer.

Suppose the series solution of (3.1.4) is;

$$y = \sum_{q=0}^{\infty} a_q x^{q+v}, \ a_0 \neq 0 \tag{3.1.5}$$

Differentiating equation (3.1.5), we have;

$$y' = \sum_{q=0}^{\infty} (q+v) a_q x^{q+v-1} \tag{3.1.6}$$

$$y'' = \sum_{q=0}^{\infty} (q+v)(q+v-1) a_q x^{q+v-2} \tag{3.1.7}$$

Substituting these values in equation (3.1.4), we get;

$$\left(1-x^2\right)\sum_{q=0}^{\infty} (q+v)(q+v-1) a_q x^{q+v-2} - 2x\sum_{q=0}^{\infty} (q+v) a_q x^{q+v-1} + n(n+1)\sum_{q=0}^{\infty} a_q x^{q+v} = 0 \tag{3.1.8}$$

$$\sum_{q=0}^{\infty} (q+v)(q+v-1) a_q x^{q+v-2} - \sum_{q=0}^{\infty} (q+v)(q+v-1) x^{q+v} - \sum_{q=0}^{\infty} 2(q+v) a_q x^{q+v} + n(n+1)\sum_{q=0}^{\infty} a_q x^{q+v} = 0 \tag{3.1.9}$$

Collecting like powers of x;

$$\sum_{q=0}^{\infty} (q+v)(q+v-1) a_q x^{q+v-2} - \sum_{q=0}^{\infty} a_q \left[(q+v)^2 + (q+v+1) - n^2 - n\right] x^{q+v} = 0 \tag{3.1.10}$$

$$\sum_{q=0}^{\infty} (q+v)(q+v-1) a_q x^{q+v-2} - \sum_{q=0}^{\infty} a_q \left[\left\{(q+v)^2 - n^2\right\} + (q+v-n+1)\right] x^{q+v} = 0 \tag{3.1.11}$$

$$\sum_{q=0}^{\infty}(q+v)(q+v-1)a_q x^{q+v-2} - \sum_{q=0}^{\infty} a_q\left[(q+v-n)(q+v+n)+(q+v-n)+1\right]x^{q+v} = 0 \qquad (3.1.12)$$

$$\sum_{q=0}^{\infty}(q+v)(q+v-1)a_q x^{q+v-2} - \sum_{q=0}^{\infty}(q+v-n)(q+v+n+1)a_q x^{q+v} = 0 \qquad (3.1.13)$$

To get the indicial roots, equate to zero the coefficient of the smallest power of x, i.e. x^{q+v-2} and put $q=0$ in equation (3.1.13);

$$\sum_{q=0}^{\infty} a_q (q+v)(q+v-1)x^{q+v-2} = 0 \qquad (3.1.14)$$

$$a_0(v-1)v = 0,\ a_0 \neq 0 \qquad (3.1.15)$$

$$\therefore v_1 = 0, v_2 = 1 \qquad (3.1.16)$$

Therefore equation (3.1.15) and (3.1.16), are the indicial equation and indicial roots respectively. The roots of the indicial equation differ by an integer.

To get the **Recurrent Relation,** we make both powers of x in equation (3.1.13) the same by putting $q = q-2$ in the second series expression. Thus, we have;

$$\sum_{q=0}^{\infty}(q+v)(q+v-1)a_q x^{q+v-2} - \sum_{q=0}^{\infty}(q+v-2-n)(q+v+n-1)a_{q-2} x^{q+v-2} = 0 \qquad (3.1.17)$$

Factorizing, equation (3.1.17), we obtain;

$$\sum_{q=0}^{\infty}\left[a_q(q+v)(q+v-1)-(q+v-2-n)(q+v+n-1)\right]x^{q+v-2} = 0 \qquad (3.1.18)$$

$$a_q(q+v)(q+v-1) - a_{q-2}(q+v-2-n)(q+v+n-1) = 0 \qquad (3.1.19)$$

$$\Rightarrow a_q = \left(\frac{(q+v-2-n)(q+v+n-1)}{(q+v)(q+v-1)}\right)a_{q-2} \qquad (3.1.20)$$

Hence, equation (3.1.20) is known as the **recurrent relation**.

Putting $v=0$ in equation (3.1.20), we have;

$$a_q = \left(\frac{(q-2-n)(q+n-1)}{q(q-1)} \right) a_{q-2} \tag{3.1.21}$$

We now express $a_2, a_4, a_6, \ldots$ in terms of a_0 and $a_3, a_5, a_7, \ldots$ in terms of a_1 by assuming that a_1 is finite.

Putting $q = 0,1,2,3,4,5,\ldots$ in equation (3.1.21), we have;

When;

$$q = 2, a_2 = -\left(\frac{n(n+1)}{2!} \right) a_0 \tag{3.1.22}$$

$$q = 3, a_3 = \left(\frac{(1-n)(2+n)}{3!} \right) a_1 = \left(-\frac{(n-1)(n+2)}{3!} \right) a_1 \tag{3.1.23}$$

$$q = 4, a_4 = \left(\frac{(2-n)(3+n)}{4!} \right) a_2 = \left(\frac{(n-2)(3+n)n(n+1)}{4!} \right) a_0 \tag{3.1.24}$$

$$q = 5, a_5 = \left(\frac{(3-n)(4+n)}{5!} \right) a_3 = \left(\frac{(n-3)(4+n)(n-1)(n+2)}{5!} \right) a_1 \tag{3.1.25}$$

and so on.

Recall from equation (3.1.5) that;

$$y = \sum_{q=0}^{\infty} a_q x^{q+v}$$

$$y = x^v \sum_{q=0}^{\infty} a_q x^q$$

$$y = x^v \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots \right) \tag{3.1.26}$$

where $v = 0$;

$$y = \left(a_0 + a_2 x^2 + a_4 x^4 + \ldots \right) + \left(a_1 + a_2 x^3 + a_5 x^5 + \ldots \right) \tag{3.1.27}$$

Input equation (3.1.22), (3.1.23) (3.1.24) and (3.1.25) into equation (3.1.27), we get;

$$
\begin{aligned}
y = a_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n-1)(n+3)}{4!}x^4-\ldots\right]+ \\
a_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5+\ldots\right] \qquad (3.1.28),
\end{aligned}
$$

which is the general series solution, a_0 and a_1 are arbitrary constants.

3.4 LEGENDRE'S FUNCTIONS OF FIRST AND SECOND KIND

Legendre's polynomial of degree n , also known as Legendre's function of the first kind is denoted by $\mathbf{P_n(x)}$ and is defined by;

$$
P_n(x)=\frac{1.3.5\ldots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}-\ldots\right] \tag{3.1.29}
$$

we can also write $P_n(x)$ in a compact form as;

$$
P_n(x)=\sum_{k=0}^{\left(\frac{n}{2}\right)}(-1)^k\,\frac{(2n-2k)}{2^n k!(n-2k)!(n-k)}x^{n-2k} \tag{3.1.30}
$$

Where $\left(\frac{n}{2}\right)=\begin{cases}\left(\frac{n}{2}\right), n \\ \frac{(n-1)}{2}, n\end{cases}$ and n is even and odd respectively.

Legendre polynomial of the second kind is denoted by $Q_n(x)$ and is defined by;

$$
Q_n(x)=\frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-n-1}+\frac{(n+2)(n+1)}{2(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5}+\ldots\right] \tag{3.1.31}
$$

3.5 GENERATING FUNCTION OF LEGENDRE POLYNOMIAL

The function $\left(1-2xh+h^2\right)^{\left(-\frac{1}{2}\right)}$ is called the generating function of $P_n(x)$, and therefore $P_n(x)$ is the coefficient of h^n in the expansion of $\left(1-2xh+h^2\right)^{\left(-\frac{1}{2}\right)}$ in ascending powers of h, i.e.,

$$
\left(1-2xh+h^2\right)^{\left(-\frac{1}{2}\right)}=\sum_{n=0}^{\infty}h^nP_n(x), |x|\le 1,\ |h|<1 \tag{3.1.32}
$$

Proof:

Since $|x| \leq 1$ and $|h| < 1$, therefore we can write;

$$\left(1 - 2xh + h^2\right)^{\left(-\frac{1}{2}\right)} = \{1 - h(2x - h)\}^{\left(-\frac{1}{2}\right)} \tag{3.1.33}$$

Expanding the right hand side by Binomial theorem;

$$\left(1 - 2xh + h^2\right)^{\left(-\frac{1}{2}\right)} = 1 + \frac{1}{2}h(2x - h) + \frac{1.3}{2.4}h^2(2x - h)^2 + \ldots + \frac{1.3\ldots(2n-3}{2.4\ldots(2n-2)}h^{n-1}\,\frac{1.3\ldots(2n-3}{2.4\ldots(2n-2)}h^n(2x - h)^n \tag{3.1.34}$$

Therefore the coefficient of h^n in the right hand side is;

$$(1 - h(2x - h))^{\left(-\frac{1}{2}\right)} = \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}(2x)^n - \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}n - 1C_1(2x)^{n-2} + \frac{1.3\ldots(2n-5)}{2.4\ldots(2n-4)} + {}^{n-2}C_2(2x)^{n-4} \tag{3.1.35}$$

$$(1 - h(2x - h))^{\left(-\frac{1}{2}\right)} = -\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}2^n\left[x^n - \frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^n} + \frac{2n(2n-2)}{(2n-1)(2n-3)}\frac{(n-2)(n-3)}{2!}\frac{x^{n-4}}{2^2} - \ldots\right] \tag{3.1.36}$$

$$(1 - h(2x - h))^{\left(-\frac{1}{2}\right)} = -\frac{1.3\ldots(2n-1)}{n!}\left[x^n - \frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^n} + \frac{2n(2n-2)}{(2n-1)(2n-3)}\frac{(n-2)(n-3)}{2!}\frac{x^{n-4}}{2^2} - \ldots\right] \tag{3.1.37}$$

$$(1 - h\left((2x - h)\right)^{\left(-\frac{1}{2}\right)} = -\frac{1.3\ldots(2n-1)}{n!}2^n\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \ldots\right] \tag{3.1.38}$$

$$(1 - h(2x - h))^{\left(-\frac{1}{2}\right)} = P_n(x) \tag{3.1.39}$$

Thus, we can say that in the expansion of $(1 - h(2x - h))^{\left(-\frac{1}{2}\right)}$, in ascending powers of h, the Legendre's Polynomials $P_0(x), P_1(x), P_2(x), \ldots$ are the coefficients of $h^0, h^1, h^2, \ldots$ respectively in the expansion given by equation(3.34).

Hence, we have;

$(1-h(2x-h))^{\left(-\frac{1}{2}\right)} = \sum_{n=0}^{\infty} h^n P_n(x)$, where $P_0(x) = 1$

This shows that $P_n(x)$ is the coefficient of h^n in the expansion of $(1-h(2x-h))^{\left(-\frac{1}{2}\right)}$

Therefore, $(1-h(2x-h))^{\left(-\frac{1}{2}\right)}$ is known as the **generating function** of the Legendre's polynomial $P_n(x)$.

CHAPTER FOUR

4.1 RODRIGUE'S FORMULA FOR $\mathbf{P_n(x)}$

The following is known as the **Rodrigue's formula for** $\mathbf{P_n(x)}$;

$$\mathbf{P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n} \tag{4.1.1}$$

Proof:

Let $y = (x^2-1)^n$ (4.1.2)

Differentiating equation (4.1.2) w.r.t , x, we have;

$$y' = 2xn(x^2-1)^{n-1} \tag{4.1.3}$$

$$\Rightarrow y' = 2xny(x^2-1)^{-1} \tag{4.1.4}$$

Multiplying both sides of equation (4.1.4) by x^2-1, we have:

$$(x^2-1)y' = 2nxy \tag{4.1.5}$$

Differentiating equation (4.1.3) $(n+1)$ times by Leibnitz's theorem, we have;

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + (n+1)\frac{d^{n+1}y}{dx^{n+1}}2x + \frac{n(n+1)}{2!}\frac{d^ny}{dx^n}2 = 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + (n+1)\frac{d^ny}{dx^n}\right]$$

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\frac{d^ny}{dx^n} = 0$$

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^ny}{dx^n} = 0 \tag{4.1.6}$$

Putting $\frac{d^ny}{dx^n} = z$ in equation (4.1.6), it becomes;

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0 \tag{4.1.7}$$

which is the Legendre's differential equation whose solution is given by ;

$$z = CP_n(x) \tag{4.1.8}$$

where C is a constant.

$$\therefore \frac{d^n y}{dx^n} = CP_n(x) \tag{4.1.9}$$

Putting $x = 1$ in equation (4.1.8), and then using $P_n(1) = 1$, we get;

$$\left[\frac{d^n y}{dx^n}\right]_{x=1} = C \text{, since } P_n(1) = 1 \tag{4.1.10}$$

Now from equation (4.1.2), we have;

$$y = (x^2 - 1)^n = (x-1)^n (x+1)^n \tag{4.1.11}$$

Differentiating equation (4.1.11) w.r.t x, n times by Leibnitz's theorem, we obtain;

$$\begin{aligned} \frac{d^n y}{dx^n} &= (x-1)^n \frac{d^n}{dx^n}(x+1)^n + n\left[\frac{d^{n-1}}{dx^{n-1}}(x+1)^n\right] n(x+1)^{n-1} + \ldots + \\ & n\left[\frac{d}{dx}(x+1)^n\right]\frac{d^{n-1}}{dx^{n-1}}(x-1)^n + (x+1)^n \frac{d^n}{dx^n}(x-1)^n \end{aligned} \tag{4.1.12}$$

$$\frac{d^n y}{dx^n} = (x-1)^n n! + n\frac{n!}{1!}(x+1)n(x-1)^n + \ldots + n.n(x+1)^{n-1}\frac{n!}{1!}(x-1) + (x+1)^n n! \tag{4.1.13}$$

Putting $x = 1$ in equation (4.1.13), we obtain;

$$\left[\frac{d^n y}{dx^n}\right]_{x=1} = (1+1)^n n! \tag{4.1.14}$$

$$C = 2^n n! \tag{4.1.15}$$

Therefore, by putting the value of C from equation (4.1.15) in (4.1.9), we get;

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2 - 1)^n \tag{4.1.16}$$

which is the required Rodrigue's formula for $P_n(x)$.

- All roots of $P_n(x)$ are real and lie between -1 and 1.

4.2 ORTHOGONAL PROPERTIES OF LEGENDRE'S POLYNOMIAL

The Legendre polynomial satisfies the condition of orthogonality as;

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, m \neq n \\ \dfrac{2}{2n+1}, m = n \end{cases} \tag{4.1.17}$$

<u>PROOF</u>:

When $\mathbf{m \neq n}$;

Therefore the Legendre differential equation is;

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{4.1.18}$$

Let $P_m(x)$ and $P_m(x)$ be Legendre polynomial satisfying Legendre differential equation;

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \tag{4.1.19}$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \tag{4.1.20}$$

Multiplying equation (4.1.19) by P_n and (4.1.20) by P_m and subtracting, we have;

$$(1-x^2)\left[P_nP_m'' - P_mP_n''\right] - 2x\left[P_nP_m' - P_mP_n'\right] + P_mP_n\left[m(m+1) - n(n+1)\right] = 0 \tag{4.1.21}$$

$$\frac{d}{dx}\left[(1-x^2)(P_nP_m' - P_mP_n')\right] + P_mP_n\left[m(m+1) - n(n=1)\right] = 0 \tag{4.1.22}$$

But;

$$\frac{d}{dx}\left(P_nP_m' - P_mP_n'\right) = P_nP_m'' + P_n'P_m' - (P_mP_n'' + P_n'P_m') \tag{4.1.23}$$

$$\frac{d}{dx}\left(P_nP_m' - P_mP_n'\right) = P_nP_m'' - P_mP_n'' \tag{4.1.24}$$

$$\Rightarrow \left[m(m+1) - n(n+1)\right]P_mP_n = -\frac{d}{dx}\left[(1-x^2)(P_nP_m' - P_mP_n')\right] \tag{4.1.25}$$

Integrating both sides of equation (4.1.25) with respect to x from -1 to +1, we obtain;

$$\left[m(m+1) - n(n+1)\right]\int_{-1}^{1} P_mP_n dx = -\int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)(P_nP_m' - P_mP_n')\right]dx \tag{4.1.26}$$

$$\int_{-1}^{1} P_m P_n = -\frac{1}{[m(m+1)-n(n+1)]}\left[(1-x^2)(P_n P_m' - P_m P_n')\right]_{-1}^{1} \tag{4.1.27}$$

Since;

$$\left[1-x^2\right]_{-1}^{1} = \left(1-(1)^2\right)-\left(1-(-1)^2\right) = 0$$

$$\Rightarrow \int_{-1}^{1} P_m(x)P_n(n) = 0, \text{ if } m \neq n \tag{4.1.28}$$

When m=n;

We know that the generating function for $P_n(x)$ is given by;

$$\left(1-2hx+h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{4.1.29}$$

Also, we have;

$$\left(1-2hx+h^2\right)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} h^m P_m(x) \tag{4.1.30}$$

Multiplying the corresponding sides of equations (4.1.29) and (4.1.30), we obtain;

$$\left(1-2hx+h^2\right)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h^{m+n} P_m(x)P_n(x) \tag{4.1.31}$$

Integrating equation (4.1.31) with respect to x between the limits -1 to +1, we have;

$$\int_{-1}^{1}\left(1-2hx+h^2\right)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[h^{m+n}\left(\int_{-1}^{1} P_m(x)P_n(x)\right)\right] \tag{4.1.32}$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}[P_n(x)]^2\,dx + \sum_{\substack{m,n=0\\ m\neq n}}^{\infty}\int_{-1}^{1} h^{m+n}P_m(x)P_n(x)dx = \int_{-1}^{1}\frac{dx}{\left(1-2hx+h^2\right)} \tag{4.1.33}$$

Now, since $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$, where $m \neq n$. Therefore, we have;

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx = \int_{-1}^{1}\frac{dx}{\left(1-2xh+h^2\right)} \tag{4.1.34}$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx \quad = -\frac{1}{2h}\left[\log(1-2hx+h^2)\right]_{-1}^{1}$$

$$(4.1.35)\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx \quad = -\frac{1}{2h}\left[\log\left(1-2h+h^2\right)-\log(1+2h+h^2)\right]$$

$$(4.1.36) \qquad \sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx = -\frac{1}{2h}\left[\log(1-h)^2-\log(1+h)^2\right]$$

(4.1.37)

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx \qquad = \frac{1}{h}\left[\frac{1}{2}\log(1+h)^2-\frac{1}{2}\log(1-h)^2\right] \tag{4.1.38}$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx \; = \frac{1}{h}\left[\log(1+h)-\log(1-h)\right] \tag{4.1.39}$$

Recall that;

$$\log(1+h) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$$
$$\log(1-h) = -(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots$$

Equation (4.1.39), becomes;

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx = \frac{1}{t}\left[h-\frac{h^2}{2}+\frac{h^3}{3}-\ldots+h+\frac{h^2}{2}+\frac{h^3}{3}+\ldots\right] \tag{4.1.40}$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx = \frac{1}{h}\left[2h+\frac{2h^3}{3}+\ldots\right] \tag{4.1.41}$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx = 2\left[1+\frac{h^2}{3}+\ldots+\frac{t^{2n}}{2n+1}\right] \tag{4.1.42}$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}\left[P_n(x)\right]^2 dx = 2\sum_{n=0}^{\infty}\left[\frac{1}{2n+1}\right]h^{2n} \tag{4.1.43}$$

Now, equating the coefficient of h^{2n} from both sides, we have;

$$\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1} \tag{4.1.44}$$

Hence;

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0, m \neq n \\ \dfrac{2}{2n+1}, m = n \end{cases}$$

4.3 RECURRENCE FORMULA FOR LEGENDRE POLYNOMIALS.

(I) $\mathbf{(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)}$.

Proof:

The generating function of $P_n(x)$ is given by;

$$\left(1 - 2hx + h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{4.1.45}$$

Differentiating both sides of equation (4.1.45) with respect to h, we have;

$$-\frac{1}{2}(1 - 2hx + h^2)^{-\frac{3}{2}}(-2x + 2h) = \sum_{n=0}^{\infty} nh^{n-1} P_n(x) \tag{4.1.46}$$

Multiplying both sides of equation (4.1.46) by $\left(1 - 2hx + h^2\right)$ and simplifying, we obtain;

$$\left(x - h\right)\left(1 - 2hx + h^2\right)^{-\frac{1}{2}} = \left(1 - 2hx + h^2\right) \sum_{n=0}^{\infty} nh^{n-1} P_n(x) \tag{4.1.47}$$

Now, from equations (4.1.45) and (4.1.47), we have;

$$(x - h) \sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2hx + h^2) \sum_{n=0}^{\infty} nh^{n-1} P_n(x) \tag{4.1.48}$$

$$x \sum_{n=0}^{\infty} h^n P_n(x) - \sum_{n=0}^{\infty} h^{n+1} P_n(x) = \sum_{n=0}^{\infty} nh^{n-1} P_n(x) - 2x \sum_{n=0}^{\infty} nh^n P_n(x) + \sum_{n=0}^{\infty} nh^{n+1} P_n(x) \tag{4.1.49}$$

Equating the general coefficients of h^n on both sides of equation (4.1.49), we have;

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) \tag{4.1.50}$$

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{4.1.51}$$

This recurrence relation is the classical three term relation for $P_n(x)$ and it is a pure recurrence relation for Legendre's polynomials.

Remarks: equating the general coefficients of h^{n-1} on both sides of equation (4.1.49), we get;

$$xP_{n-1}(x) - P_{n-2}(x) = nP_n(x) - 2x(n-1)P_{n-1}(x) + (n-2)P_{n-2}(x) \tag{4.1.52}$$

$$\mathbf{nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)} \tag{4.1.53}$$

This is a substitute recurrence relation of equation (4.1.45), and may be directly obtained by replacing n by $(n-1)$ in equation (4.1.45).

(II) $\mathbf{nP_n(x) = xP_n'(x) - P_{n-1}'(x)}$,

Proof:

The generating function of $P_n(x)$ is given by;

$$\left(1 - 2hx + h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{4.1.54}$$

Differentiating both sides of equation (4.1.54) with respect to h, we have;

$$(x-h)(1-2hx+h^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) \tag{4.1.55}$$

Differentiating equation (4.1.54) w.r.t. x, we have;

$$h\left(1 - 2hx + h^2\right)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} h^n P_n'(x) \tag{4.1.56}$$

Multiplying both sides of equation (4.1.56) by $(x-h)$, we get;

$$h(x-h)\left(1 - 2hx + h^2\right)^{-\frac{3}{2}} = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \tag{4.1.57}$$

Now, from equations (4.1.55) and (4.1.57), we obtain;

$$h\sum_{n=0}^{\infty} nh^{n-1}P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \tag{4.1.58}$$

$$\sum_{n=0}^{\infty} nh^n P_n(x) = x\sum_{n=0}^{\infty} h^n P_n'(x) - \sum_{n=0}^{\infty} h^{n+1} P_n'(x) \tag{4.1.59}$$

Equating the general coefficients of h^n on both sides of equation (4.1.59), we have;

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) \tag{4.1.60}$$

$$\Rightarrow (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{4.1.61}$$

This recurrence relation is a differential recurrence relation.

Other examples are;

(III) $\mathbf{(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$

(IV) $\mathbf{(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)}$

(V) $\mathbf{(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]}$

(VI) $\mathbf{(1-x^2)P'_n(x) = (n+1)[xP_n(x) - P_{n+1}(x)]}$

4.4 ILLUSTRATIVE EXAMPLES.

1. Using generating function for $P_n(x)$, prove the following;

a. $P_0(x) = 1$

b. $P_1(x) = x$

c. $P_2(x) = \frac{1}{2}(3x^2 - 1)$

d. $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

e. $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Solution:

Generating function for $P_n(x)$ is given by;

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2hx+h^2)^{-\frac{1}{2}} = \left[1-h(2x-h)\right]^{-\frac{1}{2}} \tag{4.1.62}$$

$$\sum_{n=0}^{\infty} h^n P_n(x) = 1 + \frac{h}{2}(2x-h) + \frac{1.3}{2.4}h^2(2x-h)^2 + \frac{1.3.5}{2.4.6}h^3(2x-h)^3 + \frac{1.3.5.7}{2.4.6.8}h^4(2x-h)^4 + \ldots \tag{4.1.63}$$

$$\begin{aligned} &P_0(x) + hP_1(x) + h^2P_2(x) + h^3P_3(x) + h^4P_n(x) + \ldots \\ &= 1 + x.h + \frac{1}{2}(3x^2-1)h^2 + \frac{1}{2}(5x^3-3x)h^3 + \frac{1}{8}(35x^4+30x^2+3)h^4 + \ldots \end{aligned} \tag{4.1.64}$$

Equating the coefficients of like powers of h, we get;

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) \end{aligned} \tag{4.1.65}$$

2. Express $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$

Solution;

We know that;

$$P_0(x) = 1 \tag{4.1.66}$$

$$P_1(x) = x \tag{4.1.67}$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \tag{4.1.68}$$

$$\Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3} \tag{4.1.69}$$

$$P_3(x)=\frac{1}{2}(5x^3-3x) \tag{4.1.70}$$

$$\Rightarrow x^3=\frac{2}{5}P_3(3)+\frac{3}{5}x \tag{4.1.71}$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3) \tag{4.1.72}$$

$$\Rightarrow x^4=\frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35} \tag{4.1.73}$$

Substituting in succession the values of $x^4.x^3,...$ in the given polynomial, we have;

$$f(x)=\frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35}+2x^3-x-3 \tag{4.1.74}$$

$$f(x)\ =\frac{8}{35}P_4(x)+2x^3+\frac{20}{7}x^2-x-\frac{108}{35}$$

$$f(x)\ =\frac{8}{35}P_4(x)+2\left[\frac{2}{5}P_3(x)+\frac{3}{5}x\right]+\frac{20}{7}x^2=x=\frac{108}{35}$$

$$f(x)\ =\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{20}{7}\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]+\frac{1}{5}x-\frac{108}{35}$$

$$f(x)\ =\frac{8}{35}P_4(x)+\frac{4}{3}P_3(x)+\frac{40}{41}P_2(x)+\frac{1}{5}x-\frac{224}{105}$$

$$f(x)\ =\frac{8}{35}P_4(x)+\frac{4}{3}P_3(x)+\frac{40}{41}P_2(x)+\frac{1}{5}x-\frac{224}{105}P_0(x) \tag{4.1.75}$$

4.5 MURPHY'S FORMULAR FOR LEGENDRE'S POLYNOMIAL $\mathbf{P_n(x)}$.

Consider the Legendre's differential equation;

$$(1-x^2)y''-2xy'+n(n+1)y=0 \tag{4.1.76}$$

where n is a non-negative integer.

It has only three singular points namely $x=1, x=-1$ and $x=\infty$ and all are regular. Therefore Legendre's differential equation is a **Fuchsian differential equation** with three regular singular points $x=1, x=-1$ and $x=\infty$.

Let us find the solution of equation (4.1.76) about the singular point $x=1$ as follows:

The substitution $t=\frac{1}{2}(1-x)$ transfers the singular point $x=1$ to $t=0$. In this case, the Legendre's differential equation in equation (4.1.76) is transformed to the following differential equation:

$$t(1-t)y''+(1+2t)y'+n(n+1)y=0 \tag{4.1.77}$$

This transformed differential equation is in the hyper-geometric form with $a=-n, b=n+1$ and $c=1$.

All solutions of the transformed differential equation (4.77) are represented by the P symbol as follows:

$$y=P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & n+1 & t \\ 0 & 0 & -n & \end{Bmatrix} \tag{4.1.78}$$

Hence, all solutions of the Legendre's differential equation (4.76) are represented the following P symbol;

$$y=P\begin{Bmatrix} 0 & -1 & \infty & \\ 0 & 0 & n+1 & x \\ 0 & 0 & -n & \end{Bmatrix} \tag{4.1.79}$$

One of the solutions of the differential equation (4.1.77) is the polynomial;

$F(-n;n+1;1;t)$.

Now, replacing t by $\frac{(1-x)}{2}$, we can have one of the solutions of the Legendre's differential equation (4.1.76) as:

$$P_n(x) = F\left(-n; n+1; 1; \frac{1-x}{2}\right) \qquad (4.1.80),$$

this is the polynomial solution of equation (4.1.76). This relation (4.1.80) for $P_n(x)$ is known as the **Murphy's formula for Legendre's polynomial $\mathbf{P_n(x)}$**.

4.6 LAPLACE'S DEFINITE INTEGRAL FOR $\mathbf{P_n(x)}$.

(I) Laplace's First Integral for $\mathbf{P_n(x)}$:

When n is a positive integer, then Laplace's first integral for $P_n(x)$ is given by;

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{(x^2-1)}\cos\theta\right]^n d\theta$$

Proof;

From integral calculus, we have;

$$\int_0^{\pi}\frac{d\theta}{a \pm \cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}, \text{ where } a^2 > b^2. \qquad (4.1.81)$$

Putting $a = 1 - hx$ and $b = h\sqrt{(x^2-1)}$ so that $a^2 - b^2 = (1-hx)^2 - h^2(x^2-1) = 1 - 2hx + h^2$.

Using these values of a, b and $a^2 - b^2$ in equation (4.1.81), we get:

$$\pi(1-2hx+h^2)^{-\frac{1}{2}} = \int_0^{\pi}\left[1 - hx \pm h\sqrt{(x^2-1)}\cos\theta\right]^{-1} d\theta \qquad (4.1.82)$$

$$\pi(1-2hx+h^2)^{-\frac{1}{2}} = \int_0^{\pi}\left[1 - h\left\{x \pm \sqrt{(x^2-1)}\cos\theta\right\} - 1\right]^{-1} d\theta$$

$$\pi(1-2hx+h^2)^{-\frac{1}{2}} = \int_0^{\pi}(1-ht)^{-1} d\theta, \text{ where } t = x \pm \sqrt{(x^2-1)}\cos\theta$$

$$\pi\sum_{n=0}^{\infty} h^n P_n(x) = \int_0^{\pi} \left(1 - ht + h^2t^2 + \ldots + h^n t^n + \ldots\right) d\theta$$

$$\pi\sum_{n=0}^{\infty} h^n P_n(x) \quad = \int_0^{\pi}\left[\sum_{n=0}^{\infty}(ht)^n\right] d\theta$$

$$\pi\sum_{n=0}^{\infty} h^n P_n(x) = \sum_{n=0}^{\infty}\left[h^n \int_0^{\pi} t^n \, d\theta\right] \tag{4.1.83}$$

Equating the coefficient of h^n on both sides of equation (4.1.83), we have;

$$\pi P_n(x) = \int_0^{\pi} t^n \, d\theta$$

$$\pi P_n(x) \quad = \int_0^{\pi}\left[x \pm \sqrt{(x^2-1)}\cos\theta\right]^n d\theta$$

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{(x^2-1)}\cos\theta\right]^n d\theta \tag{4.1.84}$$

(II) Laplace's second Integral for $\mathbf{P_n(x)}$:

When n is a positive integral, then Laplace's second integral for $P_n(x)$ is given by;

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{\left[x \pm \sqrt{(x^2-1)}\cos\theta\right]^{n+1}}$$

Proof:

From integral calculus, we have:

$$\int_0^{\pi}\frac{d\theta}{a \pm b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}, \text{ where } a^2 > b^2 \tag{4.1.85}$$

Putting $a = hx - 1$ and $b = h\sqrt{(x^2-1)}$, so that $a^2 - b^2 = 1 - 2hx + h^2$. Using these values of a, b and $a^2 - b^2$ in equation (4.1.85), we have;

$$\pi\left(1-2hx+h^2\right)^{-\frac{1}{2}} = \int_0^\pi \left[1-hx \pm h\sqrt{\left(x^2-1\right)}\cos\theta\right]^{-1} d\theta \tag{4.1.86}$$

$$\frac{\pi}{h}\left[1-2\frac{1}{h}x+\frac{1}{h^2}\right]^{-\frac{1}{2}} = \int_0^\pi \left[h\left\{x\pm\sqrt{\left(x^2-1\right)}\cos\theta\right\}-1\right]^{-1} d\theta$$

$$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^\pi \left(ht-1\right)^{-1} d\theta \tag{4.1.87}$$

where $t = x \pm \sqrt{\left(x^2-1\right)}\cos\theta$,

$$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^\pi \frac{1}{ht}\left(1-\frac{1}{ht}\right)^{-1} d\theta$$

$$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^\pi \frac{1}{ht}\left[1+\frac{1}{ht}+\frac{1}{h^2t^2}+\ldots+\frac{1}{h^nt^n}+\ldots\right] d\theta \tag{4.1.88}$$

$$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^\pi \left[\frac{1}{ht}+\frac{1}{h^2t^2}+\frac{1}{h^3t^3}+\ldots+\frac{1}{h^{n+1}t^{n+1}}+\ldots\right] d\theta$$

$$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) \quad = \sum_{n=0}^{\infty}\left[\frac{1}{h^{n+1}}\int_0^\pi \frac{1}{t^{n+1}} d\theta\right]$$

$$\pi\sum_{n=0}^{\infty}\frac{1}{h^{n+1}}P_n(x) = \sum_{n=0}^{\infty}\left[\frac{1}{h^{n+1}}\int_0^\pi \frac{d\theta}{\left\{x\pm\sqrt{\left(x^2-1\right)}\cos\theta\right\}^{n+1}}\right] \tag{4.1.89}$$

Now, equation the coefficient of $\frac{1}{h^{n+1}}$ from both sides, we have;

$$\pi P_n(x) = \int_0^\pi \frac{d\theta}{\left\{x\pm\sqrt{\left(x^2-1\right)}\cos\theta\right\}^{n+1}}$$

$$\therefore P_n(x) = \frac{1}{\pi}\int_0^\pi \frac{d\theta}{\left\{x\pm\sqrt{\left(x^2-1\right)}\cos\theta\right\}^{n+1}} \tag{4.1.90}$$

Remarks: Replacing n by $-(n+1)$ in Laplace's second integral, we have;

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi}\left\{x \pm \sqrt{\left(x^2 - 1\right)}\cos\theta\right\}^n d\theta$$

$P_{-(n+1)}(x) = P_n(x)$, {from Laplace's first .integral}

CHAPTER FIVE

SUMMARY AND CONCLUSION

5.1 SUMMARY

The Legendre's differential equation and polynomials is given by.

$$\left(1-x^{2}\right)y''(x)-2xy'(x)+n(n+1)y(x)=0 \qquad 5.1.0$$

Where n is a real number. The solutions of this equation are called Legendre functions of degree n. When n is a non-negative integer i.e. n = 0, 1, 2, 3… the Legendre functions are often referred to as Legendre polynomials $P_n(x)$.

Since Legendre's differential equation is a second order ordinary differential equation, two sets of functions are needed to form the general solution. Legendre polynomials of the second kind $Q_n(x)$ are then introduced. The general solution of a non-negative integer degree Legendre's Differential Equation can hence be expressed as:

$$Y(x) = A_n P_n(x) + B_n Q_n(x) \qquad 5.1.1$$

However, $Q_n(x)$ is divergent at $x = \pm 1$. Therefore, the associated coefficient B_n is forced to be zero to obtain a physically meaningful result when there are no sources or sinks at the boundary points $x = \pm 1$.

The generating function of a Legendre polynomials is given by:

$$\frac{1}{\sqrt{1-2tX + t^{2}}} = \sum_{n=0}^{\infty} P_n(x)t^{n} \qquad 5.1.2$$

Thus, the generating function of the Legendre polynomials can be used to investigate their properties. Generating functions are available for most orthogonal polynomials.

The orthogonality of the Legendre polynomials $P_n(x)$, n = 0, 1, 2, 3..., form a complete orthogonal set on the interval $-1 \le x \le 1$. It can be shown that:

$$\int_{-1}^{1} P_M(x) P_n(x)\,dx = \begin{cases} 0 & m \neq n \\ \dfrac{2}{2n+1} & m = n \end{cases} \quad 5.1.3$$

By using this orthogonality, a piecewise continuous function $f(x)$ in $-1 \leq x \leq 1$ can be expressed in terms of Legendre polynomials:

$$\sum_{n=0}^{\infty} C_n P_n(x) = \begin{cases} f(x) \ \textit{where} \ f(x) \ \textit{is continuous} \\ \dfrac{f(x^-) + f(x^+)}{2} \textit{at discontinuous po}\mathrm{int}\,s \end{cases} \quad 5.1.4$$

Where:

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\,dx. \quad 5.1.5$$

This orthogonal series expansion is also known as a Fourier- Legendre series expansion or a Generalized Fourier Series expansion.

However, the Legendre polynomials generate the power series that solves Legendre's differential equation:

$$\left(1 - x^2\right) p''(x) - 2xp'(x) + n(n+1)\,p(x) = 0 \quad 5.1.6$$

The series can easily be generated using the Rodrigue's formula, which is given by

$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n \quad 5.1.7$$

The first six polynomials are:

$$p_0(x) = 1$$

$$p_1(x) = x$$

$$p_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$p_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$p_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$p_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right).$$

5.2 CONCLUSION

There are much to be considered as far as **LEGENDRE** functions are concerned, and this cannot be handle in a single volume like this.

However, with some consciousness on clarity, we have systematically dealt with some important issues concerning these functions. Nevertheless, this has been with some bounds, aimed at reaching the set objectives of this work. Hence, we see treating of some basic theories of Legendre functions.

REFERENCE

Abramowitz and I. Stegun. Handbook of Mathematical functions. Washington D. C. National Bureau of Standard, Applied Mathematics Series 55, 1964.

Arfken, G. B. Weber H. J. Mathematical methods for Physics Academic Press 2001.

Chapman, S. and Bartels, J. Geomagnetism. Oxford Press Ed., 1940.

Condon, E. U. and G. H. Shortley. The theory of atomic spectra. Cambridge University Press. 1970.

Edmonds, A. R. Angular momentum in a Quantum Mechanics. Princeton University Press. 1957.

Jackson, D. Fourier series and orthogonal functions 1941.

Jackson, J. Classical electrodynamics. John Wiley and sons, Inc, New York, 1962.

Jensen, D. C. and Cain. An interim Geomagnetic field. 1962.

Lebedev, N. Special functions and their applications. Dover publications Inc New York, 1972.

National Bureau of Standard, Applied Mathematics Series 55, 1964.

.