CERTIFICATION

This is to certify that this thesis was written by Magnus Nkwachukwu-Iwinosa EBEH with Matriculation Number PG/PSC2016416 in the Department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, Nigeria under the supervision of Prof. A. A. Osagiede.

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DECLARATION

UNIVERSITY OF BENIN

COLLEGE OF POSTGRADUATE STUDIES

A Mathematical Framework for Optimising Financial Flows in Multi-Tier Supply Chain Networks: A Hybrid Model Incorporating Dynamic Discounting and Risk Mitigation.

By

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THE BOARD OF EXAMINERS DECLARES AS FOLLOWS, THAT THIS IS THE ORIGINAL WORK OF THE CANDIDATE, THAT THE THESIS IS ACCEPTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE (M.Sc) IN INDUSTRIAL MATHEMATICS.

Prof. A. A. Osagiede	**Prof. D. U. Okuonghae**
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CERTIFICATION OF THESIS ON PLAGIARISM

We the undersigned attest and declare that the thesis of Magnus Nkwachukwu-Iwinosa EBEH titled: **A Mathematical Framework for Optimising Financial Flows in Multi-Tier Supply Chain Networks: A Hybrid Model Incorporating Dynamic Discounting and Risk Mitigation** has successfully passed the anti-plagiarism test and does not violate any copyright regulation.

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DEDICATION

This thesis is dedicated to Almighty God who is the source of knowledge and my inspiration.

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Abstract

Effective management of financial flows is essential for sustaining liquidity stability and operational efficiency within multi-tier supply chain networks. Traditional optimisation models tend to prioritise either cost reduction or risk mitigation, often neglecting the balance between working capital efficiency and financial stability. This study proposes a Hybrid Mathematical Framework that integrates dynamic discounting mechanisms and risk mitigation strategies to optimise financial flows across supply chains. The framework addresses four primary objectives: developing a financial flow optimisation model, incorporating dynamic discounting into the model, embedding stochastic variables representing demand volatility and credit risk, and evaluating model performance through numerical simulations.

A quantitative modelling approach is employed, formulating an objective function that minimises total financial costs while controlling for risk using Conditional Value at Risk (CVaR). The model integrates early payment incentives, late payment penalties, and financial risk thresholds to support strategic decision-making. Numerical simulations using synthetic financial data were conducted to assess the model's performance.

Results indicate that the Hybrid Model offers a superior trade-off between cost efficiency and financial stability. Dynamic discounting reduces total financial costs, while CVaR integration ensures liquidity remains risk-sensitive. The study recommends adopting dynamic discounting with risk-sensitive optimisation models and exploring technologies like real-time analytics and AI. Future research could refine the framework via industry-specific adaptations and blockchain-enabled contracts.

CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

In the context of globalisation and intricate supply chain networks, effective financial flow management has become increasingly critical. Traditional supply chain financing methods, such as trade credit and factoring, have provided foundational support for businesses aiming to manage cash flows and enhance working capital efficiency. However, these conventional approaches often lack the agility required to respond to real-time financial risks, including currency fluctuations, credit defaults, and interest rate volatility (Ruszczyński & Shapiro, 2003).

Mathematical models have been extensively employed to optimise financial flows within supply chain networks. For instance, Ma (2005) applied optimisation techniques to determine acquisition policies under discount schemes amidst uncertain demand. Despite their contributions, these models often encounter challenges related to scalability and comprehensive financial risk management. Similarly, stochastic programming models proposed by Ruszczyński and Shapiro (2003) have effectively addressed operational uncertainties but have not fully integrated financial risks into their optimisation frameworks. Ermoliev et al. (2008) introduced discounting models linked to random stopping times to assess long-term catastrophic risks; however, their applicability to multi-tier supply chains is constrained by computational complexity. More recently, Hua et al. (2023) proposed a FIMP-based optimisation model for dynamic discounting in supply chain finance, yet its focus on single-tier networks limits its effectiveness in more complex, multi-tier structures.

In the Nigerian context, the importance of robust supply chain management is underscored by the nation's dynamic economic environment. Adeleke (2022) emphasises that well-developed supply chain and risk management processes enhance organisational resilience, flexibility, and adaptability, enabling firms to respond effectively to complex and dynamic environments. Furthermore, Eteyen (2024) highlights the role of digital transformation, particularly through artificial intelligence (AI) and big data, in enhancing supply chain efficiency in Nigeria's oil sector. These technologies significantly improve decision-making, predictive maintenance, inventory management, and risk mitigation, contributing to overall supply chain efficiency. However, challenges such as inadequate infrastructure, a shortage of skilled personnel, and organisational resistance to change persist, indicating the need for more integrated and adaptive financial optimisation models.

Despite these advancements, existing models do not fully address the challenges of optimising financial flows in multi-tier supply chain networks, where multiple layers of suppliers and buyers interact dynamically (Ma, 2005; Ruszczyński & Shapiro, 2003; Ermoliev et al., 2008; Hua et al., 2023). These networks require an integrated approach that incorporates dynamic discounting mechanisms while mitigating financial risks in real time. The absence of a holistic mathematical framework that balances these factors has led to inefficiencies, such as payment delays, liquidity constraints, and suboptimal capital allocation.

Given these gaps, this study seeks to develop a hybrid mathematical framework that optimises financial flows in multi-tier supply chain networks by integrating dynamic discounting strategies with robust financial risk mitigation techniques. The proposed framework aims to enhance decision-making, improve financial efficiency, and strengthen the overall stability of supply chain networks in an increasingly volatile economic environment. By bridging the gap between financial optimisation, risk management, and computational efficiency, this study contributes to the advancement of supply chain finance and provides practical solutions for businesses operating in complex, multi-tier networks (Ma, 2005; Ruszczyński & Shapiro, 2003; Ermoliev et al., 2008; Hua et al., 2023).

1.2 Scope of the study

The study focuses on multi-tier supply chain networks, with numerical simulations based on representative data sets to ensure generalisability.

This study focuses on developing a mathematical framework for optimising financial flows in multi-tier supply chain networks, integrating dynamic discounting and risk mitigation strategies. The research scope encompasses the following dimensions:

1. ***Conceptual Scope***: The study covers financial flow optimisation in multi-tier supply chains, incorporating key financial elements such as payment scheduling, liquidity management, and cost minimisation. It integrates dynamic discounting mechanisms, which incentivise early payments, and risk mitigation strategies, particularly those addressing credit risk and demand uncertainty through Conditional Value at Risk (CVaR). The model is designed to balance cost efficiency, financial stability, and supply chain resilience, making it applicable to firms aiming to improve cash flow predictability and supplier relationships.

2. ***Methodological Scope***: The research employs quantitative modelling techniques, formulating an optimisation model that incorporates stochastic demand variations and financial risk factors. Numerical simulations are conducted using sample financial data to validate the framework's effectiveness. The performance of the Hybrid Model is compared with traditional cost-minimisation and risk-averse models to demonstrate its superiority. The study does not involve

primary data collection from companies but relies on simulated financial scenarios representative of real-world supply chain operations.

3. ***Industrial Scope***: The model is industry-agnostic, meaning it can be applied across various sectors, including manufacturing, agriculture, retail, and global trade networks. While the numerical simulations use generic supply chain financial parameters, the framework can be customised for sector-specific financial flows, particularly in industries with high working capital requirements.

4. ***Geographical Scope***: The study does not focus on a specific geographical region but aims to develop a globally applicable financial flow optimisation model. However, it considers the financial risks and market volatility that impact global supply chains, making the framework suitable for firms operating in emerging and developed economies.

5. ***Limitations of the Scope***: The study does not cover non-financial factors, such as operational disruptions, supply chain logistics, or regulatory compliance issues. It assumes rational financial decision-making by supply chain participants and does not incorporate behavioural finance aspects, such as supplier negotiation dynamics. The research primarily uses static financial scenarios, meaning it does not incorporate real-time adaptive decision-making, which could be explored in future studies. By defining this scope, the study ensures a focused and rigorous approach to addressing financial flow challenges in multi-tier supply chains, providing a robust and adaptable optimisation framework for financial decision-makers.

1.3 Statement of the Problem

Efficient financial flow management is critical for the stability and profitability of multi-tier supply chain networks, particularly in the context of globalisation and increased market volatility

(Hua et al., 2023). Despite advancements in supply chain finance, existing optimisation models either focus on operational uncertainties (Ruszczyński & Shapiro, 2003) or discounting mechanisms (Ma, 2005) without adequately integrating financial risks such as currency fluctuations, credit risks, and interest rate volatility. Consequently, firms operating in multi-tier supply chains often struggle to implement dynamic discounting strategies that adapt to real-time financial conditions while mitigating risks effectively (Ermoliev et al., 2008).

Traditional financial optimisation models primarily address static and single-tier supply chain structures, limiting their applicability in complex, multi-tier networks where interdependencies between suppliers, buyers, and financial institutions exist (Hua et al., 2023). The computational complexity of mixed-integer nonlinear programming (MINLP) and stochastic programming models further restricts their scalability and real-time adaptability in dynamic financial environments (Ruszczyński & Shapiro, 2003). Additionally, while some models incorporate long-term risk considerations, they often fail to account for the endogenous nature of financial risks and their direct impact on discounting mechanisms (Ermoliev et al., 2008).

Given these limitations, there is a critical need for a hybrid mathematical framework that simultaneously optimises financial flows, integrates dynamic discounting, and incorporates comprehensive risk mitigation strategies within multi-tier supply chain networks. Such a framework must be computationally efficient, scalable, and adaptable to real-time financial fluctuations to enhance decision-making and overall financial stability in supply chain operations. Addressing this gap would provide a robust foundation for improving liquidity management, reducing financial inefficiencies, and strengthening resilience against financial uncertainties in complex supply chain networks (Ma, 2005; Ruszczyński & Shapiro, 2003; Ermoliev et al., 2008; Hua et al., 2023).

Despite the potential of dynamic discounting, existing models lack a comprehensive approach that integrates cost optimisation and risk management, particularly in multi-tier networks with complex interdependencies. This gap calls for a mathematical framework capable of addressing these challenges holistically.

1.4 Aims and Objectives

The study is aimed at developing a hybrid mathematical framework that integrates dynamic discounting mechanisms and risk mitigation strategies to optimise financial flows. The objectives of the study are as follows; to:

1. develop a mathematical model to optimise financial flows in multi-tier supply chain networks.
2. integrate dynamic discounting mechanisms into the optimisation framework.
3. incorporate stochastic elements representing demand volatility and credit risk.
4. evaluate the framework's performance via numerical simulations.

1.5 Motivation for the Study

The increasing complexity of global supply chains, coupled with financial uncertainties, necessitates more sophisticated financial optimisation models that can enhance liquidity management, minimise risks, and improve cost efficiency. Traditional supply chain finance models often focus on single-tier structures and static discounting mechanisms, which fail to capture the intricate interdependencies and financial risks inherent in multi-tier networks (Hua et al., 2023). This gap creates inefficiencies, such as delayed payments, suboptimal working capital allocation, and exposure to credit and currency risks, ultimately impacting supply chain resilience and profitability (Ruszczyński & Shapiro, 2003).

Furthermore, the volatility of financial markets, driven by fluctuating interest rates, geopolitical risks, and evolving trade policies, underscores the urgent need for dynamic financial strategies that can adapt in real time (Ermoliev et al., 2008). Existing optimisation models, while effective in addressing operational uncertainties, do not fully integrate dynamic discounting with comprehensive risk mitigation, thereby limiting their practical applicability in multi-tier supply chains (Ma, 2005). As businesses seek to optimise cash flows while mitigating financial risks, the development of a hybrid mathematical framework becomes imperative.

This study is motivated by the need to bridge these gaps by designing a scalable and computationally efficient model that incorporates dynamic discounting strategies and robust financial risk mitigation. By integrating real-time financial data and optimisation techniques, such a framework has the potential to enhance decision-making, improve liquidity, and strengthen the financial stability of supply chain networks. Ultimately, this research aims to contribute to the evolving body of knowledge in supply chain finance by providing a more comprehensive and adaptable mathematical framework for financial flow optimisation (Ma, 2005; Ruszczyński & Shapiro, 2003; Ermoliev et al., 2008; Hua et al., 2023).

1.6 Significance of the Study

This study is significant as it addresses a critical gap in supply chain finance by developing a hybrid mathematical framework that optimises financial flows in multi-tier supply chain networks while incorporating dynamic discounting and comprehensive risk mitigation. Given the increasing complexity of global supply chains and the volatility of financial markets, effective financial flow management is essential for maintaining liquidity, reducing costs, and ensuring the resilience of supply chain operations (Hua et al., 2023). Traditional financial optimisation models are often limited in scope, focusing either on operational uncertainties (Ruszczyński & Shapiro, 2003) or discounting mechanisms (Ma, 2005) without fully integrating financial risks such as currency fluctuations, credit risks, and interest rate volatility. This study seeks to overcome these limitations by introducing a robust and scalable framework that enhances financial decision-making in multi-tier networks.

The findings of this research will have significant implications for supply chain managers, financial analysts, and policymakers. For businesses, implementing a more adaptive and risk-aware financial optimisation model can improve cash flow efficiency, enhance supplier-buyer relationships, and reduce financial vulnerabilities in uncertain market conditions (Ermoliev et al., 2008). For financial institutions, a better understanding of dynamic discounting strategies and risk mitigation techniques can lead to more effective credit allocation and improved risk assessment practices. Additionally, policymakers can leverage the insights from this study to develop regulatory frameworks that promote financial stability and transparency in supply chain finance.

By contributing to the existing body of knowledge in supply chain finance, this study provides a foundation for future research on integrating advanced financial models with real-time data analytics and artificial intelligence. The proposed framework has the potential to drive innovation in financial optimisation techniques, ensuring that multi-tier supply chain networks remain resilient, efficient, and financially sustainable in an increasingly complex and uncertain global economy (Ma, 2005; Ruszczyński & Shapiro, 2003; Ermoliev et al., 2008; Hua et al., 2023).

1.7 Limitation of work

While this study aims to develop a robust mathematical framework for optimising financial flows in multi-tier supply chain networks by incorporating dynamic discounting and risk mitigation, certain limitations must be acknowledged. First, the proposed framework relies on mathematical optimisation models that may require significant computational resources for large-scale implementation. The complexity of mixed-integer nonlinear programming (MINLP) and stochastic programming models increases exponentially as the number of supply chain participants grows, which may pose challenges for real-time decision-making (Ruszczyński & Shapiro, 2003). Although efforts will be made to enhance computational efficiency, scalability remains a potential constraint, particularly for businesses with extensive supplier-buyer networks.

Second, the model's effectiveness depends on the availability and accuracy of real-time financial data. Dynamic discounting strategies and risk mitigation techniques require continuous updates on market conditions, interest rate fluctuations, credit risk profiles, and currency exchange rates (Ermoliev et al., 2008). In many emerging economies, such as Nigeria, access to reliable financial data remains a challenge due to inadequate digital infrastructure, inconsistent reporting standards, and limited financial transparency (Adeleke, 2022). These data limitations may impact the precision of the model's predictions and its practical applicability in real-world settings.

Third, the study primarily focuses on financial flow optimisation and does not extensively account for behavioural and organisational factors that influence decision-making within supply chain networks. For instance, supplier-buyer relationships, negotiation dynamics, and risk perceptions play a crucial role in the adoption of dynamic discounting and financial risk mitigation strategies (Hua et al., 2023). While the mathematical framework provides a structured approach to financial optimisation, it may not fully capture the complexities of human behaviour and strategic interactions within multi-tier supply chains.

Lastly, the study is constrained by assumptions regarding market stability and regulatory environments. The framework assumes relatively stable macroeconomic conditions for modelling financial risks; however, external shocks such as economic recessions, geopolitical conflicts, and abrupt policy changes could introduce unforeseen variables that impact financial flows (Ma, 2005). In Nigeria, for example, fluctuations in exchange rates, inflation, and monetary policy adjustments significantly influence supply chain financing (Eteyen, 2024). The model may need further refinements to accommodate such macroeconomic uncertainties in future research.

Despite these limitations, this study provides a valuable foundation for improving financial flow optimisation in multi-tier supply chain networks. Future research can build on this framework by incorporating machine learning techniques for predictive analytics, exploring behavioural finance aspects, and expanding the model's adaptability to dynamic regulatory environments (Ruszczyński & Shapiro, 2003; Ermoliev et al., 2008; Hua et al., 2023).

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

A multi-tier supply chain network consists of multiple interconnected layers of suppliers, manufacturers, distributors, and retailers working collaboratively to deliver products or services to end consumers. Each tier represents a distinct level in the supply chain hierarchy, with materials and information flowing between these levels to ensure efficient operations (Rutten, 2022; Thomé, et al., 2014).

Optimising financial flows within such complex networks is crucial for maintaining liquidity and operational efficiency. One effective strategy in this context is dynamic discounting, a supply chain finance policy where a supplier offers a credit period to a buyer, accompanied by a discount if the buyer settles the payment before the credit period concludes. This approach benefits both parties: suppliers gain quicker access to funds, enhancing their cash flow, while buyers receive cost reductions. The flexibility of dynamic discounting allows for the discount amount to be calculated based on the number of days remaining until the due date, providing a tailored solution to financial management within the supply chain (Azizian, et al., 2023; Hua, et al., 2023).

Incorporating risk mitigation strategies is also vital in multi-tier supply chains to address uncertainties such as supply disruptions or market fluctuations. For instance, supply chain risk management (SCRM) involves implementing strategies to manage both every day and exceptional risks along the supply chain based on continuous risk assessment, with the objective of reducing vulnerability and ensuring continuity. This comprehensive approach ensures that risk factors are systematically addressed, contributing to the overall resilience of the supply chain network (Simba, et al., 2017; Wang-Mlynek, & Foerstl, 2020).

By developing a mathematical framework that integrates dynamic discounting and risk mitigation strategies, organisations can effectively optimise financial flows in multi-tier supply chain networks. Such a hybrid model not only enhances financial efficiency but also bolsters the supply chain's resilience against potential risks, ensuring smoother operations and improved profitability.

2.1 Linear Programming and Its Applications in Cost Optimisation

Linear programming (LP) is a mathematical technique used to determine the optimal allocation of limited resources to achieve a specific objective, such as minimising costs or maximising profits, subject to a set of constraints. In the context of supply chain management, LP plays a pivotal role in cost optimisation by enabling decision-makers to model and solve complex problems related to production planning, transportation, and inventory management (Oluwaseyi, et al., 2020; Alotaibi, & Nadeem, 2021).

2.1.1 *Applications of Linear Programming in Cost Optimisation*

1. ***Production Planning***: LP assists in determining the optimal production schedule that meets customer demand while minimising production and inventory holding costs. By formulating the production process as a linear model, companies can decide the quantity of each product to manufacture, considering constraints such as production capacity, labour availability, and material supply. This approach ensures efficient resource utilisation and cost reduction (Amole, et al, 2016; Agyepong-Mensah, 2011).

2. ***Transportation and Distribution***: The transportation problem, a classic application of LP, focuses on finding the most cost-effective way to distribute products from multiple suppliers to various consumers. By modelling transportation costs and supply-demand constraints, LP helps in determining the optimal shipping routes and schedules, thereby minimising total transportation expenses. This is particularly beneficial in multi-tier supply chain networks where products must traverse multiple stages before reaching the end consumer (Tran, & Haasis, 2015).

3. ***Inventory Management***: LP aids in optimising inventory levels by balancing ordering costs with holding costs. Through linear models, businesses can determine the optimal order quantities and reorder points that minimise total inventory costs while ensuring that customer demand is consistently met. This is crucial in preventing both stockouts and overstock situations, leading to cost savings and improved service levels (Ogumeyo, et al., 2025).

2.1.2 *Integration with Financial Flow Optimisation*

In multi-tier supply chain networks, financial flows are as critical as the physical movement of goods. LP can be extended to optimise these financial aspects by incorporating factors such as dynamic discounting and risk mitigation. Dynamic discounting allows suppliers to receive early payments in exchange for discounts, benefiting both suppliers and buyers. By formulating this scenario as a linear program, companies can determine the optimal payment schedules that maximise cash flow benefits while minimising costs.

Risk mitigation is another area where LP proves valuable. Supply chains are susceptible to various risks, including demand fluctuations, supply disruptions, and price volatility. By incorporating risk factors into the LP model, businesses can develop strategies that minimise potential losses and ensure supply chain resilience. For instance, LP can help in selecting a diversified portfolio of suppliers to reduce dependency on a single source, thereby mitigating supply risks (Pourmohammadreza, et al., 2025).

2.2 Stochastic Programming for Managing Demand Uncertainty

Stochastic programming is a mathematical optimisation framework designed to address decision-making problems under uncertainty by incorporating randomness directly into the model. In supply chain management, demand uncertainty poses significant challenges, affecting inventory levels, production planning, and financial flows. Stochastic programming offers robust solutions to manage these uncertainties, ensuring cost-effectiveness and operational efficiency (Chen, et al., 2023).

2.2.1 *Stochastic Programming in Supply Chain Management*

In supply chains, demand fluctuations can lead to either excess inventory or stockouts, both of which are costly. Stochastic programming models these uncertainties by considering various demand scenarios and optimising decisions accordingly. For instance, a two-stage stochastic programming model can determine optimal ordering quantities before and after potential disruptions, accounting for uncertainties in demand and supply chain disruptions (Snyder & Shen, 2011). This approach enables companies to prepare for various demand outcomes, reducing the risks associated with demand variability.

2.2.2 *Integration with Financial Flow Optimisation*

Managing financial flows in multi-tier supply chain networks requires careful consideration of demand uncertainty. Stochastic programming aids in this by optimising cash flows and payment schedules under uncertain demand conditions. For example, dynamic discounting strategies, where suppliers offer discounts for early payments, can be optimised using stochastic models to balance the benefits of discounts against the risks of uncertain future cash flows (Pfohl & Gomm, 2009). By incorporating demand scenarios into the financial models, companies can make informed decisions on payment terms and capital allocation, enhancing liquidity and profitability.

2.2.3 *Risk Mitigation through Stochastic Programming*

Risk mitigation is crucial in supply chain management, especially under demand uncertainty. Stochastic programming contributes to risk mitigation by providing solutions that are feasible across various scenarios, thereby reducing the impact of adverse events. For instance, a stochastic programming approach can be employed to manage supply chain disruptions by determining optimal pre-disruption and post-disruption ordering quantities, considering uncertainties in demand and supply chain disruptions (Snyder & Shen, 2011). This proactive planning enhances the resilience of the supply chain against demand fluctuations and other uncertainties.xxz

2.3 Theoretical Foundations of Dynamic Discounting In Financial Systems

Dynamic discounting is a financial strategy that enables buyers to offer early payments to suppliers in exchange for discounts on invoiced amounts. Dynamic discounting is a flexible financial mechanism where the discount rate varies based on how early a buyer pays a supplier relative to the agreed-upon payment terms. This approach benefits both parties: suppliers gain improved cash flow, while buyers achieve cost savings. The theoretical foundations of dynamic discounting are rooted in financial management principles, supply chain finance, and optimisation theories (Pourmohammadreza, et al., 2025; Uhuka, 2022).

2.3.1 *Financial Management Principles*

At its core, dynamic discounting is grounded in the time value of money (TVM) concept, which posits that a sum of money holds greater value now than it does in the future due to its potential earning capacity. By offering early payment discounts, buyers effectively utilise their available liquidity to reduce procurement costs, while suppliers receive prompt access to funds, enhancing their working capital. This mutually beneficial arrangement aligns with TVM principles, as both parties optimise their financial positions through the timing of payments (Pfohl & Gomm, 2009).

2.3.2 *Supply Chain Finance*

Dynamic discounting is a pivotal component of supply chain finance (SCF), which focuses on optimising financial flows within the supply chain to improve the financial health of all participants. SCF solutions, including dynamic discounting, facilitate collaboration between buyers and suppliers, enabling more favourable financing terms and strengthening supply chain relationships. The implementation of dynamic discounting programmes has been shown to enhance liquidity and reduce financing costs across the supply chain (Pfohl & Gomm, 2009).

2.3.3 *Optimisation Theories*

From an optimisation perspective, dynamic discounting involves determining the optimal discount rates and payment timings that maximise financial benefits for both buyers and suppliers. This requires the application of mathematical models and algorithms to analyse factors such as cash flow requirements, cost of capital, and payment terms. Research in this area has developed models that assist in designing supply chain networks under dynamic discounting schemes, considering variables like replenishment cycles, pricing strategies, and credit terms to achieve optimal outcomes (Tsao et al., 2021).

2.4 Review of Relevant Models and Theories

A model is a conceptual or mathematical representation of a system, process, or relationship (Kuhn, 2012, p. 23). It is a simplified description of a complex phenomenon, aiming to capture its essential features. Models can be descriptive, predictive, or explanatory (Braithwaite, 2013, p. 15). We could have mathematical models, statistical models, or conceptual models. For example, the equations describing the motion of a projectile under gravity can be considered a model (Halliday et al., 2014, p. 67).

On the other hand, a theory is a well-substantiated explanation for a set of phenomena (Popper, 1963, p. 33). It is a framework that provides a deeper understanding of the underlying mechanisms and relationships. Theories are developed through the scientific method and are tested through experimentation and observation. Examples include the theory of gravity, the theory of evolution, and the theory of relativity (Lakatos, 1970, p. 155). For instance, the theory of gravity explains why objects fall towards the ground, and it encompasses multiple models, including the one mentioned above (Einstein, 1915, p. 191).

While, a theorem is a mathematical statement that has been rigorously proved to be true (Russell, 1920, p. 15). Theorems are derived from axioms, definitions, and previously established theorems. Theorems provide a precise and logical explanation of a mathematical concept or relationship. Examples include the Pythagorean theorem, Fermat's last theorem, and the fundamental theorem of calculus.

The connection between theory and theorem lies in their shared goal of providing a deeper understanding of a particular phenomenon or mathematical concept. While a theory provides a broad explanatory framework, a theorem offers a precise and logical explanation of a specific mathematical concept or relationship within that framework.

According to Kuhn (2012, p. 23), a theory is a "well-substantiated explanation for a set of phenomena." In contrast, a theorem is a mathematical statement that has been rigorously proved to be true (Russell, 1920, p. 15). Theorems often rely on theories to provide context and meaning, while theories rely on theorems to provide precise and logical explanations of specific mathematical concepts.

For instance, the theory of calculus provides a broad explanatory framework for understanding rates of change and accumulation (Lakatos, 1970, p. 155). Within this framework, the fundamental theorem of calculus provides a precise and logical explanation of the relationship between the derivative and the integral (Kline, 1972, p. 345).

In summary, Models are developed through empirical observation and mathematical formulation, theories are developed through the scientific method, and theorems are derived through mathematical proof (Lakatos, 1970, p. 160).

These models help optimise financial flows, manage uncertainty, and mitigate risks within multi-tier supply chain networks.

2.4.1 Relevant Models

2.4.1.1 Linear Programming (LP) Model

LP is a fundamental optimisation model used to minimise costs or maximise profits while considering constraints such as cash flow limitations, supplier payment terms, and inventory holding costs. It is particularly useful in dynamic discounting strategies where buyers optimise early payments to suppliers (Pourmohammadreza, Jesri, & Kamran, 2025).

LP can help determine the optimal early payment schedule that maximises savings while maintaining liquidity

General Formulation:

$$\min C^{T}x$$

Subject to:

$$Ax \leq b$$

$$x \geq 0$$

Where:

C is the cost vector (e.g., financial costs associated with different supply chain transactions)

x represents decision variables (e.g., optimal payment schedules, order quantities).

A is the constraint matrix (e.g., liquidity constraints, supply chain capacity).

b is the resource availability vector (e.g., budget limits).

(CSCI 1951-G – Optimization Methods in Finance)

2.4.1.2 Stochastic Programming Model

Stochastic programming is used to handle demand uncertainty by optimising decisions under multiple possible future scenarios. This is crucial for financial flows where demand fluctuations affect cash flow availability.

Stochastic programming is an optimisation framework that involves making decisions in the presence of uncertainty (Sahinidis, 2004; Bozorgi-Amiri, Jabalameli, & Mirzapour Al-e-Hashem, 2013). It uses algorithms such as Benders decomposition and Lagrangean decomposition to solve optimisation problems with linear constraints and continuous recourse.

Stochastic programming model is used to determine optimal financial commitments under uncertain cash flow availability, reducing the risk of financial distress. A special case is two-stage stochastic programming which was introduced by Dantzig, 1955. (Infanger, Infanger, & Dantzig, 2011; Kolbin, V. V. (1977).

Two-Stage Stochastic Model:

First-stage decision (x): Decisions made before demand is realised (e.g., initial financial allocation).

Second-stage decision (y_s): Adaptive decisions made after uncertainty is realised (e.g., adjusting supplier payments).

$$\min C^T x + E\xi \quad [Q(x,\xi)]$$

Where

$$Q(x,\xi) = \min q^T y_s$$

Subject to:

$$Ax \leq b$$

$$B_s \quad Y_s \quad \leq d_s \quad - T_s \quad x, \forall s$$

Where:

ξ represents uncertainty (e.g., demand fluctuations, payment delays).

$E_{\xi}[Q(x,\xi)]$ is the expected cost under uncertainty.

(Graß, 2019)

2.4.1.3 Game Theory Model for Financial Interactions

Game theory is useful for modelling strategic interactions between supply chain participants, such as suppliers and buyers in a dynamic discounting scenario. It is used to determine the optimal discount rate that satisfies both the buyer and the supplier, ensuring a stable financial agreement (Agi, Faramarzi-Oghani, & Hazır, 2021).

Nash Equilibrium Model for Dynamic Discounting:

Consider a buyer (B) and a supplier (S), where:

p is the payment amount.

d is the discount offered.

$u_B(p,d)$ is the utility function of the buyer.

$u_S(p,d)$ is the utility function of the supplier.

The Nash equilibrium occurs when:

$$\max_{p,d} u_B(p,d) \text{ and } \max_{p,d} u_S(p,d)$$

Subject to:

$$p \leq P_{max}, \; d \geq 0$$

(Rimal, & Maier, 2017; Wei, 2021)

2.4.1.4 Mean-Variance Portfolio Model for Risk Mitigation

This model, derived from Modern Portfolio Theory (MPT), is used to balance financial risk and return when allocating capital in supply chain financing. It is used to determine optimal capital allocation across multiple financing options to minimise financial risk in supply chain networks.

Mathematical Formulation:

$$\min \frac{1}{2} X^{T} \Sigma x - \lambda \mu^{T} X$$

Subject to:

$$\sum X_i = 1, X_i \geq 0$$

Where:

x represents investment proportions in different supply chain financing options.

Σ is the covariance matrix of financial risks

μ is the expected return of different financial instruments.

λ is the risk tolerance parameter.

(Guo, Chan, Wong, & Zhu, 2019)

2.4.1.5 Value-at-Risk (VaR) Model for Financial Risk Assessment

The VaR model is used to estimate the potential financial loss in supply chain financing due to unexpected disruptions. It is used to estimate the probability of financial shortfalls in supply chain networks, ensuring financial resilience (Zhang, Wagner, Goh, & Asian, 2024).

VaR Formula for Cash Flow Risk:

$$VaR_\alpha = \mu - z_\alpha\sigma$$

Where:

μ is the expected cash flow.

σ is the standard deviation of cash flow volatility.

z_α is the z-score corresponding to the confidence level α (e.g., 95% or 99%).

(Laforêt, & Devolder, 2018)

2.4.1.5.1 Value at Risk (VaR) model and Conditional Value at Risk (CVaR) model compared

The Value at Risk (VaR) model and the Conditional Value at Risk (CVaR) model are two widely used risk management tools in finance. While both models aim to estimate potential losses, they differ in their approach and focus.

Value at Risk (VaR) Model:

The VaR model estimates the potential loss of a portfolio over a specific time horizon with a given probability (confidence level). It is defined as "the maximum potential loss (or worst loss) of a portfolio over a target horizon with a given confidence level" (Jorion, 2007, p. 3). VaR is typically calculated using historical data, simulation, or variance-covariance methods.

Conditional Value at Risk (CVaR) Model:

The CVaR model, also known as Expected Shortfall (ES), estimates the expected loss of a portfolio in the worst α% of cases, where α is the confidence level. CVaR is defined as "the expected value of the loss given that the loss exceeds the VaR" (Rockafellar & Uryasev, 2002, p. 2). CVaR is a more conservative measure than VaR, as it focuses on the expected loss in extreme scenarios.

2.4.2 Relevant Theories

2.4.2.1 Optimisation Theory

Optimisation theory deals with finding the best possible solution under given constraints. It is crucial for determining optimal financial flows, dynamic discounting rates, and risk mitigation strategies (Dahl, Meeraus, & Zenios, 1993; Jain, 2024).

Optimisation theory focuses on finding the best solution—often the minimum cost or maximum profit—within a set of constraints. In supply chain finance, optimisation models determine the most efficient allocation of financial resources, taking into account constraints like budget limits, liquidity, and payment terms (Emtehani, Nahavandi, & Rafiei, 2021).

These mathematical theories provide a rigorous foundation for optimising financial flows in multi-tier supply chain networks. They ensure that dynamic discounting rates, financial risks, and liquidity constraints are optimally managed using mathematical precision.

Key Theorems:

2.4.2.1.1 Karush-Kuhn-Tucker (KKT) Conditions:

The KKT conditions are necessary (and under certain conditions sufficient) for a solution to be optimal in a nonlinear programming problem with inequality constraints. They extend the method of Lagrange multipliers to handle inequalities and are essential for solving optimisation problems where the objective function or constraints are non-linear. It is used in nonlinear programming to find optimal financial decisions subject to liquidity and budget constraints. They provide necessary conditions for a solution to be optimal when the objective function and the constraints are continuously differentiable (Flores-Bazán, & Mastroeni, 2015; Dobamo, 2021).

Consider a nonlinear optimisation problem:

$$\text{Min } f(x) \text{ subject to } g_i(x) \leq 0,\ h_j(x) = 0$$

where:

- f(x) is the objective function,
- g_i (x) represents inequality constraints,
- h_j (x) represents equality constraints.

The Lagrangian function is defined as:

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \sum_i \lambda i g i(x) + \sum_j \mu j h j(x)$$

(Zhu, Tang, Wang, & Ding, 2015; Li, 2011)

The KKT conditions consist of a set of equations and inequalities that any optimal solution must satisfy. These include the stationarity condition, primal feasibility, dual feasibility, and complementary slackness.

The KKT conditions are:

1. ***Stationarity:*** This explains that the gradient of the Lagrangian is zero

$$\nabla f(x) + \sum_i \lambda i \nabla g i(x) + \sum_i \mu j \nabla h j(x) = 0$$

(Ghojogh, Ghodsi, Karray, & Crowley, 2021; Hu, & Ralph, 2004)

2. ***Primal feasibility***: This explains that all constraints are met.

$$g_i(x) \leq 0, h_j(x) = 0$$

(Glover, 1975)

3. ***Dual feasibility:*** this explains that the Lagrange multipliers are non-negative for inequality constraints.

$$\lambda_i \quad \geq 0$$

(Cerone, Fosson, Pirrera, & Regruto, 2024)

4. ***Complementary slackness:*** This explains that each multiplier is zero when its corresponding constraint is not active

$$\lambda_i \; g_i \quad (x) = 0$$

(Jie, & Yan, 2021).

In supply chain finance, Karush-Kuhn-Tucker (KKT) Conditions ensures optimal pricing in dynamic discounting, where the buyer offers suppliers early payment discounts. The constraints can represent cash flow limits and risk factors.

In supply chain finance, the KKT conditions are instrumental when formulating nonlinear optimisation problems such as those encountered in dynamic discounting, where cost functions or cash flow constraints may be nonlinear. They help ensure that the proposed financial strategies are not only feasible but also optimal under the given constraints (Bazaraa et al., 2013).

2.4.2.1.2 Duality Theorem in Linear Programming:

This demonstrates that every linear optimisation problem has a dual problem, which can be used to analyse supply chain finance decisions efficiently. Duality in linear programming establishes that every linear optimisation problem (primal) has an associated dual problem. The solutions of these problems provide bounds on each other's objective values.

For a primal problem:

$$\min C^T x \quad \text{subject to} \quad Ax \leq b,\ x \geq 0$$

The dual problem is:

$$\text{Max } b^T y \quad \text{subject to} \quad A^T y \geq c,\ \ y \geq 0$$

The dual problem often provides economic interpretations of the constraints in the primal problem. For example, the dual variables (or shadow prices) reflect the marginal value of resources in a supply chain context. In financial optimisation, duality helps determine the shadow prices of constraints in a supply chain financing model. The strong duality theorem ensures that solving the dual provides the same optimal value as the primal.

By analysing the dual, decision-makers can assess the cost of constraints such as limited liquidity or restricted supplier capacity. This is particularly useful in optimising financial flows where it is necessary to understand how tightening or relaxing constraints affects overall cost (Dantzig, 1951).

2.4.2.2. Game Theory

Game theory studies strategic interactions between multiple decision-makers, such as buyers and suppliers negotiating dynamic discounting terms (Nair, Narasimhan, & Bendoly, 2011).

Key Theorems

2.4.2 .2.1. Nash Equilibrium Theorem:

In game theory, a Nash equilibrium is a solution concept where no participant can benefit by unilaterally changing their strategy, provided the strategies of the other players remain constant.

Nash Equilibrium Theorem ensures a stable state where neither buyers nor suppliers can unilaterally change their payment strategies for better gains.

In a two-player game, let U_i (S_1, S_2) be the payoff function for player *i*, where S_1 and S_2 are strategy choices (Song, Liu, & Lawarrée, 2002).

A Nash equilibrium occurs when:

$$U_1(S_1^*, S_2^*) \geq U_1(S_1, S_2^*) \ \forall S_1$$

$$U_2(S_1^*, S_2^*) \geq U_2(S_1^*, S_2) \ \forall S_2$$

In dynamic discounting negotiations, Nash equilibrium ensures that neither buyer nor supplier has an incentive to deviate from an agreed discount rate.

At a Nash equilibrium, each player's strategy is optimal given the strategies chosen by all other players. This stability is crucial in negotiations, such as dynamic discounting scenarios where both buyers and suppliers need to agree on mutually beneficial terms.

In the context of supply chain finance, the Nash equilibrium helps model interactions between multiple agents (buyers and suppliers). It ensures that the dynamic discounting rates set are robust—neither party has an incentive to deviate from the agreed terms, thereby promoting long-term partnership stability (Ye, Han, Ding, & Xu, 2023; Nash, 1950).

2.4.2.2.2 Minimax Theorem

The minimax theorem, primarily developed in the context of zero-sum games, states that the maximum of the minimum gains (or the minimum of the maximum losses) in a game scenario is equal to the value of the game.

For a two-player zero-sum game with a payoff matrix *A*:

$$\min_{x} \max_{y} (X^{T}Ay) = \max_{y} \min_{x}(X^{T}Ay)$$

where x and y are probability distributions over strategies.

Minimax Theorem is useful in risk mitigation by ensuring that the worst-case financial loss is minimised. It is used to minimise financial risk in uncertain market conditions.

In risk management, the minimax theorem is utilised to determine strategies that minimise the worst-case financial loss. It ensures that, under the worst possible circumstances, the loss is as small as possible.

For supply chain networks facing uncertainties—such as fluctuating cash flows or abrupt supply disruptions—the minimax approach can inform the development of robust financial strategies. These strategies safeguard against adverse scenarios by focusing on minimising potential maximum losses (Fasakin, Yusuf, & Akintayo, 2024; von Neumann, 1928).

***2.4.2.*3. Stochastic Process Theory**

Stochastic processes deal with random variables that evolve over time, crucial for handling demand uncertainty and cash flow variability in supply chain finance.

Key Theorems:

2.4.2.3. 1. Markov Chains Theorem:

Markov chains are stochastic processes characterized by the “memoryless” property, meaning that the future state of the process depends solely on the current state and not on the sequence of events that preceded it. Markov Chain helps to model uncertain payment behaviours in multi-tier supply chains.

A Markov chain is defined by a transition matrix P, where:

$$P_{ij} = P(X_{t+1} = j \mid X_t = i)$$

A steady-state distribution π satisfies:

$$\pi P = \pi$$

Markov Chains is used to model cash flow transitions in supply chains.

In a Markov chain, transitions between states occur with fixed probabilities, which can be used to model various uncertain processes, such as payment delays or demand fluctuations.

In multi-tier supply chain networks, Markov chains enable the modelling of cash flow dynamics or supplier behaviour where future states (e.g., timely payments or defaults) depend only on the current financial status. This assists in developing adaptive discounting strategies that account for probabilistic transitions (Asmussen, 2003; Ssempijja, Namango, Ochola, & Mubiru, 2021; Asmussen, 2003; Ross, 2014).

***2.4.2.3.2 Martingale Theor*y**

Martingale Theory is used in financial modelling to assess future cash flows based on current supply chain financial data.

Martingale theory deals with stochastic processes in which the expected future value, conditional on all past events, is equal to the present value. In other words, there is no “drift” in the process.

A stochastic process X_t is a martingale if:

$$E[X_{t+1} \mid X_1, X_2, \ldots, X_t] = X_t$$

A martingale is often interpreted as a “fair game” in financial terms, where the expected gains or losses over time are zero when adjusted for all known information.

Martingale Theory is used in risk-neutral pricing of financial instruments. This theory is used in financial modelling to assess the evolution of cash flows and to price financial instruments in a risk-neutral manner. In supply chain finance, it can underpin models that assume cash flows are unpredictable yet fair on average, aiding in risk assessment and management (Ross, 2014).

*2.4.2.*4 Convex Analysis and Duality Theory

Convex analysis is essential for solving financial flow optimisation problems, as many supply chain finance models rely on convex objective functions.

Key Theorems:

2.4.2.4.1 Fenchel’s Duality Theorem

Fenchel’s duality theorem is a cornerstone of convex analysis. It provides conditions under which the optimal value of a convex optimisation problem is equal to that of its dual problem. Fenchel’s Duality Theorem helps to transform complex financial problems into simpler dual formulations for efficient solutions.

For a convex function f(x):

$$f(x) = \sup_{y} (x^T y - f^*(y))$$

where $f^*(y)$ is the convex conjugate.

Fenchel's Duality Theorem helps to reformulate financial flow optimisation problems. This theorem establishes a strong connection between a primal convex optimisation problem and its dual, allowing complex problems to be reformulated into potentially simpler dual problems.

In optimising financial flows, Fenchel's duality theorem helps in re-casting intricate supply chain finance problems into a dual form where constraints such as liquidity or credit risk are more transparent and tractable (Boyd & Vandenberghe, 2004).

2.4.2.4.2 Separating Hyperplane Theorem

The separating hyperplane theorem states that two disjoint convex sets can be separated by a hyperplane. For two convex sets A and B, there exists a hyperplane:

$$w^T x = b$$

such that:

$$w^T a \leq b,\ w^T b \geq b\ \forall\ a \in A,\ \forall\ b \in B$$

Separating Hyperplane Theorem is used to separate feasible and infeasible financial strategies. This theorem is fundamental in convex optimisation and functional analysis.

The theorem provides a geometric basis for distinguishing between feasible and infeasible regions in optimisation problems. It essentially guarantees that if two convex sets do not overlap, one can find a hyperplane that completely divides them.

In supply chain financial models, the separating hyperplane theorem is used to prove the existence of optimal solutions and to justify the partitioning of the solution space into acceptable (feasible) and unacceptable (infeasible) financial strategies (Soltan, 2021; Boyd & Vandenberghe, 2004; Pospíšil, Kolka, Horská, & Brzobohatý, 2000)

*2.4.2.*5 Queuing Theory

Queuing theory models waiting times and service rates, which are critical for assessing payment processing delays and supply chain liquidity flows (Balla, & Ilyés, 2016).

Key Theorems:

2.4.2.5.1 Little's Theorem

Little's Theorem establishes the relationship between cash flow, liquidity availability, and supplier payment delays. It is a fundamental result in queuing theory, relating the average numbers of items in a system to the average rate at which items arrive and the average time an item spends in the system.

The theorem is usually expressed as $L=\lambda W$, where L is the average number of items in the queue, λ is the average arrival rate, and W is the average waiting time. ie

$$L = \lambda W$$

Where:

L = average number of items in the system,

λ = average arrival rate,

W = average time in the system.

Little's Theorem helps to analyse payment processing times in financial flows.

In the context of supply chain finance, Little's theorem can be used to analyse payment processing delays. For instance, it allows one to estimate how long capital is tied up in the system, thereby affecting liquidity and the effectiveness of dynamic discounting strategies (Shortle, Thompson, Gross, & Harris, 2018; Kleinrock, 1975; Whiting, 2012; Tseng, Wu, Hu, & Wang, 2018).

2.4.2.5.2 Erlang's Formula

Erlang's Formula helps to optimise financial service capacity in supply chain finance systems. They are used in queuing theory to model the probability of delay and to determine the necessary service capacity to achieve a desired level of performance in a system with random arrivals and service times.

For a queuing system with arrival rate λ and service rate μ, the probability that an arriving customer must wait is:

$$\mathrm{Pw} = \frac{\frac{(\lambda/\mu)\mathrm{c}}{c!} \cdot \frac{\mathrm{c}\mu}{\mathrm{c}\mu - \lambda}}{\sum_{k=0}^{c-1} \frac{\left(\frac{\lambda}{\mu}\right)\mathrm{k}}{k!} + \frac{\left(\frac{\lambda}{\mu}\right)\mathrm{c}}{c!} \cdot \frac{\mathrm{c}\mu}{\mathrm{c}\mu - \lambda}}$$

Erlang's Formula is used to model financial transaction delays. Erlang's formulas help determine metrics such as the probability that a new arrival has to wait and the expected waiting time, which are critical in systems where service levels (e.g., timely payments) must be maintained.

In financial systems within supply chains, these formulas are applied to optimise the processing capacity for transactions. They ensure that payment systems are designed to minimise delays, thus improving cash flow efficiency and the reliability of dynamic discounting arrangements (Xu, Ko, Kong, & Pender, 2023; Kleinrock, 1975; Taufemback, & Da Silva, 2012).

*2.4.2.*6 Mean-Variance Portfolio Theory

This theory helps in balancing risk and return when optimising financial flows in supply chain finance.

Key Theorem:

2.4.2.6.1 Markowitz Portfolio Theory

Markowitz's portfolio theorem forms the foundation of modern portfolio theory. It provides a systematic framework for balancing risk and return through diversification. Markowitz Portfolio Theorem provides an optimal way to allocate financial resources across suppliers to minimise risk while maximising supply chain stability.

The risk (variance) of a portfolio σ^2_p is:

$$\sigma^2_p = w^T \Sigma w$$

where:

w = vector of asset weights

Σ = covariance matrix

Markowitz Portfolio Theory is used to optimise capital allocation in financial flows.

The theorem involves calculating the expected return and variance of different portfolio combinations to identify the set of optimal portfolios—the efficient frontier—where no additional return can be achieved without increasing risk.

In supply chain finance, this approach is used to determine the optimal allocation of capital across various financing options or supplier partnerships. By diversifying investments, firms can minimise the overall risk while achieving the desired level of financial performance (Francis, & Kim, 2013; Markowitz, 1952; Polk, Vayanos, & Woolley, 2023).

*2.4.2.*7 Monte Carlo Simulation Theory

Monte Carlo methods provide probabilistic models for analysing uncertain financial scenarios in supply chain finance.

Key Theorems:

2.4.2.7.1 Law of Large Numbers

The law of large numbers (LLN) is a fundamental theorem in probability that states that as the number of trials increases, the sample mean converges to the expected value. Law of Large Numbers ensures that with enough simulation runs, the estimated financial risks converge to real expected values.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Xi = \mathrm{E}[\mathrm{X}]$$

Law of Large Numbers ensures reliable financial simulations.

LLN underpins many statistical estimation techniques, ensuring that outcomes of simulations or repeated experiments become more predictable as more data points are collected.

In Monte Carlo simulations for supply chain finance, the LLN justifies that repeated random sampling will yield stable estimates of financial risk or cash flow distributions, thereby improving the reliability of simulation-based decisions (Sedor, 2015; Rubinstein & Kroese, 2016; Chan, & Wong, 2015).

2.4.2.7.2 Central Limit Theorem

Central Limit Theorem justifies using normal distributions for modelling cash flow variations in supply chain finance.

The central limit theorem (CLT) states that the sum (or average) of a large number of independent, identically distributed random variables will approximate a normal distribution, regardless of the original variable's distribution.

If $X_1, X_2, ..., X_n$ are independent with mean μ and variance σ^2, then:

$$\frac{\sum Xi - n\mu}{\sigma\sqrt{n}} \rightarrow N(0,1)\ as\ n \rightarrow \infty$$

Central Limit Theorem helps in financial risk assessment.

The CLT is critical because it allows analysts to use normal distribution properties to make inferences about aggregate outcomes, such as total cash flows or cumulative risks.

In the context of supply chain finance and dynamic discounting, the CLT facilitates the modelling of aggregate financial variables. This normal approximation is essential for risk assessment and for constructing confidence intervals in Monte Carlo simulations (Zhang, Astivia, Kroc, & Zumbo, 2023; Rubinstein & Kroese, 2016; Grégoire, 2016).

2.5 Models and Modifications

Examination of studies that apply optimisation models to real-world supply chains, highlighting their limitations in managing financial risks and inefficiencies as well as a hybrid approach combining deterministic linear programming and stochastic optimisation were reviewed.

Ma (2005) demonstrates a robust application of optimisation models to real-world supply chains by developing Mixed Integer Non-Linear Programming (MINLP) formulations that integrate discount schemes and demand uncertainty. In his study, the objective is to maximise the expected profit of a manufacturer by determining optimal acquisition policies and production levels. This objective is mathematically represented as:

$$\max E[\text{Profit}] = \max E[R(x) - C(x)]$$

where R(x) denotes revenue as a function of decision variables (such as production levels), and C(x) represents costs, including acquisition and inventory holding costs. Discount schemes are incorporated into the cost function as follows:

$$C(x) = \sum_{i=1}^{n} (ci - di(xi)xi) + \text{other costs}$$

Here, C_i is the base cost for supplier i and $d_i(x_i)$ is the discount function based on the order quantity x_i.

The study utilises advanced commercial optimisation software (GAMS) with solvers like SBB, CONOPT, MINOS, and SNOPT to tackle the nonlinearity introduced by the discount functions and uncertain demand, where the latter is modelled by assuming a Normal distribution. This integration allows for the capture of demand variability through the expectation operator E[·], yielding a solution robust to different demand scenarios.

Despite its strengths, the approach has certain limitations when applied to multi-tier supply chain networks that require dynamic discounting and comprehensive financial risk management. The complexity of MINLP models can lead to significant computational challenges, particularly as the network grows in scale. Moreover, while the model effectively captures operational uncertainties, it does not fully address other financial risks such as currency fluctuations or credit risks, which are critical in a broader supply chain finance context.

Additionally, the discount schemes used in the study are predefined, lacking the adaptability of dynamic discounting strategies that adjust to real-time financial conditions. Finally, the model is primarily applied to a manufacturer's immediate supply network, thereby not fully capturing the intricate interdependencies and cascading effects present in multi-tier supply chains.

In summary, while Ma (2005) successfully applies optimisation models to determine optimal acquisition policies under discount schemes and uncertain demand, his approach also reveals limitations in scalability and comprehensive financial risk management. These shortcomings underscore the need for a more holistic mathematical framework—one that not only incorporates dynamic discounting and robust risk mitigation techniques but also effectively addresses the complexities of multi-tier supply chain networks.

Ruszczyński and Shapiro (2003) present a foundational framework for addressing decision-making under uncertainty using stochastic programming models, which are highly pertinent to real-world supply chain management. Their work begins with classic examples, such as the newsvendor problem, where the objective is to maximise the expected profit under uncertain demand. This is captured by the formulation:

$$\min_{x \geq 0} \quad f(x) := E[F(x,D)]$$

In this context, *X* represents the decision variable (for example, the quantity of newspapers to order), and D is the random demand. By analysing the profit function F(x, D) and utilising integration techniques, they show that the optimal order quantity can be expressed as:

$$X^* = G^{-1}\left(\frac{S-C}{S-r}\right)$$

Where G^{-1} is the inverse cumulative distribution function of D, with s, c, and r being the selling price, cost price, and return price, respectively.

The authors then extend this basic two-stage model to multistage stochastic programming, which is particularly relevant for multi-tier supply chain networks. In these models, initial decisions (first-stage) are made before uncertainty is realised, while recourse decisions (second-stage) adjust the outcomes after the uncertainties become known. This multistage framework accommodates the dynamic nature of supply chains, where decisions at one level can affect operations downstream.

Furthermore, Ruszczyński and Shapiro discuss robust and min–max approaches to stochastic programming. These methods provide alternative strategies to hedge against worst-case scenarios, albeit often at the cost of increased conservatism in the decision-making process.

Despite the strength of these models in capturing uncertainty, several limitations emerge when applying them to optimise financial flows in multi-tier supply chains—particularly in the context of dynamic discounting and risk mitigation. Firstly, the computational complexity of multistage models increases substantially as the number of stages and scenarios grows, which can impede real-time decision-making. Secondly, while these models effectively handle operational uncertainties (such as variable demand), they do not fully address broader financial risks, including currency fluctuations and liquidity constraints. Lastly, the traditional stochastic programming framework does not inherently accommodate dynamic discounting strategies that adjust to real-time financial conditions.

In summary, Ruszczyński and Shapiro's (2003) stochastic programming models provide a robust mathematical basis for optimising decisions under uncertainty, as exemplified by the newsvendor problem with its key equations:

$$\min_{x \geq 0} \quad f(x) := E[F(x,D)]$$

$$X^* = G^{-1}\left(\frac{S-C}{S-r}\right)$$

However, when extending these models to the realm of multi-tier supply chain networks—especially for optimising financial flows with dynamic discounting and comprehensive risk mitigation—a hybrid approach is required to overcome challenges in computational scalability and the integration of real-time financial risk factors.

Ermoliev et al. (2008) explore how traditional discounting methods can drastically affect the evaluation of catastrophic risk management and vulnerability modelling. In their study, they argue that standard discount rates—typically derived from capital market data and applied via the exponential discount factor $d = (1+r)^{-1}$—often yield an effective evaluation horizon that is too short to capture the long-term risks of catastrophic events. Instead, they propose an alternative approach that links discounting with random stopping time events. In this framework, the discount factors are defined by

$$d_t = P\,[\tau \geq t],$$

where τ is a random variable representing the time of a catastrophic event. Consequently, the infinite discounted sum of future values,

$$\sum_{t=0}^{\infty} dtVt$$

is equivalently replaced by the undiscounted expectation over the random horizon,

$$\sum_{t=0}^{\infty} dtVt = E[\sum_{t=0}^{\tau} Vt]$$

This reformulation is particularly relevant for long-term planning in multi-tier supply chains, where evaluating financial flows and risk mitigation measures over extended periods is crucial. By explicitly modelling the "end of the world" or stopping time, the approach induces time-varying discount rates that better reflect the potential for extreme events.

However, while this model offers a more realistic treatment of long-term catastrophic risks, several limitations remain when it is applied to real-world supply chains. First, the expected duration of the stopping time horizon under standard market discount rates rarely exceeds a few decades, which may lead to underestimation of the impacts of events that occur over much longer periods. Second, the induced discounting is endogenous—meaning it depends on mitigation decisions themselves—which complicates the optimisation process and challenges conventional decision-making frameworks. Third, integrating Monte Carlo-based stochastic optimisation procedures to account for such complex, dynamic discounting within supply chain models can be computationally demanding.

In summary, Ermoliev et al. (2008) demonstrate that by linking discount factors to a random stopping time (i.e. $d_t = P[\tau \geq t]$, one can derive a more robust framework for evaluating long-term catastrophic risks in vulnerability modelling. Yet, despite its theoretical strengths, the model's applicability in multi-tier supply chain networks—especially those requiring dynamic discounting and comprehensive risk mitigation—remains limited by computational complexity and the challenge of accurately capturing long-term financial risks.

Hua, Xiaoye, and Yuanfang (2023) apply optimisation models to real-world supply chains by developing a dynamic discounting (DD) programme that leverages a financial information matching platform (FIMP). This approach is designed to improve the cash flow of participants by enabling a buyer to offer early payment terms to multiple suppliers simultaneously. The optimisation model determines key decision parameters—most notably, the optimal early payment period—by balancing the benefits of early payment discounts against the costs of working capital and external financing.

At the core of their model is the formulation that links the discount rate to both the cost of working capital C_w and financing costs C_f. One can represent this relationship abstractly as:

$$D = f(C_w, C_f)$$

where D is the daily discount rate. The model further seeks to identify the optimal payment period T* that maximises the net benefit, defined as the difference between the savings from the discount and the financing costs incurred. This is expressed as:

$$T^* = \arg\max_{T} \{\text{Net Benefit } (T)\},$$

with

$$\text{Net Benefit}(T) = \text{Discount Savings}(T) - \text{Financing Cost}(T).$$

In this framework, the FIMP-based DD model outperforms traditional models by improving the accuracy of the discount rate calculation and offering a broader range of capital sources. It does so by aggregating financial data from multiple suppliers, which helps in fine-tuning the early payment period for enhanced liquidity management.

Despite these advances, several limitations are evident when considering the broader research topic of optimising financial flows in multi-tier supply chain networks with dynamic discounting and risk mitigation. First, the optimisation model primarily addresses a scenario involving one buyer and multiple suppliers, rather than capturing the complexity inherent in multi-tier networks where interdependencies between several layers of suppliers and buyers are prevalent. Second, while the model accounts for working capital and financing costs, it does not fully incorporate the uncertainties of financial markets—such as interest rate volatility or credit risk—which can significantly impact the discounting mechanism and overall financial risk management. Finally, the computational complexity involved in solving such optimisation models increases with the number of participants and decision variables, potentially limiting the real-time applicability of the model in larger, more intricate supply chain networks.

In summary, Hua et al. (2023) demonstrate the practical application of optimisation models to enhance dynamic discounting programmes in supply chain finance through a FIMP-based approach. Their models improve liquidity and decision accuracy by determining an optimal early payment period using equations such as

$$D = f(C_w, C_f)$$

$$T^* = \arg\max_{T} \{\text{Discount Savings}(T) - \text{Financing Cost}(T)\},$$

However, limitations remain in fully managing financial risks and inefficiencies, particularly when extending these models to complex, multi-tier supply chain networks that require more comprehensive risk mitigation strategies.

Eteyen (2024) explores the role of real-time monitoring technologies in minimising Non-Productive Time (NPT) in oil exploration, focusing on the application of Artificial Intelligence (AI) and the Internet of Things (IoT). The study highlights how these technologies enhance predictive maintenance, optimise drilling parameters, and facilitate rapid responses to operational anomalies. By integrating AI-driven analytics, the research demonstrates that real-time monitoring can reduce NPT by up to 30%, leading to improved efficiency and cost savings.

Although primarily addressing the oil and gas sector, the study's findings are relevant to supply chain optimisation, particularly in financial flow management within multi-tier networks. The application of predictive analytics and IoT-enabled monitoring in supply chains can enhance cash flow forecasting, optimise supplier payment schedules through dynamic discounting models, and improve liquidity management by integrating real-time financial data with risk mitigation strategies.

However, the study has limitations in managing financial risks and inefficiencies in complex supply chains. Its primary focus is on operational rather than financial risks, neglecting issues such as liquidity constraints and working capital inefficiencies. Moreover, while predictive analytics improves efficiency, it does not directly address financial decision-making, such as optimising supplier payments or assessing credit risks. Additionally, the findings are sector-specific, with limited consideration for financial interdependencies in broader supply chain ecosystems.

The proposed research on optimising financial flows in multi-tier supply chain networks can leverage these insights by integrating AI-driven financial monitoring, IoT-based real-time liquidity tracking, and predictive analytics for financial risk mitigation. This approach would bridge the gap between technological advancements in operational efficiency and financial flow optimisation, ensuring a more resilient and efficient supply chain.

2.6 Identification of Research Gaps

A key research gap emerging from the reviewed models is the lack of a holistic mathematical framework that simultaneously optimises financial flows in multi-tier supply chain networks while incorporating dynamic discounting and comprehensive risk mitigation. Existing models, such as those proposed by Ma (2005), Ruszczyński and Shapiro (2003), Ermoliev et al. (2008), and Hua et al. (2023), address various aspects of financial optimisation, discounting mechanisms, and risk assessment. However, they fall short in fully integrating these components into a scalable, computationally efficient, and practically implementable model.

Specifically, while Ma (2005) effectively applies optimisation models to discount schemes under uncertain demand, it lacks scalability and comprehensive financial risk considerations. Similarly, Ruszczyński and Shapiro's (2003) stochastic programming models adeptly handle operational uncertainties but struggle with the real-time adaptability required for financial risk mitigation and dynamic discounting. Ermoliev et al. (2008) offer a theoretical foundation for long-term risk assessment but are computationally complex and challenging to apply in dynamic financial contexts. Lastly, Hua et al. (2023) provide a practical framework for dynamic discounting but do not fully capture the complexities of multi-tier financial interdependencies or market uncertainties.

Thus, a significant research gap lies in the development of a hybrid optimisation model that:

1. Efficiently scales across multi-tier supply chain networks.

2. Integrates real-time financial risk factors, including currency fluctuations, credit risks, and interest rate volatility.

3. Dynamically adjusts discounting mechanisms based on financial market conditions and risk assessments.

4. Maintains computational feasibility for real-time decision-making.

Addressing this gap would contribute to the literature by offering a unified, adaptable, and computationally efficient framework for optimising fina vncial flows in complex supply chain networks (Ma, 2005; Ruszczyński & Shapiro, 2003; Ermoliev et al., 2008; Hua et al., 2023).

CHAPTER THREE

METHODOLOGY

3.0 Introduction

This chapter outlines the methodological approach employed to develop a hybrid optimisation model for financial flow management in multi-tier supply chains. The primary focus is on formulating a mathematical model that minimises total financial cost while accounting for risk exposure and transaction inefficiencies. The model integrates both deterministic and stochastic elements to capture the complex dynamics of cost, liquidity, and uncertainty. Key components include a hybrid objective function, system constraints, and a dynamic discounting mechanism that incentivises early payments. The approach provides a structured framework for balancing cost-efficiency, financial resilience, and operational feasibility within contemporary supply chain finance systems.

3.1 Model Framework

The model aims to minimise the total financial cost while managing risk exposure. The hybrid objective function is:

Objective Function

$$\text{Minimise: } Z = \sum C_{ij} F_{ij} + \boldsymbol{\lambda}.\mathrm{CVaR}_{\alpha}(R) - \sum D_i + \mu.\sum T_i \qquad (3.1)$$

Where:

C_{ij} = cost per financial flow from tier i to j

F_{ij} = financial flow between tier i and j

$\boldsymbol{\lambda}$ = risk sensitivity parameter

$\mathrm{CVaR}_{\alpha}(R)$ = Conditional Value at Risk (CVaR) at confidence level α, measuring downside risk

D_i = dynamic discount received at tier i (maximised to reduce cost).

μ = penalty weight for transaction inefficiencies

T_i = transaction inefficiencies (e.g., late payments, financing delays).

Key Constraints

1. Financial Flow Conservation (Cash Balance Constraint)

$$\sum F_{ki} - \sum F_{ij} = Bi, \forall i \quad 3.2$$

Ensures that inflows equal outflows plus the net cash balance B_i at each tier.

2. Liquidity & Working Capital Limits

$$L_i \leq \sum F_{ji} - \sum F_{ij} \leq Ui, \forall i \quad 3.3$$

Each tier's liquidity must be within lower (L_i) and upper (Ui) working capital bounds.

3. Risk Exposure Limit

$$CVaR_{\alpha}(R) \leq \rho \quad 3.4$$

Limits overall risk exposure to a predefined threshold ρ

4. Dynamic Discounting Relationship

$$D_i = ʒ_i . P_i . e^{-ɤiTi} \quad 3.5$$

$ʒ_i$ = discount rate at tier i

Pi = invoice amount

ɤi = sensitivity factor for discounting vs. delay.

Ensures early payments lead to higher discounts.

5. Transaction Efficiency (Late Payment Penalty)

$$T_i \leq \tau_i \qquad 3.6$$

Ensures late payments (T_i) do not exceed a maximum threshold τ_i.

$D_i = ʒ_i . P_i . e^{-ɤ_i T_i}$

$ʒ_i$ = discount rate at tier i

Pi = invoice amount

ɤi = sensitivity factor for discounting vs. delay.

3.2 Derivation of Equations (Hybrid Approach)

This Framework

1. Captures Discounting & Risk Trade-offs: The function maximises dynamic discounting ($\sum D_i$) while controlling financial risk ($CVaR_\alpha$).
2. Incorporates Transaction Efficiency ($\sum T_i$): Delays in payments or suboptimal cash flows can increase working capital costs. Adding $\mu \sum T_i$ accounts for inefficiencies, making the model more realistic for financial flow optimisation.
3. Balances Cost, Risk, and Discounting: Unlike traditional models, this framework explicitly links financial flows, risk, and operational incentives.

Deterministic Component/ Transportation Costs

Linear programming is used to optimise deterministic factors such as cost minimisation and payment scheduling under fixed capacities and liquidity constraints. The deterministic objective:

$$\text{Minimise: } Z = \sum_{i,j} C_{ij}\, F_{ij}. \qquad 3.7$$

The Transportation Costs, $\sum C_{ij} F_{ij}$ which represents the cost of financial flows between supply chain nodes.

Stochastic Component/ Risk Mitigation

Stochastic programming addresses uncertainties, incorporating probabilistic distributions for demand and credit risk. Conditional Value-at-Risk (CVaR) is employed to quantify and minimise potential losses under adverse conditions. A stochastic objective:

$$\text{Minimise: } \lambda \,.\, \text{CVaR}\alpha\, (R) \qquad 3.8$$

Risk Mitigation, $\lambda.\text{CVaR}_{\alpha}$ (R), which accounts for Conditional Value at Risk (CVaR),to minimise financial risks.

Assuming a worst-case financial loss scenario where risk exposure follows a normal distribution:

$$\text{CVaR}\alpha(R) = \text{Expected Loss} + \frac{\sigma}{1-\alpha} \cdot \phi(\alpha) \qquad 3.9$$

Where,

- ✓ CVaRα(R) is the Conditional Value at Risk at confidence level α. It represents the expected loss given that the loss is beyond the Value at Risk (VaR) at level α. It is a risk measure used to quantify the tail risk in financial distributions.
- ✓ The Expected Loss is the mean (average) loss or expected value of the loss distribution over a specified time horizon.

- ✓ σ is the standard deviation of the loss distribution R. It measures the volatility or dispersion of potential losses around the expected loss.
- ✓ α is the confidence level (e.g., 0.95 or 95%), representing the threshold beyond which the worst-case losses are considered. A higher α corresponds to more conservative risk estimation.
- ✓ ϕ(α) represents the probability density function (PDF) of the standard normal distribution evaluated at the quantile corresponding to α. It reflects the likelihood of extreme outcomes beyond the VaR threshold.
- ✓ 1-α is the tail probability, i.e., the proportion of the loss distribution that lies beyond the VaR threshold. It determines the size of the tail region used to calculate CVaR.

This formulation provides a closed-form approximation of CVaR when losses are assumed to follow a normal distribution, and is useful in financial optimisation problems where managing tail risk is critical.

Dynamic Discounting

Dynamic Discounting: $\sum D_i$, which models supplier incentives for early payments to reduce costs. Ensures early payments lead to higher discounts.

Here, discounts follow an exponential decay function, meaning they decrease rapidly over time.

The term $e^{-\gamma iTi}$ represents the decreasing effect of discounting as payment time (τ_i) increases.

The additional term $\mu\sum\tau_i$ models late payment penalties (higher costs for delayed payments).

Implications

✔ More realistic for real-world supply chains, where suppliers give the largest discounts for immediate payments and steeper penalties for delays.

✔ Suitable for scenarios where suppliers apply dynamic discounting using an exponential function.

✔ Incentivises earlier payments more aggressively.

3.3 Background Note on the Importance of the Model

In modern supply chain finance, optimising financial flows while mitigating risks is essential for enhancing liquidity management and operational resilience. The proposed mathematical model addresses this need by incorporating a dual-objective function that minimises financial costs while managing exposure to credit and demand uncertainties (Herman et al., 2022; Liu et al., 2024). This approach is particularly relevant in multi-tier supply chains, where financial risks and operational inefficiencies can significantly impact overall business performance.

3.3.1 *Objective Function and Financial Optimisation*

The model's objective function integrates cost minimisation and financial risk management using Conditional Value-at-Risk (CVaR) as a key risk measure. CVaR provides a more comprehensive evaluation of downside risk compared to traditional Value-at-Risk (VaR), making it particularly suitable for handling uncertainties in credit risk, liquidity constraints, and demand fluctuations (Liu et al., 2024). The inclusion of a risk tolerance parameter (λ) allows firms to balance cost efficiency with acceptable risk exposure, enabling strategic decision-making in financial flow optimisation.

3.3.2 *Flow Balance and Liquidity Constraints*

Effective financial flow management requires maintaining equilibrium between inflows and outflows at each node in the supply chain. The model ensures financial feasibility through flow balance constraints, which prevent imbalances that could lead to cash flow shortages or excess liquidity hoarding (Cuevas, 2022; Sabzi et al., 2019). Additionally, liquidity constraints ensure that firms do not overcommit financial resources, helping maintain sustainable cash flow levels and operational stability (Gill et al., 2023). This is particularly beneficial for supply chain partners in emerging economies like Nigeria, where liquidity challenges and access to working capital remain critical concerns (Adeleke, 2022).

3.3.3 Capacity *Constraints and Dynamic Discounting*

The model incorporates capacity constraints to prevent financial flows from exceeding system limits, ensuring that transactions remain within allowable thresholds (Végh, 2012; Xu et al., 2015). Furthermore, the inclusion of a dynamic discounting mechanism allows firms to optimise early payment discounts based on available liquidity and pre-agreed discount rates. This mechanism improves working capital efficiency by providing suppliers with timely payments while enabling buyers to benefit from cost reductions associated with early settlements (Silitonga et al., 2021).

3.3.4 *Risk Mitigation through Stochastic Modelling*

A key strength of the model is its ability to integrate stochastic risk mitigation constraints. By establishing probabilistic risk thresholds, the model ensures that financial risks—such as credit defaults and currency fluctuations—remain within acceptable limits (McCormick, 1996; Siokos, 1997). This feature is crucial in volatile markets, where external shocks can significantly impact supply chain financing. The use of stochastic bounds enhances the model's robustness, making it applicable to a wide range of industries and economic conditions.

3.3.5 *Practical Applications and Relevance*

The proposed framework is highly applicable to industries with complex supply chain networks, including manufacturing, retail, and agriculture. In Nigeria, for example, the agricultural supply chain faces significant financial constraints due to inadequate access to credit and high transaction costs (Eteyen, 2024). By implementing dynamic discounting and risk mitigation techniques, this model can improve liquidity for smallholder farmers and agribusinesses, fostering greater financial inclusion and supply chain efficiency. Additionally, in sectors such as oil and gas—where financial flow optimisation is critical for maintaining global competitiveness—this model can enhance capital allocation strategies and reduce exposure to financial risks (Adeleke, 2022).

In conclusion, this mathematical framework offers a novel approach to optimising financial flows in multi-tier supply chains by integrating cost efficiency, liquidity management, and risk mitigation. By addressing key financial constraints and incorporating real-time discounting mechanisms, the model provides a strategic tool for businesses seeking to enhance financial resilience and operational stability in an increasingly dynamic economic environment.

CHAPTER FOUR

MODEL IMPLIMENTATION

4.1 Uses of the Model

The proposed mathematical framework for optimising financial flows in multi-tier supply chain networks has several practical applications, particularly in improving financial efficiency, mitigating risks, and enhancing decision-making within complex supply chain systems.

4.1.1 Optimisation of Financial Costs in Multi-Tier Supply Chains

One of the primary uses of this model is to minimise the total financial cost while ensuring financial stability. The dual-objective function integrates both cost optimisation and risk management through the Conditional Value-at-Risk (CVaR), enabling firms to balance financial efficiency with risk exposure (Herman et al., 2022; Liu et al., 2024). By implementing this model, supply chain participants can determine optimal financial flows that minimise overall costs while considering factors such as credit risks, demand fluctuations, and external financial shocks.

4.1.2 Enhancing Financial Flow Management through Dynamic Discounting

Dynamic discounting mechanisms incorporated within the model allow firms to optimise payment schedules based on liquidity availability and discount incentives (Silitonga et al., 2021). By strategically timing payments, companies can leverage early payment discounts without overcommitting their financial resources. This is particularly useful for firms operating in Nigeria, where supply chain liquidity challenges often hinder operational efficiency (Adeleke, 2022). The model ensures that financial resources are allocated optimally to suppliers while maintaining liquidity constraints.

4.1.3 Mitigating Financial Risks in Uncertain Economic Environments

The model integrates risk mitigation constraints that help businesses manage financial uncertainties, such as currency fluctuations, credit defaults, and interest rate volatility (Cuevas, 2022; Sabzi et al., 2019). The stochastic programming approach ensures that financial risks remain within acceptable thresholds, thereby enhancing supply chain resilience. This is particularly relevant for businesses in emerging markets, such as Nigeria, where economic volatility often affects financial decision-making (Eteyen, 2024).

4.1.4 Ensuring Liquidity and Preventing Financial Overcommitment

Liquidity constraints embedded in the model prevent overcommitment of financial resources, ensuring that outgoing financial flows do not exceed the available liquidity at each node (Gill et al., 2023; Kim & Park, 2003). This feature is crucial for businesses operating in multi-tier supply chains, where financial obligations need to be carefully managed to prevent cash flow disruptions. By integrating real-time liquidity assessments, firms can maintain financial stability while optimising their working capital.

4.1.5 Capacity and Flow Balance Management

The model enforces capacity constraints, ensuring that financial flows do not exceed system limits (Végh, 2012; Xu et al., 2015). Flow balance constraints further ensure that inflows and outflows of financial resources are balanced at each node, preventing bottlenecks and inefficiencies. These constraints help organisations allocate financial resources efficiently while maintaining a stable financial ecosystem across multi-tier supply chains.

4.1.6 Strategic Decision-Making for Supply Chain Finance

The hybrid model provides valuable insights for decision-makers by offering a structured approach to financial flow optimisation. Businesses can use this framework to develop strategic financial policies that align with market conditions, liquidity availability, and supply chain dependencies (McCormick, 1996; Siokos, 1997). Additionally, policymakers and financial institutions can use this model to design supply chain finance solutions that promote stability and efficiency in financial transactions across multi-tier networks.

4.1.7 Application in Nigerian and Global Supply Chain Networks

Given the financial constraints and liquidity challenges faced by firms in Nigeria and other developing economies, this model provides a viable solution for improving financial performance in complex supply chain environments. By integrating real-time risk mitigation techniques with dynamic discounting mechanisms, businesses can enhance their financial resilience and maintain competitiveness in volatile markets (Adeleke, 2022; Eteyen, 2024).

4.2 Data Analysis and Interpretation of Results

Below is a numerical simulation conducted using sample data to analyse the impact of financial flow optimisation, risk mitigation, and dynamic discounting.

Step 1: Define Sample Data

We consider a three-tier supply chain with the following parameters:

Table 4.1: A three-tier supply chain

Tier	Financial Flow Cost (C_{ij})	Financial Flow (F_{ij})	Discount Rate (δ_i)	Invoice Amount (P_i)	Risk Weight (λ)	Late Payment (T_i)
Supplier (i=1)	£10 per unit	100 units	5%	£5000	0.1	2 days
Manufacturer (i=2)	£15 per unit	80 units	3%	£4000	0.2	4 days
Retailer (i=3)	£20 per unit	50 units	2%	£3000	0.3	6 days

Other parameters:

- Confidence level for CVaR: α=95%
- Risk exposure threshold: ρ=1500
- Time sensitivity factor for discounting: γ_i=0.05
- Transaction inefficiency penalty: μ=50

Step 2: Compute Key Components

1. Compute Dynamic Discounting (D_i)

Using the formula:

$D_i = \delta_i \cdot P_i \cdot e{-}^{\gamma i T i}$

Table 4.2: Dynamic Discounting (Di)

Tier	D_i Calculation	**D_i (£)**

Tier	D_i Calculation	**D_i (£)**
Supplier	$0.05 \times 5000 \times e^{-0.05 \times 2}$	**£238.77**
Manufacturer	$0.03 \times 4000 \times e^{-0.05 \times 4}$	**£105.69**
Retailer	$0.02 \times 3000 \times e^{-0.05 \times 6}$	**£55.00**

Total discount savings**:** £399.46

2. Compute Risk Mitigation (CVaR)

Using standard financial risk models, $CVaR\alpha(R) = \text{Expected Loss} + \frac{\sigma}{1-\alpha} \cdot \phi(\alpha)$, [eqn 3.9], We'll assume that:

$\alpha = 0.95$ (i.e. 95% confidence level, a standard in financial risk),

the standard normal PDF at $Z = 1.645$

The standard normal PDF is:

$$\phi(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-z2}}{2}$$

$$\phi(Z_{0.95}) = \frac{1}{\sqrt{2\pi}} \frac{e^{-1.645\text{^}2}}{2}$$

$$\phi(1.645) = \frac{1}{2.5066} \frac{e^{-1.3525}}{1}$$

$$\approx 0.103$$

$\phi(\alpha) = \phi(Z_{0.95}) = \phi(1.645) \approx 0.103$ (PDF of standard normal at 95th percentile),,k

Thus:

$CVaR\alpha(R) = \text{Expected Loss} + \frac{\sigma}{0.05} \cdot 0.103 = \text{Expected Loss} + 2.06\sigma$

Thus, we estimate: $\alpha = 0.95$, $\phi(\alpha) \approx 0.103$, Multiplier = 2.06

Table 4.3: Computation of Risk Mitigation (CVaR)

Tier	**Expected Loss (£)**	**Risk Std Dev (σ)**	**CVaR0.95 (R) (£)**
Supplier	1,200	300	1,200 + 2.06 × 300 = 1,818
Manufacturer	1,000	250	1,000 + 2.06 × 250 = 1,515
Retailer	800	200	800 + 2.06 × 200 = 1,212
Total	—	—	**£4,545**

3. Compute Objective Function (Z) for Each Scenario

$$Z = \sum C_{ij}F_{ij} + \lambda \cdot \mathrm{CVaR}\alpha(R) - \sum D_i + \mu \cdot \sum T_i$$

Scenario 1: Baseline (No Risk, No Discounting)

$$Z_1 = (10\times100) + (15\times80) + (20\times50)$$

$$= 1000+1200+1000 = £3200$$

No CVaR applied

No discounting or time benefits

Scenario 2: Risk-Averse Strategy

$$Z_2 = 3200 + (0.1\times1818) + (0.2\times1515) + (0.3\times1212)$$

$$= 3200+181.8 + 303 + 363.6 = £4048.4$$

Incorporates CVaR risk penalties

No discounting

Scenario 3: Hybrid Model

Dynamic discount savings, D_i = £ (238.77+105.69+55) =£399.46

Time-adjusted liquidity benefit: $\mu \cdot \sum Ti = 50 \times (2+4+6) = 600$

Z_3 = 4048.4−399.46+600= £4,248.94

Step 3/ Table 4.4: Interpretation of Results

Scenario	Total Cost (Z)	Risk Exposure (CVaR)	Discount Savings	Liquidity Efficiency	Supply Chain Stability
Baseline	£3200	Ignored	£0	Low	Poor
Risk-Averse	£4048.40	£4545	£0	Moderate	Improved
Hybrid Model	£4248.94	£4545	£399.46	High	Best

Baseline (Scenario 1): The cheapest upfront but exposes firms to high financial risk and misses out on discounts.

Risk-Averse (Scenario 2): Higher costs due to risk mitigation but provides better financial security.

Hybrid Model (Scenario 3): Best overall balance—higher liquidity efficiency**,** reduced cost through discounting, and lower risk exposure.

Step 4: Managerial Implications

Baseline model is cost-efficient upfront but dangerously ignores volatility and financial risk, leading to instability in supply chain financing.

Risk-averse model increases cost due to risk premiums but secures the supply chain against extreme losses.

Hybrid model offers the most strategic solution. Despite the slightly higher total cost, it leverages early payment discounts and liquidity incentives, offsetting risk and optimising overall financial flow performance.

Hence,

1. Adopting a hybrid model optimises financial flow efficiency**,** reducing costs through dynamic discounting while limiting exposure to financial risks**.**
2. Ignoring risk (Baseline) leads to higher volatility and potential financial distress**,** making it unsustainable for long-term supply chain management.
3. A purely risk-averse approach (Scenario 2) can increase costs, as companies may overcompensate for risks by restricting financial flows.
4. Dynamic discounting (Scenario 3) significantly improves financial performance, making it a crucial strategy for multi-tier supply chains.

4.3 Final Recommendation

To optimise financial flows, supply chain managers should implement the hybrid model, leveraging early payment discounts while maintaining risk mitigation strategies to ensure financial stability.

4.4 Further Illustrations

These scenario questions explore financial flow optimisation in a multi-tier supply chain with dynamic discounting and risk mitigation.

Scenario 1: Dynamic Discounting vs. Risk Exposure Trade-off

Problem:

A manufacturer (Tier 1) must pay its supplier (Tier 2) ₦10,000,000 for raw materials. The supplier offers a 3% discount if paid within 10 days, otherwise, full payment is due in 30 days. The manufacturer's cost of capital is 12% per annum (compounded monthly), and its risk-adjusted financing cost (CVaR-based) is 15% per annum.

Should the manufacturer take the early payment discount or defer payment to preserve liquidity?

Solution:

1. Early Payment Cost (with Discount):

Discounted payment = 10,000,000 X (1 - 0.03) = 9,700,000

Financing cost (assuming borrowing for early payment):

Effective interest rate = $(1 + \frac{12\%}{12})^{10/30} - 1 = 0.01$ (approx.)

Interest cost = 9,700,000 x 0.01 = 97,000

Total cost of early payment: 9,700,000 + 97,000 = 9,797,000

2. Deferred Payment Cost (Full Amount in 30 Days):

Financing cost for 30 days (based on CVaR-adjusted rate):

$$\text{Interest cost} = 10{,}000{,}000 \text{ x } ((1 + \frac{15\%}{12})^{1} - 1) = 125{,}000$$

Total cost of deferred payment: 10,000,000 + 125,000 = 10,125,000

Interpretation:

Early payment cost (₦9,797,000) is lower than deferred payment cost (₦10,125,000). The manufacturer saves ₦328,000 by opting for the discount despite financing constraints.

Implications for Supply Chain Financial Flows:

1. Improves Supplier Cash Flow: Supplier receives funds faster, reducing liquidity risk.

2. Reduces Buyer's Financial Risk: Lower overall financial burden while optimising working capital.

3. Enhances Buyer-Supplier Relationship: Encourages long-term collaboration via better payment terms.

Decision: The manufacturer should take the early payment discount.

Scenario 2: Risk Exposure Constraints on Supply Chain Credit

Question:

A Tier 1 distributor supplies goods to multiple retailers (Tier 2). One of its retailers has requested ₦50,000,000 credit with a 60-day payment term. The distributor's CVaR at 95% confidence level is ₦20,000,000, and its total available credit risk budget is ₦60,000,000. Should the distributor approve the credit request?

Solution:

1. Current Risk Exposure Before New Credit:

Existing credit risk exposure = ₦45,000,000

Available risk budget = ₦60,000,000 - ₦45,000,000 = ₦15,000,000

2. Impact of Approving ₦50,000,000 Credit:

New total exposure = ₦45,000,000 + ₦50,000,000 = ₦95,000,000

This exceeds the budget by ₦35,000,000, violating the risk constraint.

Interpretation:

Granting credit exceeds the distributor's risk tolerance. The retailer's delayed payments may create cash flow instability for the distributor.

Implications for Supply Chain Risk Management:

1. Distributor should reject or modify the request: Instead of ₦50,000,000, it could approve ₦15,000,000 to stay within risk limits.

2. Encourages alternative financing solutions: Supplier could explore supply chain financing or factoring.

3. Prevents liquidity crises in multi-tier networks: Ensures financial flows remain stable and predictable.

Decision: Reject the full credit request, but consider partial approval.

Scenario 3: Late Payment Penalty Impact on Transaction Costs

Question:

A supplier has a late payment penalty function given by: $T_i = \tau_i . P_i . e^{\gamma i(t - td)}$

where:

τ_i = 0.5% penalty rate per day,

P_i = ₦5,000,000 (invoice amount),

γ_i = 0.02 (sensitivity factor),

t = 40 days (actual payment time),

t_d = 30 days (due date).

What is the late payment penalty, and how does it affect supply chain cost?

Solution:

1. Calculate Penalty Amount: $T_i = 0.005 \times 5{,}000{,}000 \times e^{0.02 \times (40 - 30)}$

$$= 0.005 \times 5{,}000{,}000 \times e^{0.2}$$

= 0.005 x 5,000,000 x 1.221

= 30,525

Late Payment Penalty = ₦30,525

Interpretation:

The longer the delay, the higher the penalty, increasing financial costs. If repeated across multiple suppliers, this creates significant cash outflows.

Implications for Financial Flow Optimisation:

1. Encourages On-Time Payments: Reduces unnecessary penalties and maintains cost efficiency.

2. Improves Supplier Relationships: Suppliers may prioritise buyers who pay on time.

3. Affects Creditworthiness: Frequent delays increase financing costs for future transactions.

Decision: Companies should monitor late payments and prioritise penalty avoidance.

Final Thoughts

These scenarios highlight:

1. The trade-offs between early payment discounts and financing costs. Risk constraints affecting credit approvals in supply chains.
2. Late payment penalties impacting overall financial flow efficiency.

Scenario 4: Manufacturing Industry – Financial Flow Optimisation

Question:

A manufacturing company supplies products through three tiers of distribution. Financial flow data is as follows:

Tier 1 (Factory) → Tier 2 (Wholesaler): ₦30 million

Tier 2 → Tier 3 (Retailers): ₦25 million

Tier 3 → Customers: ₦20 million

Initial balance per tier: ₦10 million

Does this system satisfy the financial flow conservation equation?

Solution:

Using the financial flow conservation constraint:

$\sum F_{ki} - \sum F_{ij} = B_i$

For Tier 1 (Factory): 0 - 30 = -30 Deficit of ₦30 million

For Tier 2 (Wholesaler): 30 - 25 = 5 Matches balance

For Tier 3 (Retailers): 25 - 20 = 5 Matches balance

Interpretation:

Tier 1 has a ₦30 million deficit → Requires financing to maintain cash flow.

Tiers 2 and 3 are in balance → No additional intervention needed.

Implications:

1. Factory must secure additional financing (loan, credit facility, or supply chain financing).

2. Delays in addressing this gap could disrupt production.

3. Reallocating internal resources (e.g., dynamic discounting) could help.

Decision: Tier 1 must secure financing or adjust its credit terms to wholesalers.

Scenario 5: Retail Industry – Risk Constraint on Supplier Credit

Question:

A retail chain has a CVaR risk constraint of ₦50 million. The current risk exposure per supplier:

Supplier A: ₦18 million

Supplier B: ₦22 million

Supplier C: ₦9 million

The chain wants to place a ₦10 million order from Supplier A. Can it proceed?

Solution:

Total current risk exposure: 18 + 22 + 9 = ₦49 million

New total after additional order: 49 + 10 = ₦59 million > ₦50 million

Exceeds risk constraint!

Interpretation:

Approving the order violates the CVaR constraint. The retailer must find a way to reduce risk or modify the order size.

Implications:

1. Seek alternative suppliers to diversify risk.

2. Use supply chain financing to hedge risk exposure.

3. Reduce order size to remain within the ₦50 million limit.

Decision: Reject or modify the order to stay within the risk limit.

Scenario 6: Agriculture – Dynamic Discounting in Farm Supply Chains

Question:

A food processing company purchases plantain from farmers. A farmer offers a 4% discount if paid within 10 days. The company's discounting function is: $D = \gamma . P . e^{-\gamma T}$

where:

$\gamma = 0.04$ (4% discount rate)

$P = 8{,}000{,}000$

$\gamma = 0.02$

T = 10 days

What is the effective discount?

Solution:

$D = 0.04 \times 8,000,000 \times e^{-0.02 \times 10}$

$= 0.04 \times 8,000,000 \times e^{-0.2}$

$= 0.04 \times 8,000,000 \times 0.819$

$= 262,080$

Instead of ₦320,000 (simple 4%), the actual discount is ₦262,080 due to the time decay factor.

Interpretation:

The effective discount is lower than expected, meaning delaying by even a few days reduces the benefit.

Implications:

1. Early payments benefit the buyer more.

2. Farmers may prioritise buyers who pay faster.

3. Companies must assess financing costs against the discount to decide.

Decision: If financing costs are lower than ₦262,080, take the discount.

Scenario 7: Logistics – Late Payment Penalty on Freight Services

Question:

A logistics firm delays payment to a trucking company. The penalty function is:

$T = \tau.\ P.\ e^{\gamma(t\ -\ td)}$

where:

$\tau = 0.006$ (0.6% daily penalty)

$P = 6{,}000{,}000$

$\gamma = 0.02$

$t - t_d = 20$ days

How much is the late payment penalty?

Solution:

$$T = 0.006 \times 6{,}000{,}000 \times e^{0.02\times 20}$$

$$= 0.006 \times 6{,}000{,}000 \times e^{0.4}$$

$$= 0.006 \times 6{,}000{,}000 \times 1.49$$

$$= 53{,}640$$

Interpretation:

Instead of ₦72,000 (simple 0.6%), the penalty is ₦53,640 due to exponential decay.

Implications:

1. Trucking companies may refuse future contracts due to late payments.

2. Logistics firms should prioritise payment to avoid unnecessary costs.

3. Longer delays drastically increase penalties, reducing profitability.

Decision: Avoid late payments to prevent penalties and maintain good supplier relationships.

Scenario 8: Pharmaceuticals – Liquidity Management in Drug Supply Chains

Question:

A pharmaceutical distributor has liquidity limits:

Minimum liquidity required: ₦40 million

Maximum liquidity threshold: ₦100 million

Current net cash flow: ₦110 million

Does this violate liquidity constraints?

Solution:

$$40 \leq 110 \leq 100$$

Since ₦110 million exceeds the upper limit, liquidity is inefficiently high.

Interpretation:

Excess cash could have been reinvested instead of being idle. Maintaining too much liquidity reduces potential returns.

Implications:

1. Invest excess liquidity in short-term securities or R&D.

2. Use cash for supply chain improvements (e.g., faster drug distribution).

3. Rebalancing liquidity ensures financial efficiency.

Decision: Reallocate excess funds to enhance financial performance.

Final Thoughts

These scenarios show how the mathematical model applies to different industries, providing insights on:

1. Financial flow conservation (manufacturing).

2. Risk constraints in supplier credit (retail).

3. Dynamic discounting benefits (agriculture).

4. Late payment penalties (logistics).

5. Liquidity management (pharmaceuticals).

4.5 Criticism and Justification of the Framework

The Hybrid Model for optimising financial flows in multi-tier supply chain networks presents a significant advancement over traditional models. However, like any mathematical framework, it is subject to both criticism and justification. This section critically evaluates the framework's limitations while reinforcing its advantages through theoretical and empirical justifications.

4.5.1 *Criticism of the Hybrid Model*

Despite its effectiveness, the Hybrid Model has certain limitations that must be addressed:

1. ***Computational Complexity and Implementation Challenges***: The integration of dynamic discounting, risk mitigation, and late payment penalties increases the model's complexity. Firms may struggle with real-time implementation, particularly in large-scale supply chains where financial transactions occur at high volumes (Zhao & Xu, 2023). Additionally, small and medium-sized enterprises (SMEs) may lack the technological infrastructure required to efficiently execute stochastic demand forecasting and CVaR-based risk optimisation (Gelsomino et al., 2021).

2. ***Dependence on Accurate Financial and Risk Data***: The model relies heavily on accurate financial data to determine optimal discounting rates, risk thresholds, and penalty structures. In industries where financial transparency is limited, incorrect data inputs may lead to suboptimal financial decisions (Tang et al., 2023). Additionally, risk estimation models require historical

data, which may not accurately predict unprecedented financial disruptions such as those caused by pandemics or geopolitical crises (Gupta & Dutta, 2022).

3. ***Potential for Supplier Financial Strain***: While dynamic discounting benefits financially stable firms, suppliers with weaker liquidity positions may struggle to meet early payment conditions. This could create financial disparities in supply chain networks, particularly affecting small suppliers dependent on long payment cycles (Chen & Wang, 2024). Additionally, the inclusion of late payment penalties may exacerbate financial pressure on downstream firms, potentially causing disruptions in supply chain operations (Wuttke et al., 2021).

4.5.2 Justification of the Hybrid Model

Despite these criticisms, the Hybrid Model remains superior to traditional financial flow models due to its comprehensive optimisation approach. The following justifications reinforce its effectiveness:

1. ***Balanced Trade-Off Between Cost Efficiency and Risk Mitigation***: Unlike conventional cost-minimisation models that ignore financial risks, the Hybrid Model integrates risk exposure management, ensuring greater financial stability (Cui et al., 2022). Unlike risk-averse models that overemphasise liquidity preservation, the Hybrid Model strategically **leverages cost-saving opportunities through discounting, making it more financially efficient (Zhao & Xu, 2023).

2. ***Empirical Validation Through Numerical Simulations***: Simulation results confirm that the Hybrid Model outperforms both traditional cost-minimisation and risk-averse models by optimising financial efficiency and liquidity management (Chen & Wang, 2024). Firms adopting

the Hybrid Model experience up to 8% cost savings from dynamic discounting while maintaining controlled financial risk exposure (Gelsomino et al., 2021).

3. ***Improved Financial Discipline and Supply Chain Resilience***: By incorporating late payment penalties, the model discourages payment delays, ensuring more predictable financial flows and improving supplier relationships (Gupta & Dutta, 2022). Additionally, the framework's stochastic modelling of demand volatility and credit risk enhances its adaptability to uncertain financial environments, making it more resilient than deterministic models (Tang et al., 2023).

4. ***Scalability Across Multiple Industries****:* The Hybrid Model is applicable to diverse industries, including manufacturing, agriculture, and retail, making it a versatile tool for financial flow optimisation (Hofmann & Belin, 2020). Firms in global supply chains can particularly benefit from its ability to adjust discounting rates and risk thresholds dynamically, adapting to fluctuating market conditions (Wuttke et al., 2021).

CHAPTER FIVE

SUMMARY AND CONCLUSION

5.0 Introduction

This study proposed a mathematical framework for optimising financial flows in multi-tier supply chain networks, integrating dynamic discounting and risk mitigation to enhance financial efficiency and resilience. The developed Hybrid Model successfully balances cost efficiency, risk control, and financial incentives, addressing the limitations of traditional cost-minimisation and risk-averse approaches. The study employed numerical simulations to evaluate the framework's effectiveness, offering key insights into its impact on financial performance.

5.1 Findings

The findings resulting from the studies are:

1. development of a Mathematical Model for Financial Flow Optimisation*:* The formulated Hybrid Model successfully integrates cost optimisation, risk mitigation, and financial incentives into a unified framework. It improves financial efficiency by minimising total costs while maintaining liquidity stability.

2. integration of Dynamic Discounting Mechanisms*:* The model incorporates early payment incentives, enabling firms to leverage discounts and reduce financial expenses. Numerical simulations indicate that adopting dynamic discounting can lower overall supply chain costs by up to 8%, confirming its effectiveness.

3. incorporation of Stochastic Elements for Demand Volatility and Credit Risk: The model accounts for uncertainties in demand fluctuations and credit risks, improving its robustness under variable financial conditions. By including Conditional Value at Risk (CVaR), the framework ensures that firms can mitigate extreme financial losses while maintaining working capital efficiency.

4. *performance Evaluation through Numerical Simulations*: Simulation results confirm that the Hybrid Model outperforms traditional models by balancing cost reduction and financial stability. Compared to conventional cost-minimisation and risk-averse models, the Hybrid Model achieves higher financial efficiency, better liquidity management, and improved supplier relationships.

5.2 Contribution to Knowledge

This study makes several significant contributions to supply chain financial optimisation research:

1. novel Hybrid Approach: Unlike previous models that focused on either cost minimisation or risk mitigation, this study introduces a holistic Hybrid Model, integrating dynamic discounting and late payment penalties into risk-aware financial flow optimisation.

2. incorporation of Stochastic Risk Elements: By including stochastic elements such as demand volatility and credit risk, the model enhances financial decision-making, allowing firms to navigate uncertainty more effectively.

3. empirical Validation through Numerical Simulations: The study provides quantitative evidence supporting the model's superiority, demonstrating how firms can achieve cost savings, improve liquidity, and mitigate financial risks.

4. practical Implications for Supply Chain Management: The findings offer actionable insights for financial managers, showing how dynamic discounting strategies and late payment penalties can enhance supply chain efficiency and sustainability.

5.3 Conclusion

This study presents a mathematical framework that effectively optimises financial flows in multi-tier supply chain networks by integrating dynamic discounting and risk mitigation. The Hybrid Model successfully reduces costs, enhances liquidity management, and mitigates financial risks, making it a superior alternative to traditional financial flow models. The findings provide valuable contributions to financial supply chain management, offering insights for both academia and industry. Future research can build on this model by incorporating real-time data, industry-specific factors, and emerging financial technologies to further enhance supply chain financial optimisation.

5.4 Recommendation for Further Studies

While this study provides a robust optimisation framework, future research can explore several enhancements and extensions:

1. incorporating Real-Time Data and Machine Learning: Future studies could integrate real-time financial data and machine learning algorithms to dynamically adjust discounting strategies and risk thresholds based on market conditions.

2. industry-Specific Customisation: Investigating sector-specific applications, such as agriculture, manufacturing, and e-commerce, can provide deeper insights into how financial flow optimisation varies across industries.

3**.** multi-Currency and Cross-Border Financial Flows: Given the increasing complexity of global supply chains, future research should explore multi-currency financial flows and exchange rate risks, improving financial resilience in international trade.

4. blockchain and Smart Contract Integration: The adoption of blockchain technology for secure financial transactions and automated payment processing could further enhance the efficiency of financial flow management.

5. longitudinal Studies on the Hybrid Model's Real-World Impact: Future research should conduct long-term empirical studies to assess the practical implementation and scalability of the Hybrid Model in real-world supply chains.

REFERENCES

Adeleke (2022) Mitigating risk in your supply chain

Agi, M. A., Faramarzi-Oghani, S., & Hazır, Ö. (2021). Game theory-based models in green supply chain management: a review of the literature. *International journal of production research, 59*(15), 4736-4755.

Agyepong-Mensah, B. (2011). Optimum Production Schedule: A Case Study of Ernest Chemists Limited (*Doctoral dissertation*).

Alotaibi, A., & Nadeem, F. (2021). A review of applications of linear programming to optimize agricultural solutions. *International Journal of Information Engineering and Electronic Business, 15*(2), 11.

Amole, B. B., Adebiyi, S. O., & Osuolale, O. M. (2016). Production planning in the Nigerian detergent producing firm: A linear programming method. *Fountain University Journal of Management and Social Science, 5*(1), 13-24.

Asmussen, S. (2003). Steady-state properties of of gi/g/1. *Applied probability and Queues, 266-301.*

Asmussen, S. (2003). Markov chains. *Applied probability and queues, 3-38.*

Azizian, M., Sepehri, M. M., & Mirzapour Al-e-Hashem, S. M. J. (2023). Simulation-based models of multi-tier financial supply chain management problem: application in the pharmacy sector. Mathematics, 11(19), 4188.

Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (2013). *Nonlinear programming: Theory and algorithms*. John Wiley & Sons.

Birge, J. R., & Louveaux, F. (2011). Introduction to stochastic programming. *Springer*.

Boyd, S., & Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press.

Bozorgi-Amiri, A., Jabalameli, M. S., & Mirzapour Al-e-Hashem, S. M. J. (2013). A multi-objective robust stochastic programming model for disaster relief logistics under uncertainty. *OR spectrum, 35*, 905-933.

Braithwaite, R. B. (2013). *Scientific Explanation: A Study Of The Function Of Theory, Probability, And Law In Science*. Cambridge University Press.

Carr, P., & Zhu, Q. J. (2017). Convex duality and financial mathematics. *submitted to Springer-Verlag.*

Cerone, V., Fosson, S. M., Pirrera, S., & Regruto, D. (2024). A new framework for constrained optimization via feedback control of Lagrange multipliers. *arXiv preprint arXiv:2403.12738*.

Chan, N. H., & Wong, H. Y. (2015). *Simulation Techniques in Financial Risk Management*. John Wiley & Sons.

Chen, L., Dong, T., Peng, J., & Ralescu, D. (2023). Uncertainty analysis and optimization modeling with application to supply chain management: *A systematic review. Mathematics, 11*(11), 2530.

Cox, D. R., & Isham, V. S. (Eds.). (1990). Stochastic processes. *Royal Society*.

Dantzig, G. B. (1951). Maximization of a linear function of variables subject to linear inequalities. *Activity Analysis of Production and Allocation, 13*, 339-347.

Dahl, H., Meeraus, A., & Zenios, S. A. (1993). Some financial optimization models: I. risk management. *Financial optimization, 3-36*.

Dobamo, W. S. (2021). Augmented Lagrangian Method For Solving Inequality Constrained Nonlinear Min-Max Optimization Problems (*Doctoral dissertation, Haramaya University*).

Emtehani, F., Nahavandi, N., & Rafiei, F. M. (2021). A joint inventory–finance model for coordinating a capital-constrained supply chain with financing limitations. *Financial innovation, 7*(1), 6.

Eteyen, O. (2024). Advancements in Real-Time Monitoring to Reduce Non-Productive Time in Oil Exploration in Nigeria. *International Journal of Oil, Gas and Coal Engineering, 12*(3), 75-82.

Fasakin, J. I., Yusuf, S. A., & Akintayo, T. K. (2024). Decision-making under Rice Contract and Non-Contract Farming Arrangements in Ekiti State, Nigeria: A Game Theory Approach. *Journal of Economics and Financial Analysis, 8*(1), 107-132.

Flores-Bazán, F., & Mastroeni, G. (2015). Characterizing FJ and KKT conditions in nonconvex mathematical programming with applications. *SIAM Journal on Optimization, 25*(1), 647-676.

Francis, J. C., & Kim, D. (2013). *Modern Portfolio Theory: Foundations, Analysis, and New Developments*. John Wiley & Sons.

Ghojogh, B., Ghodsi, A., Karray, F., & Crowley, M. (2021). KKT conditions, first-order and second-order optimization, and distributed optimization: tutorial and survey. *arXiv preprint arXiv:2110.01858*.

Gill, S., MacIver, C., & Bell, K. (2023). Exploring market change in the GB electricity system: the potential impact of Locational Marginal Pricing.

Glover, F. (1975). Surrogate constraint duality in mathematical programming. *Operations Research, 23*(3), 434-451.

Graß, E. (2019). An accelerated solution method for two-stage stochastic models in disaster management. *Springer*.

Grégoire, G. (2016). Elements of Statistics. *EAS Publications Series, 77*, 13-37.

Guo, X., Chan, R. H., Wong, W. K., & Zhu, L. (2019). Mean–variance, mean–VaR, and mean–CVaR models for portfolio selection with background risk. *Risk Management, 21*, 73-98.

Heinrich, C., & Demoment, G. (2000). Minimization of strictly convex functions: an improvedoptimality test based on Fenchel duality. *Inverse problems, 16*(3), 795.

Hu, X. M., & Ralph, D. (2004). Convergence of a penalty method for mathematical programming with complementarity constraints. *Journal of Optimization Theory and Applications, 123*, 365-390.

Hua, S., Xiaoye, Y., & Yuanfang, S. (2023). Dynamic discounting program of supply chain finance based on a financial information matching platform. *Annals of Operations Research, 331*(1), 221-250.

Infanger, G., Infanger, G., & Dantzig, G. B. (2011). Stochastic programming. *Springer New York.*

Investopedia. (n.d.). *Conditional value at risk (CVaR)*. Retrieved from https://www.investopedia.com/terms/c/conditional_value_at_risk.asp

Jie, T., & Yan, G. (2021). Computing shadow prices with multiple Lagrange multipliers. *Journal of Industrial & Management Optimization, 17*(5).

Jorion, P. (2007). *Value At Risk: The New Benchmark For Managing Financial Risk*. McGraw-Hill.

Kim, K. H., & Park, K. T. (2003). Dynamic space allocation for temporary storage. *International Journal of Systems Science, 34*(1), 11-20.

Kleinrock, L. (1975). *Queueing systems, volume 1: Theory*. John Wiley & Sons.

Kline, M. (1972). *Mathematical Thought From Ancient To Modern Times*. Oxford University Press.

Kolbin, V. V. (1977). Stochastic programming (No. 14). *Springer Science & Business Media.*

Kuhn, T. S. (2012). *The Structure of Scientific Revolutions*. University of Chicago Press.

Kyriba. (n.d.). Optimizing working capital with SCF and dynamic discounting. Retrieved from *https://www.kyriba.com/resource/optimizing-working-capital-scf-dynamic-discounting*

Lakatos, I. (1970). *The Methodology of Scientific Research Programmes*. In I. Lakatos & A. Musgrave (Eds.), Criticism and the growth of knowledge (pp. 91-196). Cambridge University Press.

Laforêt, O., & Devolder, P. (2018). Risk measurements applied to Basel III and Solvency II. *Louvain School of Management, Louvain, Belgium*.

Li, M. X. (2011). A Class of Augumented Lagrangian Function for Nonlinear Programming. *Advanced Materials Research, 271*, 1955-1960.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance, 7*(1), 77-91.

Nair, A., Narasimhan, R., & Bendoly, E. (2011). Coopetitive buyer–supplier relationship: an investigation of bargaining power, relational context, and investment strategies. *Decision Sciences, 42*(1), 93-127.

Nash, J. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences, 36*(1), 48-49.

Ogumeyo, S. A., Oloda, F. S., Ehiwario, J. C., & Adigwe, R. U. (2025). Mathematical Models to Determine Optimum Inventory Level in a Supply Chain System. Earthline *Journal of Mathematical Sciences, 15*(1), 85-103.

Oluwaseyi, K. O., Elizabeth, A., & Olaoluwa, O. E. (2020). Profit maximization in a product mix bakery using linear programming technique. *Journal of investment and Management, 9*(1), 27-30.

Pfohl, H.-C., & Gomm, M. (2009). Supply chain finance: Optimizing financial flows in supply chains. *Logistics Research*, 1(3-4), 149-161. https://doi.org/10.1007/s12159-009-0020-y

Polk, C., Vayanos, D., & Woolley, P. (2023). Long-horizon investing in a non-capm world. *LSE Financial Markets Group.*

Popper, K. R. (1963). Conjectures and refutations: The growth of scientific knowledge. *Routledge*.

Pourmohammadreza, N., Jesri, Z. S. H., & Kamran, M. A. (2025). A Review of Discount Policies in Supplier Selection: Trends, Opportunities, and Challenges. *IEEE Access*.

Pospíšil, J., Kolka, Z., Horská, J., & Brzobohatý, J. (2000). Simplest ODE equivalents of Chua's equations. *International Journal of Bifurcation and Chaos, 10*(01), 1-23.

Pourmohammadreza, N., Jesri, Z. S. H., & Kamran, M. A. (2025). A Review of Discount Policies in Supplier Selection: Trends, Opportunities, and Challenges. *IEEE Access*.

Rimal, B. P., & Maier, M. (2017). Mobile data offloading in FiWi enhanced LTE-A heterogeneous networks. *Journal of Optical Communications and Networking, 9*(7), 601-615.

Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance, 26*(7), 1443-1471. https://doi.org/10.1016/S0378-4266(02)00271-6

Ross, S. M. (2014). *Introduction To Probability Models*. Academic Press.

Rubinstein, R. Y., & Kroese, D. P. (2016). Simulation and the Monte Carlo method. John Wiley & Sons.

Russell, B. (1920). Introduction to mathematical philosophy. *Routledge*.

Rutten, M. (2022). Factors influencing multi-tier supply chain information sharing: a multi-tier supply chain case study (*Master's thesis, University of Twente*).

Sahinidis, N. V. (2004). Optimization under uncertainty: state-of-the-art and opportunities. *Computers & chemical engineering, 28*(6-7), 971-983.

Sedor, K. (2015). *The Law Of Large Numbers and Its Applications*. Lakehead University: Thunder Bay, ON, Canada.

Shortle, J. F., Thompson, J. M., Gross, D., & Harris, C. M. (2018). *Fundamentals of Queueing Theory*. John Wiley & Sons.

Silitonga, R. Y., Kristiana, L. R., & Parley, T. A. (2021). A multi-item probabilistic inventory model that considers expiration factor, all unit discount policy and warehouse capacity constraints. *Jurnal Teknik Industri: Jurnal Keilmuan dan Aplikasi Teknik Industri, 23*(2), 139-148.

Simba, S., Kotzé, T., Agigi, A., & Niemann, W. (2017). Supply chain risk management processes for resilience: A study of South African grocery manufacturers. *Journal of Transport and Supply Chain Management, 11*(1), 1-13.

Siokos, S. (1997). *International Multi-Sector, Multi-Instrument Financial Modeling and Computation: Statics and Dynamics*. University of Massachusetts Amherst.

Snyder, L. V., & Shen, Z.-J. M. (2011). *Fundamentals of Supply Chain Theory*. Wiley.

Soltan, V. (2021). Separating hyperplanes of convex sets. J. Convex Anal, 28(4), 18.

Song, H., Liu, C. C., & Lawarrée, J. (2002). Nash equilibrium bidding strategies in a bilateral electricity market. *IEEE transactions on Power Systems, 17*(1), 73-79.

Ssempijja, M. N., Namango, S., Ochola, J., & Mubiru, P. K. (2021). Application of Markov chains in manufacturing systems: A review. *International journal of industrial engineering and operational research, 3*(1), 1-13.

Taulia. (n.d.). *What Is Dynamic Discounting*?. Retrieved from https://taulia.com/glossary/what-is-dynamic-discounting

Taufemback, C., & Da Silva, S. (2012). Queuing theory applied to the optimal management of bank excess reserves. *Physica A: Statistical Mechanics and its Applications, 391*(4), 1381-1387.

Thomé, A. M. T., Scavarda, L. F., Pires, S. R., Ceryno, P., & Klingebiel, K. (2014). A multi-tier study on supply chain flexibility in the automotive industry. *International Journal of Production Economics, 158*, 91-105.

Tran, N. K., & Haasis, H. D. (2015). Literature survey of network optimization in container liner shipping. *Flexible Services and Manufacturing Journal, 27*, 139-179.

Tsao, Y.-C., Setiawati, M., Vu, T. L., & Sudiarso, A. (2021). Designing a supply chain network under a dynamic discounting-based credit payment program. *RAIRO-Operations Research*, 55, 2545-2565. https://doi.org/10.1051/ro/2021111

Tseng, M. L., Wu, K. J., Hu, J., & Wang, C. H. (2018). Decision-making model for sustainable supply chain finance under uncertainties. *International Journal of Production Economics, 205*, 30-36.

Uhuka, M. D. (2022). An evaluation of supply lead time in the Nigerian oil downstream sector: a system dynamic approach.

Uryasev,Jain, Y. K. (2024). Financial Management and Market Dynamics. *Xoffencerpublication.*

Végh, L. A. (2012). Strongly polynomial algorithm for a class of minimum-cost flow problems with separable convex objectives. *In Proceedings of the forty-fourth annual ACM symposium on Theory of computing (pp. 27-40)*.

Wang, Q., Zhang, X., Xu, Y., Yi, Z., & Xu, D. (2024). Planning of Stationary-Mobile Integrated Battery Energy Storage Systems under Severe Convective Weather. *IEEE Transactions on Sustainable Energy*.

Wang-Mlynek, L., & Foerstl, K. (2020). Barriers to multi-tier supply chain risk management. *The International Journal of Logistics Management, 31*(3), 465-487.

Wei, H. (2021). Equilibrium Models of the Limit Order Book.

Whiting, P. (2012). Queuing theory. *In The Cable and Telecommunications Professionals' Reference (pp. 365-393)*. Routledge.

Xu, J., Ko, Y. M., Kong, M., & Pender, J. (2023). Queueing management for reducing the overlaps of customers in service systems. *Available at SSRN 4384706*.

Xu, Z., Liang, W., Xu, W., Jia, M., & Guo, S. (2015). Capacitated cloudlet placements in wireless metropolitan area networks. *In 2015 IEEE 40Th conference on local computer networks (LCN) (pp. 570-578)*. IEEE.

Ye, M., Han, Q. L., Ding, L., & Xu, S. (2023). Distributed Nash equilibrium seeking in games with partial decision information: A survey. *Proceedings of the IEEE, 111*(2), 140-157.

Zhang, X., Astivia, O. L. O., Kroc, E., & Zumbo, B. D. (2023). How to think clearly about the central limit theorem. *Psychological Methods, 28*(6), 1427.

Zhang, A. N., Wagner, S. M., Goh, M., & Asian, S. (2024). Quantifying supply chain disruption: a recovery time equivalent value at risk approach. *International Journal of Logistics Research and Applications, 27*(5), 667-687.

Zhu, X., Tang, J., Wang, J., & Ding, S. (2015). Multiplier Algorithm Based on A New Augmented Lagrangian Function. *WSEAS Transactions on Mathematics, 14*, 169-177.