ITERATIVE TECHNIQUES FOR SOLVING LINEAR SYSTEMS

BY

OKOH FAITH OSELUOLEMEN

(PSC1909079)

A PROJECT SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE AWARD OF (BSC HONS) IN PURE AND APPLIED MATHEMATICS, OF THE FACULTY OF PHYSICAL SCIENCES, UNIVERSITY OF BENIN, BENIN CITY, EDO STATE, NIGERIA.

APRIL, 2024

UNDERTAKING

This project work was carried out by me, **OKOH FAITH OSELUOLEMEN** with matriculation number **PSC1909079.** I have not plagiarized the work of any other author(s). All published works used in this project have been duly cited and acknowledged.

OKOH FAITH OSELUOLEMEN _____

Name of Student **Signature and Date**

CERTIFICATION

This is to certify that this project titled "ITERATIVE TECHNIQUES FOR SOLVING LINEAR SYSTEMS" was carried out by **OKOH FAITH OSELUOLEMEN** under the supervision of **Dr. (Mrs.) G.C. Nwachukwu** in accordance with the rules and regulations of the University of Benin.

_____ _____

Dr. (Mrs.) G.C. Nwachukwu
(Project Supervisor)

Prof. Robert I. Okuonghae
(Head of Department)

Date _____ **Date _____**

DEDICATION

This project is dedicated to God Almighty for His unfailing love, strength and grace throughout this journey and has made this work a success and to my dad (late) Mr. Friday Samson Okoh.

ACKNOWLEDGEMENT

My profound gratitude goes to God Almighty, for preserving me throughout my stay in the University of Benin and the completion of this project.

My most sincere appreciation goes to my supervisor Dr. (Mrs.) G.C. Nwachukwu for her contribution towards the completion of this project. I appreciate all the lecturers in the department of mathematics.

I am forever grateful to my mum (Mrs. Maria Okoh), my aunt (Madam Mary Benson), my uncle (Mr. Lucky Okoh) and my sister (Mrs. Dominica Onasanya) for their love, care and financial support. May God reward you abundantly.

TABLE OF CONTENT

ABSTRACT

This project focuses on exploring iterative techniques for solving linear systems. The goal is to examine the efficiency and accuracy of iterative methods such as Jacobi and Gauss-seidel method used in solving linear systems commonly found in scientific applications.

CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF STUDY

Linear systems are essential in numerous fields including computer science, physics, engineering and mathematical disciplines as it serves as an important tool from computational fluid dynamics to structural analysis. Efficiently solving large linear system is essential for maximizing computations and resources. Conventional approaches such as direct method like Gaussian elimination can require significant computational resources and memory, especially dealing with large scale systems.

The system of linear equations which was introduced with coordinate geometry now called cartesian geometry started in Europe by Rene Descartes [1637]. In the branch of geometry known as cartesian geometry, lines and planes are expressed through linear equations. Determining the point at which these lines or planes intersect involves solving systems of linear equations (Okuonghae, 2021).

Iterative methods offer a different approach, where a sequence of estimates to the solution is produced. These methods are particularly effective for handling large, sparse matrices commonly found in practical settings. Their ability to efficiently handle extensive systems has sparked significant interest in fields such as data

science, engineering and physics simulations. The study of iterative techniques for solving linear systems is relevant for optimizing the resolution of problems and advancing computational science as it involves both practical and theoretical aspects, with the goal of providing efficient and exact solutions to linear systems in a wide range of field.

When the number of the unknowns (n) is very large, direct methods cannot be used due to rounding errors from the multitude of operations involved thereby allowing the use of iterative techniques. Iterative techniques are rarely employed to solve small- dimensional linear systems because the time needed for enough accuracy surpasses that of direct methods and are suited for large systems with many zero entries. They have built-in error correction mechanisms and at times can be utilized to minimize round-off errors in the solution compared to that of direct methods. We are focused on how to discover iterative methods that are time efficient and provide the level of sufficient accuracy needed for mathematical problems.

Iterative method includes the Jacobi method, Gauss-Seidel method, Successive Over-Relaxation (SOR) and Conjugate Gradients method used in mathematics and computational science which allows us to find precise numerical solutions to our systems of equations without excessive effort and within a reasonable amount of time.

1.2 STATEMENT OF PROBLEM

This study is to examine how efficient and feasible it is to use iterative methods for solving linear systems and to understand the effectiveness of different iterative techniques when addressing linear systems commonly encountered in scientific and engineering fields. Iterative techniques offer another strategy that are more effective for certain kinds of linear system. The challenge of solving large system of linear equations is a common occurrence in real-world scenarios like analyzing structures, image manipulation and other applications.

These techniques aim to discover the solution by starting with an initial estimation and enhancing it through a series of successive steps until a desirable solution is achieved.

1.3 AIM AND OBJECTIVES OF THE STUDY

The aim of this study is to analyze and compare two iterative methods (the Jacobi method and the Gauss-seidel method) used in solving linear systems, with a precise focus on their theoretical aspect and computational effectiveness.

The objectives of the study are as follows:

i. To analyze the theoretical aspects of the two iterative methods for resolving linear systems and it also involves understanding the fundamental concept of such method.

ii. To conduct a thorough analysis on the comparism and contrast of the iterative approaches focused on computational effectiveness.

1.4 BASIC DEFINITIONS IN LINEAR SYSTEMS

1. **Linear systems:** many linear equations with the same variables constitute a linear system. The expression **Ax=b** can be represented in matrix notation, where **A** is the coefficient matrix, **x** is the variable containing vector, and **b** is the vector of constants.
2. **Iterative techniques:** these are methods for improving a first estimate of a linear system solution iteratively. Large systems that are computationally difficult to solve using direct approaches, such as Gaussian elimination, frequently employ these techniques.
3. **Error:** The discrepancy between an approximation at a specific iteration and the linear system's actual solution.
4. **Convergence**: is the characteristic of an iterative process whereby, as the number of iterations grows, the series of approximations becomes closer to the actual solution of the system.

5. **Residual:** is the difference between the vector on the right side, **b**, and the product of the current approximation vector, **x,** and the coefficient matrix, **A**. It is frequently employed to calculate the error in the present estimate.

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

There are two types of numerical methods for solving systems of linear equations which are the direct method and the iterative approaches. Using direct approaches would require limited number of basic algebraic operations before providing an exact solution to the problem. Systems with nonsingular matrices can be solved using direct techniques like the Cramer's rule, Inverse method and Gaussian elimination method

An approximate solution (X) to a given system of linear equations is found using indirect methods (known as iterative methods). These iterative techniques include the Jacobi's iterative method and the Gauss Seidel method.

2.2 DIRECT METHODS FOR SOLVING LINEAR SYSTEMS

a) Cramer's Rule: This is a linear algebra method for solving systems of linear equations using determinants. The rule is named after the Swiss mathematician Gabriel Cramer, who introduced it first in his book in 1750. Gabriel Cramer used them for giving explicit solutions of linear system. This method gives a unique solution to a system of equations, if it exists. However, if the system has no solution or an infinite number of solutions, this will be indicated by a determinant of zero. Although Cramer's rule offers a sophisticated and theoretical technique for solving

linear systems, its computational complexity limits its practicality. The extensive calculations involving determinants and ratios of determinants can be particularly burdensome for large systems, making Cramer's Rule less efficient than other direct and iterative methods for solving linear systems. Despite these drawbacks, Cramer's rule remains used in specific mathematical scenarios where the coefficient matrix is small and easily invertible. Its application extends to theoretical realms within linear algebra and matrix theory, providing valuable insights into the structure and properties of systems of linear equations.

b) Inverse method: The inverse method is another method for solving systems of linear equations in various field. It involves calculating the inverse of the coefficient matrix of the linear system to obtain the solution. The inverse method is based on the concept of matrix inversion, where $Ax = B$ is gotten by multiplying the inverse of matrix A with vector B. The solution can be represented as $x = A^{-1}B$, where A^{-1} is the inverse of matrix A. The inverse method is particularly useful when the coefficient matrix is non-singular (that is, it has an inverse) and well-conditioned. One of the main advantages of the inverse method is its ease of implementation and gives a direct solution to the linear system without the need for iterative approximation. When applied to small medium-sized linear systems, it is computationally efficient.

The utilization of matrix inversion for solving systems of linear equations has been a longstanding practice and a basic element of linear algebra. While it is difficult to attribute the invention of the inverse method for solving linear systems of equations to a single individual, the evolution and systemization of matrix inversion as a mathematical tool can be traced back to many mathematicians over time.

An early illustration of matrix inversion can be identified in the works of the esteemed mathematician and physicist Carl Friedrich Gauss, who played a vital role in advancing the field of linear algebra during the late 18^{th} and early 19^{th} centuries. He also introduced matrix inversion as an integral component of his studies on systems of linear equations and determinant theory. In addition to Gauss's contributions, other mathematicians like Augustin-Louis Cauchy and Authur Cayley also made some important contributions in refining the theory of matrix inversion and its importance in solving linear equations. Since the 19^{th} and 20^{th} centuries, the concept of inverse matrix as a mechanism for solving linear systems became formalized and embraced within academic circles.

c) Gaussian Elimination Method: This simple technique for solving systems of linear equations, is accredited to the German mathematician Carl Friedrich Gauss, who is recognized for introducing this method in the latter part of the 18^{th} century. Gauss's significant contributions to the field of linear algebra and his formulation of

the procedure for systematically reducing a matrix to row-echelon form through a sequence of elementary row operations led to the method being named after him. Carl Friedrich Gauss presented the method of Gaussian elimination in his publication “Methodus nova integralium” outlining its application for solving linear systems of equations.

This method involves the transformation of the augmented matrix of the system into row-echelon form by executing elementary row operation, followed by back substitution to determine variable solutions. Gaussian elimination provides a structured and efficient approach to solving linear systems of equations, making it a fundamental tool in the mathematical field and its interdisciplinary applications.

The method serves as the foundation for numerous numerical algorithms utilized for solving linear systems, encompassing iterative techniques such as the Gauss-Seidel method and so on.

2.3 ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

We have talked about the direct methods for solving systems of linear equations namely Crammer's Rule, Inverse method and Gaussian elimination method. Now, we focus on the iterative method (indirect method) for the computation of linear systems of equations which includes: Jacobi's iterative method and Gauss-Seidel method. These approaches can be faster in some situations and are therefore preferable over direct methods, particularly in scenarios when the coefficient matrix contains a large number of zeros(sparse). Additionally, they could be cost effective in terms of a computer's basic storage requirements.

In a letter written on December 26, 1823 from Gauss to his pupil Gerling, the first mention of an iterative method for solving **Ax =B** appears to be made in relation to least squares issues using the normal equations. In the 1826 supplement to his well-known work on least squares, Gauss provided a block variant of the method. The solution of normal equations by iteration became standard in Germany in the 19^{th} century among geodesists and astronomers including Gerling, Bessel, etc. Bodewig claims that Gauss was required to solve systems with 20-30-40 unknowns. These systems had rapid convergence and were diagonally dominant. In triangulation of saxony in 1890, Bagel employed iteration to solve a system of 159 unknowns.

In 1845, an iterative method for solving normal equations for least squares problems emerging in astronomical calculations was introduced by Jacobi. In the same publication, he increases the diagonal dominance of the coefficient matrix by utilizing well selected plane rotations. This may be the first instance of preprocessing a linear system to hasten the iterative method's convergence. He gave a 3×3 example. In a subsequent paper (1846) he used plane rotations to diagonalize a symmetric matrix. Jacobi took a linear system that appeared in Gauss' Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium (1809). In contemporary notation, Jacobi wants to solve **Ax=b**.

Again in the context of least squares, in 1874 another German, Seidel, publishes his own iterative method. The paper contains what we now call the Gauss-Seidel method, which he describes as an improvement over Jacobi's method. Seidel notes that the unknowns do not have to be processed cyclically. He seems to be unaware that this is precisely Gauss' method. In the same paper, Seidel mentions a block variant of his scheme. He also notes that the calculations can be computed to variable accuracy, using fewer decimals in the first iterations. His linear system had up to 72 unknowns.

a) **Jacobi's iterative method:** The Jacobi's iterative method was derived from Carl Gustav Jacobi (1804-1851). This approach was the first numerical methodology for

solving a system of linear equations numerically. Since the most recent set of x-values is used to update each question simultaneously, this procedure is also known as the method of simultaneous displacement.

The Jacobi method is an iterative technique in numerical linear algebra that finds the solutions of a strictly diagonally dominant system of linear equations. After finding the estimated value for each diagonal element, the result is entered. After that, the procedure is repeated until it converges. The following presumptions are made by this method:

i) The system **Ax = B** has a unique solution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_n \end{bmatrix} \qquad (2.3.1)$$

ii) There are no zeros on the major diagonal of the coefficient matrix A.

If any of the diagonal entries a_{11}, a_{22}, a_{33},….., a_{nn} are zero, then a coefficient matrix with non-zero entries on the major diagonal can only be created by switching the rows and columns.

Begin by solving the first equation for x_1 the second equation for and x_2 so on as follows:

$$x_1 = \frac{1}{a_{11}} (b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n),$$

$$x_2 = \frac{1}{a_{22}} (b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n), \qquad (2.3.2)$$

.

.

.

$$x_n = \frac{1}{a_{nn}} (b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}).$$

Two counters are introduced in order to implement the simultaneous displacement methods in (2.3.2) by superscripting variables in the procedure to signify successive iterates. That means, the input variables on the right hand side of (2.3.2) becomes, $x_1^{(k)}$, $x_2^{(k)}$, …. , $x_{n-1}^{(k)}$, $x_n^{(k)}$ and the output variables on the left hand side of (2.3.2) becomes, $x_1^{(k+1)}$, $x_2^{(k+1)}$, …. , $x_{n-1}^{(k+1)}$, $x_n^{(k+1)}$. The method in (2.3.2) with these counters becomes ,

$$x_1^{(k+1)} = \frac{1}{a_{11}} \left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)} \right),$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)} \right),$$

.

.

. (2.3.3)

.

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots - a_{n,n-1}x_{n-1}^{(k)} \right)$$

Where k = 0, 1, 2, …. and $x_1^{(0)}, x_2^{(0)}, \ldots, x_{n-1}^{(0)}$ are the initial guess values to start up the method in (2.3.3) . Every first estimate can be assumed to be equal to zero if there are no better starting values available, substituting this into the right hand set of equations generates new approximations $x_1^{(1)}, x_2^{(1)}, \ldots, x_{n-1}^{(1)}$ that are closer to the true value thereby substituting the updated values into the right hand side yields a second approximation $x_1^{(2)}, x_2^{(2)}, \ldots, x_{n-1}^{(2)}$, then the procedure is carried out again until each variable successive values are sufficiently alike (converge).

b) Gauss-Seidel iterative method: The Gauss-Seidel method is a further technique for numerically approximating systems of linear equations. Carl Friedrich Gauss (1777 – 1855) was the first to offer this method; however, Phillip L. Seidel (1821- 1896) made significant improvements. Gauss-Seidel method is a slight modification of the Jacobi method.

By solving each equation for one variable in terms of the others, we can reorder the set of equations using this procedure used above in the Jacobi method and the iterative method becomes;

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right),$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right)$$

$$\vdots \qquad (2.3.4)$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots - a_{n,n-1}x_{n-1}^{(k)}\right)$$

Where k = 0, 1, 2, … and $x_1^{(0)}, x_2^{(0)}, \ldots, x_{n-1}^{(0)}$ are the initial guess values to start up the method in (2.3.4) . This algorithm in (2.3.4) is easy to use and it often requires fewer iterations to yield the same level of precision with Jacobi method. First, we utilize the value of $x_1^{(1)}$ from the first equation to compute the new $x_2^{(1)}$ which is then used in the second equation. In a similar manner, the third equation uses $x_1^{(0)}, x_1^{(0)}, and\ the\ new\ x_2^{(1)}\ to\ get\ the\ new\ x_3^{(1)}$ and so on. It is actually important to notice that the Gauss-Seidel method rate of convergence is faster than the Jacobi method.

2.4 IMPORTANCE OF SOLVING LINEAR SYSTEMS

The ability to identify the values of unknown variables that simultaneously satisfy all of the provided equations makes solving linear systems of equations crucial. In many disciplines, including mathematics, economics, engineering and many more, this is essential. Among the factors indicating the significance of solving linear systems are:

1) In geometry, solving linear systems can assist in locating the intersection of two or more lines or planes.
2) In engineering, linear systems are utilized to address issues with circuits, control systems, structural analysis and many other areas.
3) To represent the interactions between variables in a system, linear systems are frequently used in economics. By solving these systems, economists can make predictions and analyze the impact of different factors.
4) Linear systems are used in statistics and data analysis to model relationships between variables.

Overall, solving linear systems is a fundamental skill that is used in various fields to understand and solve complex problems. It offers a systematic way to find solutions and make decisions based on the relationships between variables.

CHAPTER THREE

METHODOLOGY

ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

3.1 INTRODUCTION

Numerical techniques more commonly involve an iterative method. In this chapter we will introduce the iterative methods for approximating the solution of a system of linear equations.

3.2 THE JACOBI METHOD

The first iterative technique is called the **Jacobi method,** after Carl Gustav Jacob Jacobi (1804–1851). This method makes two assumptions: (1) that the system given by

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

Has a unique solution and (2) that the coefficient matrix *A* has no zeros on its main diagonal. If any of the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ are zero, then rows or

columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

The Jacobi method: For each k ≥ 1, generate the components $x_i^{(k)}$ of $x^{(k)}$ from $x^{(k-1)}$ by

$$\boldsymbol{x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right)}$$

For i = 1,2,…….,n

To begin the Jacobi method, we solve the first equation for x_1, the second equation for x_2, and so on, as follows.

$$x_1 = \frac{1}{a_{11}} \cdot (b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}} (b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n)$$

.
.
.

$$x_n = \frac{1}{a_{nn}} (b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})$$

Then we make an initial approximation of the solution,

$(x_1, x_2, x_3, \ldots, x_n)$, . To get the initial approximation, enter these values of x_i into the right-hand side of the modified equations.

Once this process is finished, we declare that one iteration has been carried out.

Similarly, the first approximation's x-values are substituted into the right-hand side of the equations to generate the second approximation. By repeated iterations, we form a sequence of approximations that often **converges** to the actual solution. This procedure is illustrated in the following example.

E X A M P L E 3.2.1

Applying the Jacobi

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits

Solution

To begin, we write the system in the form

$$x_1 = -\frac{1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$
$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$
$$x_3 = -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$$

Since we have no idea of the actual solution, we choose $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

as a convenient initial approximation. This means that the first approximation is

$$x_1 = -\frac{1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$
$$x_2 = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$
$$x_3 = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

Continuing this procedure, we obtain the sequence of approximations shown in the Table 3.2.1

TABLE 3.2.1

N	x_1	x_2	x_3
0	0.000	0.000	0.000
1	-0.200	0.222	-0.429
2	0.146	0.203	-0.517
3	0.192	0.328	-0.416
4	0.181	0.332	-0.421
5	0.185	0.329	-0.424
6	0.186	0.331	-0.423
7	0.186	0.331	-0.423

Because the last two rows in Table (3.2.1) are identical, we conclude that to three significant digits the solution is

$$x_1 = 0.186,\ x_2 = 0.331,\ x_3 = -0.423.$$

For the system of linear equations given in Example (3.2.1), the Jacobi method is said to converge**.** In other words, numerous iterations are successful in yielding an accurate estimate to three significant digits. More iterations would be necessary for increased precision, as is typically the case with iterative approaches.

3.3 THE GAUSS-SEIDEL METHOD

A modification of the Jacobi method called the Gauss-Seidel method, named after Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). This modification is just as simple to apply as the Jacobi method, and it frequently takes less iterations to produce the same level of precision.

With the Jacobi method, the values of x_i obtained in the nth approximation remain unchanged until the entire $(n + 1)$th approximation has been calculated. With the Gauss-Seidel method, on the other hand, we use the new values of each x_i as soon as they are known. That is, once we have determined x_1 from the first equation, its value is then used.

Gauss-Seidel method: for each k ≥ 1, generate the components $x_i^{(k)}$ of $x^{(k)}$ from $x^{(k-1)}$ by

$$\boldsymbol{x_i^{(k)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} \left(a_{ij} x_j^{(k)} \right) - \sum_{j=i+1}^{n} \left(a_{ij} x_j^{(k-1)} \right) + b_i \right]}$$

For i = 1,2,……,n

in the second equation to obtain the new x_2. Similarly, the new x_1 and x_2 are used in the third equation to obtain the new x_3, and so on. This procedure is demonstrated in example 3.3.1

EXAMPLE 3.3.1

Applying the Gauss-seidel method

Use the Gauss-Seidel iteration method to approximate the solution to the system of equations given in example 3.2.1

Solution

The first computation is identical to that given in example (3.2.1). That is, using (x_1, x_2, x_3) = (0, 0, 0) as the initial approximation, we obtain the following new value for x_1.

$$x_1 = -\frac{1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$

Now that we have a new value for x_1, however, we use it to compute a new value for x_2.

$$x_2 = \frac{2}{9} + \frac{3}{9}(-0.200) - \frac{1}{9}(0) \approx 0.156$$

Similarly, we use x_1 = -0.200 and x_2 = 0.156 to compute a new value for x_3.

$$x_3 = -\frac{3}{7} + \frac{2}{7}(-0.200) - \frac{1}{7}(0.156) \approx -0.508$$

Thus the first approximation is x_1 = -0.200, x_2 = 0.156 and x_3 =-0.508. The series of estimates displayed in the table was created through further iterations.

TABLE 3.3.1

N	x_1	x_2	x_3
0	0.000	0.000	0.000
1	-0.200	0.156	-0.508
2	0.167	0.334	-0.429
3	0.191	0.333	-0.422
4	0.186	0.331	-0.423
5	0.186	0.331	-0.423

After only five iterations of the Gauss-Seidel method, we achieved the same accuracy as was obtained with seven iterations of the Jacobi method in Example (3.2.1).

It is possible to apply the Jacobi method or the Gauss-Seidel method to a system of linear equations and obtain a divergent sequence of approximations. In such cases, we say that the method diverges**.**

EXAMPLE 3.3.2

An example of divergence

Apply the Jacobi method to the system

$$x_1 - 5x_2 = -4$$

$$7x_1 - x_2 = 6$$

Using the initial approximation (x_1, x_2)= $(0, 0)$ and show that the method divergences.

Solution

As usual, we begin by rewriting the given system in the form

$$x_{1=} - 4 + 5x_2$$

$$x_2 = -6 + 7x_1$$

Then the initial approximation (0,0) produces

$$x_1 = - 4 + 5(0) = -4$$
$$x_2 = -6 + 7(0) = -6$$

as the first approximation. Repeated iterations produce the sequence of approximations shown in the Table (3.3.2)

TABLE 3.3.2

N	x_1	x_2
0	0	0
1	-4	-6
2	-34	-34
3	-174	-244
4	-1,244	-1,244
5	-6,124	-8,574
6	-42,874	-42,874
7	-214,376	-300,124

For this particular system of linear equations we can determine that the actual solution is $x_1 = 1$ and $x_2 = 1$. Thus we see from Table that the approximations given by the Jacobi method become progressively *worse* instead of better, and we conclude that the method diverges.

The problem of divergence in Example (3.3.2) is not resolved by using the Gauss-Seidel method rather than the Jacobi method. In fact, for this particular system the Gauss-Seidel method diverges more rapidly, as shown in the table (3.3.2)

TABLE 3.3.2

N	x_1	x_2
0	0	0
1	-4	-34
2	-174	-1224
3	-6,124	-42,874
4	-214,374	-1,500,624
5	-7,503,124	-52,521,874

With an initial approximation of $(x_1, x_2) = (0, 0)$, neither the Jacobi method nor the Gauss-Seidel method converges to the solution of the system of linear equations given in Example (3.3.2)

EXAMPLE 3.3.3

Interchanging rows to obtain convergence

Interchange the rows of the system

$$x_1 - 5x_2 = -4$$
$$7x_1 - x_2 = 6$$

to obtain one with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to four significant digits.

Solution

We begin by interchanging the two rows of the given system to obtain

$$7x_1 - x_2 = 6$$
$$x_1 - 5x_2 = -4$$

Note that the coefficient matrix of this system is strictly diagonally dominant. Then we solve for x_1 and x_2 as follows.

$$x_1 = \frac{6}{7} + \frac{1}{7}x_2$$
$$x_2 = \frac{4}{5} + \frac{1}{5}x_1$$

Using the initial approximation $(x_1, x_2) = (0,0)$, we obtain the sequence of approximations shown in the Table (3.3.3)

TABLE 3.3.3

n	x_1	x_2
0	0.000	0.000
1	0.8571	0.9714
2	0.9959	0.9992
3	0.9999	1.000
4	1.000	1.000
5	1.000	1.000

Thus we conclude that the solution is $x_1 = 1$ and $x_2 = 1$.

CHAPTER FOUR

SUMMARY AND CONCLUSION

This project study, consisting of four chapters, presents a systematic study of the iterative techniques for solving linear systems.

In chapter one, a general introduction of the project includes the background of the study, statement of problem, aim and objectives and some basic definitions in linear systems.

An up- to- date literature review of the development of linear systems, methods used in solving linear systems specifically on the iterative methods, the steps on how to carry out each method and the importance of solving linear systems is given in chapter two. Throughout the course of this project study, our aim was to analyse and compare different iterative methods used in solving linear systems, which led us to chapter three where we showed examples on how to solve problems using the Jacobi method and the Gauss-Seidel method. The Gauss-Seidel method converges faster than the Jacobi method.

Depending on the number of iterations, one or both of these may occasionally be unable to yield a fair approximation to the solution. The approximations are said to diverge in these situations. Lastly, in chapter four, we have summary and conclusion.

REFERENCES

Kelly, C.T. (1995), *Iterative methods for linear and non-linear equations.* Society for Industrial and Applied Mathematics. Raleigh, North Carolina.

Okuonghae, R.I. (2021), *Introductory numerical analysis first edition.* Tundey Folasmark Nigeria Enterprise, Benin City.

Vishal, V. M. and Aditya, S. (2019), *Jacobi and Gauss-Seidel Iterative methods for the solution of systems of linear equations comparison,* volume 6, issue 7, Bharati Vidyapeeth Deemed University, College of Engineering, Pune, India.

Saad, Y. (2003), *Iterative methods for sparse linear systems second edition. S*ociety for Industrial and Applied Mathematics, Pennsylvania.