ITERATIVE METHOD FOR SOLVING INITIAL AND BOUNDARY VALUE PROBLEMS

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CERTIFICATION

This to certify that this project report was carried out by OLKALEMWEN EROMOSELE MOSES in the Department of Mathematics, University of Benin, Benin City.

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DECLARATION

I OLKALEMWEN EROMOSELE MOSES, hereby declare that the project work entitled "Iterative method for solving initial and boundary value problems" was carried out by me. I have not copied the work of any author. All work has been duly cited and acknowledged.

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Olkalemwen Eromosele Moses Date

DEDICATION

This project is dedicated firstly to the almighty God for guiding me throughout the course of my study. Secondly to my parents MR and MRS OGBEIDE for their ending words of encouragement and the constant push for the achievement of academic success.

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I would like to express my sincere appreciation and gratitude to the almighty God, whose grace and guidance have been the cornerstone of my life's journey. Your blessings and providence have illuminated my path and filled my heart with gratitude and humility.

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Thank you all for being part of this incredible chapter in my life.

ABSTRACT

Numerical methods for solving initial and boundary value problems play a crucial role in various fields of science and engineering.The objective of this project is to present a numerical iterative method for solving initial and boundary value problems to ordinary differential equations . This iterative method is based on the use of the Euler's method and the finite difference method (FDM) in solving initial and boundary value problems respectively. The project begins with a comprehensive literature review on numerical methods for solving IVPs and BVPs, emphasizing the theoretical foundations and practical applications of Euler's and Finite difference methods. The mathematical formulations and algorithmic procedures of both methods are discussed in detail, highlighting their similarities, and differences. Furthermore, the Euler's method and the finite difference method enables us to approximate the solutions of an ordinary differential equation at a given initial value problem and boundary value problem respectively.

Indeed, two numerical examples are provided to illustrate the effectiveness of the Euler's and Finite difference methods. Results obtained show that the numerical method is very effective and convenient for solving ordinary differential equations with initial and boundary value problems.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

In order to solve initial and boundary value problems, differential equations are essential. The study of differential equations is a wide field in pure and applied mathematics, physics, and engineering (Makukula, 2012). Differential equations first came into existence with the invention of calculus by Isaac Newton, a physicists and Gottfried Wilhelm Leibniz. Different kinds of differential equations are:

$$\frac{dy}{dx} = f(x) \tag{1.1}$$

$$\frac{dy}{dx} = f(x, y) \tag{1.2}$$

He solved these differential equations and others using infinite series and discussed the non-uniqueness of solutions. The Bernoulli differential was proposed by Jacob Bernoulli in 1695 is expressed as

$$y' + P(x) = Q(x)y^n \tag{1.3}$$

Which was later simplified and solved by Leibniz. Numerous scientists like Jean Le Rond d' Alembert, Leonhard Euler, Joseph-Louis Lagrange (Lagrangian mechanics) also worked on improving more on the differential equations. Differential equations are used to model problems in science and engineering where one variable changes in relation to another. Most of the problems require the solutions of initial and boundary value problems.

There are two approaches to approximate the solution of differential equations that model real-life mathematical problems when the problems are too complicated to be solved.

The first approach involves changing the problem by making the differential equation simpler such that it can be solved, and then using the differential equation solution to solve the problem.

To solve such a problem for initial value problems (IVPs), let consider a first order differential equation (ODE) such as:

$$\frac{dy}{dx} = 6x \qquad (1.4)$$

With an initial condition $y(0) = 1$. This forms an initial value problem, solving this IVPs involves finding the function of $y(x)$ that satisfies the differential equation and the given initial condition.

Form equation (1.4) integrate w.r.t dx

$$y(x) = 3x^2 + k \qquad (1.5)$$

Where k is the constant applying the initial condition $y(0) = 1$ we find $k = 1$ and the solution is given as

$$y(x) = 3x^2 + 1 \qquad (1.6)$$

For Boundary value problems (BVPs), let consider a second order differential equation such as:

$$\frac{d^2y}{dx^2} + y = 0 \qquad (1.7)$$

With a boundary condition $y(0) = 0$ and $y\left(\frac{\pi}{2}\right) = 2$. This forms a BVPs solving this, involves finding for $y(x)$ that satisfies the differential equation and the given conditions at both $x = 0$ and $x = \frac{\pi}{2}$. The general solution to equation (1.7) is given as

$$y(x) = Asinx + Bcosx \qquad (1.8)$$

Where A and B are constant so applying the boundary condition we find $B = 0$ when $y(0) = 0$ and $A = 2$ when $y\left(\frac{\pi}{2}\right) = 2$. Therefore, the solutions to the BVPs is given as

$$y(x) = 2sinx \tag{1.9}$$

The other approach that we will examine in this project is the use of iterative methods for approximating the solution to the original problem. This is the approach that is most commonly taken because the approximation methods give more accurate results and reliable error information. The Iterative method used for solving initial and boundary value problems are the Euler's method, the Runge-Kutta method, the finite difference method (FDM), and finite element method (FEM). The Euler's method and the Runge-Kutta method are iterative methods for solving initial value problems, while the FDM method and the FEM method are iterative methods for solving boundary value problems. Euler's method is a simple numerical technique that is easy to implement, but it has limited accuracy. When it comes to accuracy, the Runge-Kutta method outperforms the Euler method.

FDM discretizes the differential equation by using finite difference approximations to approximate the derivatives. The domain is divided into grids, and at each grid point, the differential equation is substituted with finite difference approximations. As a result, the continuous issue is reduced to a set of algebraic equations that have numerical solutions. However, FEM divides the domain into smaller, distinct subdomains known as elements. It shows a piecewise approximation of the solution over the elements, and then by integrating over each element, the differential equation is converted into a system of algebraic equations.

1.2 Statement of the Problem

Initial and boundary value problems cannot in general be solved analytically. In this case, therefore solutions to these problems must be approached using numerical methods, including iterative techniques and computational approaches are often employed. The principle of these methods of solving consists in starting from an arbitrary point-the closest possible point to the solution gradually through successive tests. With this project, I aim at analyzing and comparing different numerical methods of solving initial and boundary value problems and getting a deeper understanding of the concept by reading and reviewing literature and journals.

1.3 Aim and objectives of the Study

The aim of the study is to investigate and apply iterative techniques such as Euler's method and the finite difference method (FDM) method for finding approximate solutions to differential equations with initial and boundary conditions.

The objectives of the study are to:

1. Understand the concept of initial and boundary value problems
2. Solve questions using the Euler's and the FDM methods
3. Compare the result obtained by this method with the result obtained by the analytical method.

1.4 Scope of the Study

The scope of this study is to examine the application of iterative methods for solving initial and value boundary value problems methods provide. Iterative method is used to solve complex mathematical equations that describe physical phenomena. These methods provide numerical approximations to the solutions of differential equations which are essential in the field of engineering, physics, biology etc.

Initial value problems involve finding the solution to a differential equation given an initial condition. On the other hand, boundary value problems require finding the solution to a differential equation subject to specified boundary conditions. Both types of problems are encountered in various engineering applications, such as structural analysis, fluid dynamics, and heat transfer.

1.5 Some Basic Definitions

1.5.1 Differential Equation

A differential equation is a mathematical equation that relates some functions with its derivatives, in other word a differential is an equation that as a derivative $y^{'}(x), ..., y^{n}(x)$, independent variable (x) and dependent variable (y). In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Mathematically, it is defined as

$$f\left(x, y(x), y^{'}(x), ..., y^{n}(x)\right) = 0 \quad (1.10)$$

Examples of differential equations are:

$$\frac{dy}{dx} = 3x^{2} - 6x^{2} + 5 \quad (1.11)$$

$$x^{2}\frac{dy}{dx} = 5x + 4 \quad (1.12)$$

and

$$\frac{dy}{dx} = \frac{2x}{y+1} \quad (1.13)$$

1.5.2 Order of Differential Equation

Order of the differential equation is determined by the terms with the highest derivatives. An equation containing only first derivatives is called a first-order differential equation, an equation containing the second order derivatives is called a second-order differential equation, and so on. Otherwise, the order of the differential equation is the highest derivative present in the differential equation. Some examples are:

$$\frac{dy}{dx} = 6x + 3 \quad (1.14)$$

$$\frac{d^2y}{dx^2} = 2x \quad (1.15)$$

and

$$\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \quad (1.16)$$

are of order 1,2, and 3 respectively.

1.5.3 Degree of a Differential Equation

Degree of a differential equation is the power of the highest derivatives of the differential equation. Examples are:

$$\frac{d^2y}{dx^2} + 3y\left(\frac{dy}{dx}\right)^7 = 5x \quad (1.17)$$

and

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} + 8y = 0 \quad (1.18)$$

are differential equations of degree 1 and 2 respectively.

1.5.4 Ordinary Differential Equation (ODE)

An ordinary differential equation (ODE) is an equation containing one independent variable and its derivatives. The term "ordinary" is used in contrast with the term partial differential equation which may be with respect to more than one independent variable. The first order differential equation is given as:

$$y' = f(x, y(x)) \quad (1.19)$$

Where x is the independent variable and y is the dependent variable, the types of first order ordinary differential equation include **separable variable differential equation**, **homogeneous differential equation**, **linear differential equation, Bernoulli**

differential equation, and **exact differential equation**. These equations are of different forms and their solutions are obtained uniquely.

1.5.5 Linear Ordinary Differential Equation

An nth order ODE is said to be linear if it can be written as:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \qquad (1.20)$$

If $f(x) = 0$ the equation is called homogeneous and if $f(x) \neq 0$ is called inhomogeneous. It should be observed that:

1 The dependent variable y and all its derivatives $y', y'', y''' \dots, y^n$ are of the first degree, that is the power of the highest derivative of each term is 1

2 The coefficient $a_0, a_1, a_2, \dots, a_n$ of $y', y'', y''', \dots, y^n$ depend on the independent variable x.

3 No transcendental functions of the dependent variable y is involve like $siny, logy, e^y$.

Examples are:

$$\frac{dy}{dx} - 4xy = 1 \qquad (1.21)$$

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 5xy =- 25 \qquad (1.22)$$

1.5.6 Non-linear Differential Equation

A differential equation is said to be a nonlinear ordinary differential equation if it is not linear. In other words, differential equations that are not linear are called nonlinear differential equations.

Examples are;

$$\left(\frac{dy}{dx}\right)^3 - 6xy = 1 \qquad (1.23)$$

Since the degree of the highest derivatives is 3

$$\frac{d^2y}{dx^2} + 10y\frac{dy}{dx} - 4y = log\, x \quad (1.24)$$

Since the coefficient variable is not depending on x.

1.5.7 Initial and Boundary Value Problems

An ordinary differential equation, together with an initial condition, is called an **initial value problem** (I.v.p). Thus, if the ordinary differential equation is explicit, $y' = f(x, y)$, the initial value problem is of the form

$$y' = f(x, y),\ y(x_0) = y_o \quad (1.25)$$

A different type of problem arises when the auxiliary conditions are specified at two different points, say $x = a$ and say $x = b$. Conditions of this type are called boundary conditions, because in such problems x usually represents a space variable. The solution is required to be determined between two boundaries located at $x = a$ and $x = b$ where boundary conditions are prescribed. A **boundary value problem** (b.v.p) involves finding a solution of a differential equation that satisfies prescribed boundary conditions.
Example of boundary value problem

$$6y'' + 7y' = 0\ \ y(0) = 1, y(1) = 2 \quad (1.26)$$

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

Ali H. Nayfeh (21 December 1933 – 27 March 2017) was a Palestinian-Jordanian mathematician, mechanical engineer and physicist. He is regarded as the most influential scholar and scientist in the area of applied nonlinear dynamics in mechanics and engineering. Ali H. Nayfeh (1973), has made significant contributions to the development of approximate solutions to initial and boundary value problems, particularly in the history of nonlinear dynamics. His book, "Introduction to Approximate Solutions to Initial and Boundary Value Problems," provides a comprehensive overview of various numerical techniques used in engineering applications. His research often involves analytically techniques to find approximate solutions for complex mathematical models.

Most real-world problems are modeled using partial differential equations or ordinary differential equations. The process of modeling physical problems results in an equation that may have variable coefficients and nonlinear boundary conditions (Nayfeh, 1973). Most of these differential equations are highly nonlinear and exact solutions are not always possible. For those cases where exact solutions are not possible, numerical methods often provide approximate solutions.

The basis of any numerical method is the discretization of the time and space variables in the governing equations. The discretization process approximates the differential equation by a system of algebraic equations. Brief description of a few common numerical methods is given in the section that fellows.

2.1 Discretization approaches for IVPs and BVPs

They are different types of discretization approaches for solving initial and boundary value problems. In this section, we will discuss some of the common approaches.

For initial value problems (IVPs), the discretization method includes

1 The Euler's method
2 The Runge-Kutta method

And for boundary value problems (BVPs), the discretization method includes

3 The finite difference method
4 The finite element method

2.1.1 The Euler's Method

In mathematics and computational science, the Euler method (also called the forward Euler method) is an iterative method for solving first order differential equations (ODEs) with a given initial value problem. The Euler method is named after Swiss mathematician Leonhard Euler, who first proposed it in his book "Institutionum calculi integralis" (published 1768–1770). This method played a crucial role in the field of mathematics and has found extensive applications in various scientific disciplines. This project aims to provide an overview of the history of the Euler method and its development in solving initial value problems.

Leonhard Euler born in Basel Switzerland in 1707 was one of the most influential mathematicians of the 18th century. He made significant contributions to various branches of mathematics including calculus number theory and graph theory. In the 1730s, Euler focused his attention on solving problems related to celestial mechanics, particularly those involving the motion of celestial bodies. These investigations led him to formulate the Euler method to approximate the solutions of differential equations, which describe how quantities change with respect to an independent variable.

Euler's interest in solving differential equations led him to develop the Euler method as a numerical method for approximating solutions to initial value problems (IVPs). It involves the approximation of the solution curve by constructing a sequence of tangent lines. These tangent lines are connected to form a polygonal curve that approximates the solution.

According to Hairer Norsett and Wanner (2008) the tangent line in the Euler method is derived from the differential equation itself. One of the fundamental principles in the Euler method is the concept of the tangent line. The tangent line at a given point on the solution curve represents the instantaneous slope or the derivative of the curve at that point. It provides an approximation of the slope of the solution curve and is used to estimate the value of the solution at the next time step.

The Euler method starts with an initial value problem of the form:

$$\frac{dy}{dx} = f(x, y) \quad y\left(x_o\right) = y_0 \tag{2.1}$$

where $f(x, y)$ represents the rate of change of y with respect to x. It divides the $\left[x_0, x_n\right]$ into small step of size called h , the smaller the step size the more accurate the approximation. The iterative formulae for the Euler's method are:

$$y_{n+1} = y_n + f\left(x_{n,} y_n\right)h \text{ and } x_{n+1} = x_n + h \tag{2.2}$$

We discuss more about deriving the formula in chapter 3

2.1.2 The Runge-Kutta Method

In numerical analysis, the Runge–Kutta methods are a family of implicit and explicit iterative methods, which include the Euler method, used in temporal discretization for the approximate solutions of simultaneous nonlinear equations. The method is named after two German mathematicians Carl David Tolme Runge and Wilhelm Martin Kutta who independently proposed the technique in the late 19th century.

In 1895 Carl David Tolme Runge published a paper titled "Uber die numerische Auflosung von Differentialgleichungen" (On the Numerical Solution of Differential Equations) where he introduced the concept of using iteratively calculated slopes to approximate the solution of differential equations. This work laid the foundation for what would later become the Runge-Kutta method.

Wilhelm Kutta in 1901 further expanded on Runge's work and presented a more general and accurate method for solving differential equations. Kutta's contribution was published in the paper "Beitrag zur näherungsweisen Integration totaler Differentialgleichungen" (Contribution to the Approximate Integration of Total Differential Equations). Kutta's method involved the use of a set of coefficients to calculate the weighted average of multiple slopes resulting in higher accuracy compared to Runge's original approach. Since their initial proposals the Runge-Kutta method has undergone several advancements and modifications. In the early 20th century mathematicians like Arthur Cayley, Ludwig Lorenz and Alexander Aitken made notable contributions to the method's development.

Cayley introduced a more efficient version of the method in 1905, while Lorenz proposed a simplification of the method in 1912. Aitken in 1932 derived a three-stage variant of the Runge-Kutta method known as "Aitken's delta-squared process," which offered increased accuracy. Its accuracy, stability, and efficiency make it a preferred choice for solving a wide range of differential equations encountered in various scientific and engineering fields, such as control systems, fluid dynamics, structural analysis, and electrical circuits.

In the mid-20th century, the New Zealand mathematician John C. Butcher made significant contributions to the theory and practicality of the Runge-Kutta method. Butcher developed a systematic approach for deriving the coefficients of the method known as the Butcher tableau. This approach allowed for the construction of higher-order Runge-Kutta methods with improved accuracy and stability.

Butcher's work led to the development of several widely used variants of the Runge-Kutta method, such as the classical fourth-order method (RK4) and the adaptive step-size Runge-Kutta methods. These variants provided more flexibility in solving initial value problems as they allowed for efficient error control and a better approximation of the solution. The Runge-Kutta method has been particularly useful in solving complex systems of differential equations such as those encountered in control systems, fluid dynamics, and electrical circuit analysis.

2.1.3 The Finite Difference Method (FDM)

The FDM is believed to have been first used in 1768 by Leonhard Euler. At that time, it was used to find numerical solutions of differential equations using pen and paper. Leonhard Euler used the method to approximate the solutions to a second-order differential equation. However, in the 19^{th} century the method gained more attention and started to be recognized as a valuable tool for solving engineering problems.

During the 19^{th} century significant advancements were made in the field of numerical analysis which further contributed to the development of finite difference methods. A German mathematician Carl Gustav Jacob Jacobi introduced the concept of finite difference method in 1825. He used this method to approximate the solutions of partial differential equations or ordinary differential equations by discretizing the domain into a grid of points. Taylor series expansions are then used to compute the finite difference approximations to derivatives in the governing equations at each nodal point of the grid.

In the following decades, other mathematicians and engineers made notable contributions to the finite difference method. In 1851, Jean-Baptiste Joseph Fourier proposed a numerical method for solving heat conduction problems using finite differences. His work played a crucial role in the development of numerical heat transfer analysis. Furthermore, in 1888, Thomas Craig introduced the concept of central differences, which significantly improved the accuracy of the finite difference method.

The finite difference method has great flexibility in handling problems that are defined in complex geometries and finite difference methods are relatively easy to implement and are computationally efficient. However, they also suffer from inherent discretization errors which may lead to poor accuracy (Makukula,2012).

2.1.4 The Finite Element Method (FEM)

Unlike the FDM, in the finite element method (FEM) the computational domain is viewed as a collection of simple geometric shapes called finite elements. The FEM is a generalization of the classical variational and weighted residual methods. The history of the FEM dates back to the beginning of the twentieth century.

In the early 1940s, when engineers and mathematicians started exploring numerical methods for solving partial differential equations. Richard Courant, a renowned mathematician, introduced the concept of variational calculus, which laid the foundation for the finite element method (FEM). However, it was not until the 1950s and 1960s that the method gained significant attention. The development of the finite element method as we know it today can be attributed to engineers and researchers from various disciplines.

In the 1950s, J.C. Turner and R.W. Clough developed the stiffness method, which forms the basis of the finite element method. They introduced the concept of subdividing a structure into smaller elements to simplify the analysis process. In the 1960s, engineers like O.C. Zienkiewicz and Y.C. Cheung further developed the finite element method making it more adaptable and suitable to a wide range problem. Zienkiewicz introduced the concept of isoparametric elements which allowed for more accurate representation of complex geometries and material properties.

Unlike the FDM, the FEM is efficient in solving problems with complex geometries and boundary conditions. This is because the meshes created can easily be adapted to almost any type of domain. However, the FEM also suffers from low accuracy. Mistakes by users including, for example the use of wrong or distorted elements, may lead to a very serious error. The FEM uses a variational formulation that automatically accommodates the boundary conditions (Thomee,1984).

CHAPTER THREE

METHODOLOGY

3.1 Existence and Uniqueness of Solutions to O. D. E.

Consider the first order differential equation with an initial value problem given as

$$\frac{dy}{dx} = f(x,\ y), \quad y(x_0) = y_0 \tag{3.1}$$

To solve this solution, first we need to know if the solution exists, that's, if the initial value problem can be solved. Secondly, if the solution exists, is it unique? This gives the following theorem, which tells us that solutions to first-order differential equations exist and are unique under specific reasonable conditions.

Theorem 3.1 (Existence and Uniqueness Theorem): If $f(x,\ y)$ is continuous on $\left(x_0, y_0\right)$, then the solution exists. If both $f(x,\ y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous on $(x_{0,} y_0)$ then the solution is unique.

Proof of Theorem 3.1

Suppose $f(x,\ y)$ is a continuous function in a rectangle region $R = \{(x, y) :\ a < x < b,\ c < y < d\}$ in xy-plane if (x_0, y_0), is a point in this rectangle, then there exists an $\varepsilon > 0$ and a function $y(x)$ defined for

$x_0 - \varepsilon < x < x_0 + \varepsilon$ that solves the initial value problem (3.1). if both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous on R then there is a unique solution, defined on an interval $x_0 - \varepsilon < x < x_0 + \varepsilon$ for some $\varepsilon > 0$, that solves the initial value problems (3.1).

3.2 Examples on Existence and Uniqueness Theorem

Example 1. Check the existence and uniqueness of the solution

$$y' = x - y + 1,\ y(1) = 2 \quad (3.2)$$

Solving equation (3.2), Form theorem 3.1 If both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous on $(x_{0,} y_0)$ then the solution exists and is unique.

Where $f(x, y) = x - y + 1$ from IVP $y(1) = 2$ then $f(1, 2) = 0$ is continuous on $(1, 2)$ and its partial derivative with respect to y, $\frac{\partial}{\partial y} f(x, y) =- 1$, is continuous on $(1, 2)$. Hence equation (3.2) as an existence and uniqueness solution on the interval $(1, 2)$. Then the solution to the given ODE is to have one unique solution.

Now solve equation (3.2), which can be written as

$$y' + y = x + 1 \quad (3.3)$$

Using linear equation $y' + p(x)\, y = Q(x)$ (3.4)

Comparing equation (3.3) with equation (3.4)

$P = 1\ and\ Q = x + 1$ (3.5)

Integrating factor (IF) = $e^{\int pdx} = e^{\int 1dx} = e^{x}$
(3.6)

The solution of the linear equation given as $y.\ If = \int Q.\ If\ dx$ (3.7)

Substitute equation (3.5) and equation (3.6) in equation (3.7) we have,

$$ye^{x} = \int (x + 1)e^{x}\, dx \tag{3.8}$$

Equation (3.8) becomes

$$ye^{x} = \int xe^{x}dx + \int e^{x} \tag{3.9}$$

from integration by part to get $\int xe^{x}dx$

$$\int xe^{x}dx = xe^{x} - e^{x} \tag{3.10}$$

Substitute equation (3.10) into equation (3.9)

$$ye^{x} = \; = xe^{x} + c \tag{3.11}$$

Divide both side by e^{x}

$$y = x + ce^{-x} \tag{3.12}$$

To get C, given the initial condition $y(1) = 2$

From equation (3.12) we get $c = e$. Thus, the solution of given ODE is $y = x + e^{1-x}$

3.3 Euler's Method

One of the simplest numerical methods for solving initial and value problems (IVPs) in ordinary differential equations is the Euler's method. Euler's method is also called tangent line (Slope) method, and is particularly helpful for quick programming, which was originated by Leonhard Euler in 1768.

Euler's method is subdivided into three, namely:

1 Forward Euler's method

2 Backward Euler's method

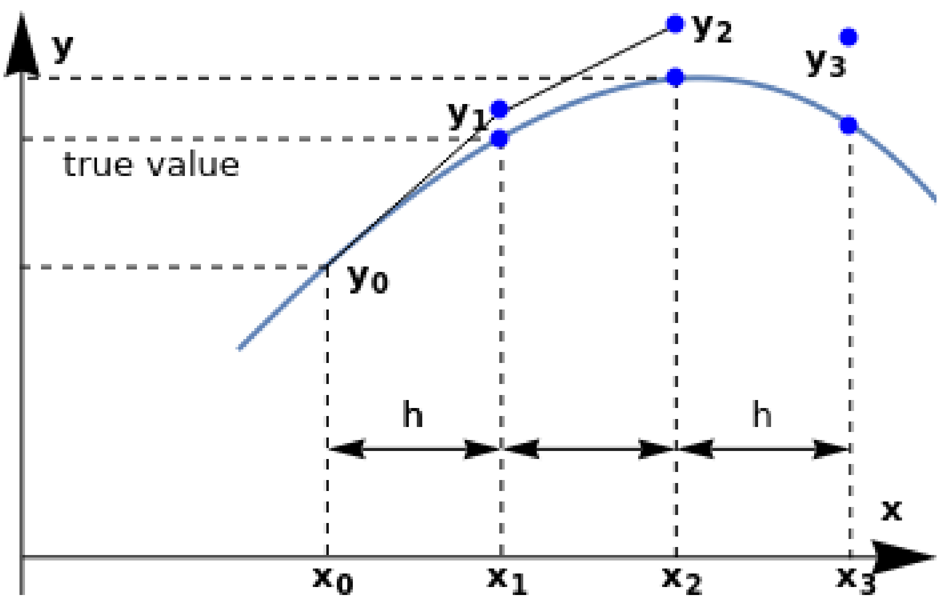
In this project, our main concern is the forward Euler's method.

3.4 Derivation of Euler's Method

Consider the first order differential equation with an initial value problem given as

$$y' = f(x, y(x)) \;, \quad y(x_0) = y_0 \tag{3.13}$$

Where x_0 and y_0 are the initial values for x and y respectively. Our aim is to determine the approximate solution for the unknown function $y(x)$ for $x \geq x_0$. We will now describe the methods which give the solution in the form of a set of tabulated values.



The solution of this differential equation subject to the given condition represents a curve $y = f(x)$ whose slope at any point (x, y) is $f(x, y(x))$. We take note that the curve $y = f(x)$ passes through $(x_{0,} y_0)$ and the slope of the curve at $(x_{0,} y_0)$ is $f(x_{0,} y_{0)}$.

The equation of the tangent at $(x_{0,} y_0)$ is

$$y_1 - y_0 = m(x_1 - x_0) \tag{3.14}$$

By solving equation (3.14), making y_1 subject formula which give us

$$y_1 = y_0 + m(x_1 - x_0) \tag{3.15}$$

Where m is the slope and is given by $m = y' = f(x, y)$, it depends on $(x_{0,}y_0)$. Hence

$m = y' = f(x_{0,}y_0)$

Substituting m in equation (3.15) which give us

$$y_1 = y_0 + f(x_0, y_o)(x_1 - x_0) \tag{3.16}$$

Where $h = x_1 - x_0$ and is called the step size

Equation (3.16) becomes,

$$y_1 = y_0 + hf\left(x_{0,}y_0\right) \tag{3.17}$$

Repeating the above process to get y_2, given the initial condition $y\left(x_1\right) = y_1$

$$y_2 - y_1 = m(x_2 - x_1) \tag{3.18}$$

Making y_2 subject of the formula which give us

$$y_2 = y_1 + m(x_2 - x_1) \tag{3.19}$$

Where m is the slope and is given by $m = y' = f(x, y)$, it depends on $(x_{1,}y_1)$ then

$m = y' = f(x_{1,}y_1)$

Substituting m in equation (3.19) which give us

$$y_2 = y_1 + f(x_{1,}y_1)(x_2 - x_1) \tag{3.20}$$

Where $h = x_2 - x_1$

$$y_2 = y_1 + hf(x_1, y_1) \qquad (3.21)$$

Similarly, to get for y_3 given the initial condition $y(x_2) = y_2$

$$y_3 = y_2 + hf(x_2, y_2) \qquad (3.22)$$

''

''

''

''

In general, we obtain a recurrence relation as

$$y_{n+1} = y_n + hf(x_{n,} y_n), n = 0, 1, 2, 3...$$

(3.23)

Equation (3.23) is called the Euler's method and can be used recursively to evaluate $y_{1,} y_{2,} ....$ starting from initial condition $y(x_0) = y_0$. We take note that from equation (3.23) the term $hf(x_0, y_0)$ represents the incremental value of y and $f(x_{n,} y_n)$ is the slope of y at $(x_{n,} y_n)$.

3.5 Finite Difference Method (FDM)

In numerical analysis, finite-difference methods (FDM) are a class of numerical techniques for solving boundary value problems in differential equations by approximating derivatives with finite differences.

Derivatives for Finite Difference Approximation

Consider Taylor series expansions for $y(x + h)$ and $y(x - h)$ given as

$$y(x + h) = y(x) + y'(x)h + y''(x)\frac{h^2}{2!} + y'''(x)\frac{h^3}{3!} + ... \qquad (3.24)$$

$$y(x - h) = y(x) - y'(x)h + y''(x)\frac{h^2}{2!} - y'''(x)\frac{h^3}{3!} + ... \qquad (3.25)$$

Equation (3.24) can be written as

$$y(x + h) = y(x) + y^{'}(x)h + 0(h^{2}) \tag{3.26}$$

Making $y'(x)$ subject of formulae

$$y^{'}(x) = \frac{y(x+h)-y(x)}{h} - 0(h) \tag{3.27}$$

The forward difference approximation of the first derivative is given as

$$y^{'}(x) = \frac{y(x+h)-y(x)}{h} \tag{3.28}$$

Similarly, we take equation (3.25) which can be written as

$$y(x - h) = y(x) - y^{'}(x)h + 0(h^{2}) \tag{3.29}$$

Making $y'(x)$ subject of formulae

$$y^{'}(x) = \frac{y(x-h)-y(x)}{h} - 0(h) \tag{3.30}$$

The backward difference approximation of the first derivative is given as

$$y^{'}(x) = \frac{y(x-h)-y(x)}{h} \tag{3.31}$$

Subtracting equation (3.24) and equation (3.25) and make $y'(x)$ subject of formulae

$$y^{'}(x) = \frac{y(x+h)-y(x-h)}{2h} - 0(h^{2}) \tag{3.32}$$

The centered difference approximation of the first derivative is given as

$$y^{'}(x) = \frac{y(x+h)-y(x-h)}{2h} \tag{3.33}$$

Adding equation (3.24) and equation (3.25) and make $y''(x)$ subject of formulae

$$y''x) = \frac{y(x+h)-2y(x)+y(x-h)}{h^{2}} - 0(h^{2}) \tag{3.34}$$

The centered difference approximation of the second derivative is given in equation (3.35)

$$y''x) = \frac{y(x+h)-2y(x)+y(x-h)}{h^{2}} \tag{3.35}$$

Equation (3.33) gives

$$y'_{i} = \frac{y_{i+1}-y_{i-1}}{2h} \tag{3.36}$$

And equation (3.35) gives

$$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \tag{3.37}$$

Equation (3.36) and (3.37) are called derivatives of finite difference approximation

3.6 Finite Difference Method to Solve BVP in an Ordinary Differential Equation

Consider a second order linear differential equation of the form

$$f\left(x, y, y', y''\right) = 0 \tag{3.38}$$

subject to the boundary conditions $y\left(x_0\right) = y_0$ & $y\left(x_n\right) = y_n$

STEP 1: find $h = \frac{x_n - x_0}{n}$ by choosing n. Thus, the x values are $x_0, x_1, x_2 \ldots x_{n-1}, x_n$

AIM: We need to find y values at $x_1, x_2, \ldots. x_{n-1}$

STEP 2: Replace the derivatives y', y'' in equation (3.38) by the finite difference approximation which are derived from the Taylor's series expansion

$y'_i = \frac{1}{2h}\left[y_{i+1} - y_{i-1}\right]$

$y''_i = \frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right]$

Replace x by x_i and y by y_i in equation (3.38) hence equation (3.38) is in the form of

$$f\left(x_i, y_i, y'_i, y''_i\right) = 0 \tag{3.39}$$

by giving $i = 1, 2, \ldots., n - 1$ in equation (3.8), a simultaneous system of $(n - 1)$ a linear equation will be generated. Thus, using known methods that's using substitution method, gaussian method or matrix method to solve the system of linear equations for $y_1, y_2, \ldots, y_{n-1}$.

CHAPTER FOUR

NUMERICAL EXAMPLES

In this chapter, we apply the Euler's method and the Finite difference method to obtain an approximate solution to initial and boundary value problems of ordinary differential equations.

Example 4.1: Solve by Euler's method, $y' = 3x + y$, $y(0) = -1$ and find $y(0.2)$taking step size $h = 0.04$. Compare the result obtained by this method with the result obtained by the analytical method.

Solution

Consider the differential equation in the form of $y' = f(x, y)$, $y(x_0) = y_0$

Here $f(x, y) = 3x + y,\ x_0 = 0,\ y_0 = -1$ and $h = 0.04$

recall from the Euler's formula from equation (3.23) we have ,

$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, 3....$

Taking $n = 0$

$$y_1 = y_0 + hf(x_0, y_o) \quad (4.1)$$

$$f(x_o, y_0)$$

substitute for y_0, h and

$$y_1 = -1 + 0.04f(0, -1)$$

$$y_1 = -1 + 0.04[3(0) + (-1)]$$

$$y_1 = -1.04 \quad (4.2)$$

Next, we have $x_n = x_{n-1} + h$ (4.3)

Taking into equation (4.3)

$$x_1 = x_0 + h \quad (4.4)$$

substitute x_o for and h into equation (4.4)

$$x_1 = 0 + 0.004$$

$$x_1 = 0.04 \quad (4.5)$$

Therefore $x_1 = 0.04$, $y_1 = -1.04$

Hence, taking n= 1 into the Euler's formula

$$y_2 = y_1 + hf\left(x_1, y_1\right) \quad (4.6)$$

substitute y_1, h for $f\left(x_1, y_1\right)$ and into equation (4.4)

$$y_2 = -1.04 + 0.04f(0.04, -1.04)$$

$$y_2 = -1.04 + 0.04[3(0.04) + (-1.04)]$$

$$y_2 = -1.0768 \quad (4.7)$$

Next, we have $x_n = x_{n-1} + h$ (4.8)

Taking $n = 2$ into equation (4.8)

$$x_2 = x_1 + h \quad (4.9)$$

substitute x_1 for h and into (4.9)

$$x_2 = 0.04 + 0.04$$

$$x_2 = 0.08 \tag{4.10}$$

Therefore $x_2 = 0.08$, $y_2 = -1.0768$

Hence, taking n=2 into Euler's formula

$$y_3 = y_2 + hf\left(x_2, y_2\right) \tag{4.11}$$

substitute y_2, h and $f\left(x_2, y_2\right)$ into equation (4.11)

$$y_3 = -1.0768 + 0.04f\,(0.08, -1.0768)$$

$$y_3 = -1.0768 + 0.04\left[3(0.08) + (-1.0768)\right]$$

$$y_3 = -1.1103 \tag{4.12}$$

Next, we have $x_n = x_{n-1} + h$ (4.13)

Taking $n = 3$ into equation (4.13)

$$x_3 = x_2 + h \tag{4.14}$$

substitute x_2 for h and into equation (4.14)

$$x_3 = 0.08 + 0.04$$

$$x_3 = 0.12 \tag{4.15}$$

Therefore $x_3 = 0.12$, $y_3 = -1.1103$

Hence, taking $n = 3$ into Euler's formula

$$y_4 = y_3 + hf\left(x_3, y_3\right) \tag{4.16}$$

substitute y_3, h for and $f\left(x_3, y_3\right)$ into equation (4.16)

$$y_4 = -1.103 + 0.04f\,(0.12, -1.1103)$$

$$y_4 = -1.103 + 0.04\left[3(0.12) + (-1.1103)\right]$$

$$y_4 = -1.1403 \tag{4.17}$$

Next, we have $x_n = x_{n-1} + h$ (4.18)

Taking $n = 4$ into equation (4.18)

$x_4 = x_3 + h$ (4.19)

substitute x_3 *and* h for into equation (4.19)

$x_4 = 0.12 + 0.04$

$x_4 = 0.16$ (4.20)

Therefore $x_4 = 0.16$, $y_4 = -1.1403$

Hence, taking $n = 4$ into Euler's formula

$y_5 = y_4 + hf(x_4, y_4)$ (4.21)

substitute y_4, h for $f(x_4, y_4)$ and into equation (4.21)

$y_5 = -1.1403 + 0.04f(0.16, -1.1403)$

$y_5 = -1.1403 + 0.04[3(0.16) + (-1.1403)]$

$y_5 = -1.1667$ (4.22)

Next, we have $x_n = x_{n-1} + h$ (4.23)

Taking $n = 5$ into equation (4.23)

$x_5 = 0.16 + 0.04$

$x_5 = 0.2$ (4.24)

Therefore $x_5 = 0.2$, $y_5 = -1.1667$

1.e $y(0.2) = -1.1667$

To compare with exact solution :

let us now find the exact solution of the given differential equation, the equation given as

$$y' = 3x + y \tag{4.25}$$

equation (4.25) can be written as

$$\frac{dy}{dx} - y = 3x \tag{4.26}$$

comparing equation (4.26) with linear differential equation in $\frac{dy}{dx} + P(x)y = Q(x)$ the $P = -1$ and $Q = 3x$ form of

The integrating factor $(IF) = e^{\int p dx}$

$$(IF) = e^{\int -dx} = e^{-x}$$

The general solution of equation (4.25) is $y(IF) = \int Q \times (IF)\, dx$

$$ye^{-x} = \int 3xe^{-x} dx$$

$$ye^{-x} = 3\int xe^{-x} dx + c \tag{4.27}$$

To get $\int xe^{-x} dx$, using integration by part we have

$$\int u dv = uv - \int v du$$

$$u = x,\ du = dx,\ dv = e^{-x},\ v = -e^{-x}$$

$$\int xe^{-x} dx = -xe^{-x} - \int -e^{-x} dx$$

$$\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx$$

$$\int xe^{-x} dx = -xe^{-x} - e^{-x} \tag{4.28}$$

substitute equation (4.28) into equation (4.27)

$$ye^{-x} = -3xe^{-x} - 3e^{-x} + c \tag{4.29}$$

To get c, using the IVP $y(0)=-1$

from equation (4.29) we have,

$$-1e^{-0}=-3\times 0e^{-0}-3e^{-0}+c$$
$$-1=0-3+c$$
$$-1=-3+c$$
$$c=2 \tag{4.30}$$

substitute for c into equation (4.29) we have,

$$ye^{-x}=-3xe^{-x}-3e^{-x}+2$$

divide both side by e^{-x} hence we have ,

$$y(x)=2e^{x}-3x-3 \tag{4.31}$$

Equation (4.31) gives the general solution, taking $x=0, 0.04, 0.08, 0.12, 0.16, 0.2$ into equation (4.31) . Hence we have,

$$y(0)=2e^{0}-3(0)-3=-1$$
$$y(0.04)=2e^{0.04}-3(0.04)-3=-1.0384$$
$$y(0.08)=2e^{0.08}-3(0.08)-3=-1.0734$$
$$y(0.12)=2e^{0.12}-3(0.12)-3=-1.1050$$
$$y(0.16)=2e^{0.16}-3(0.16)-3=-1.1330$$
$$y(0.2)=2e^{0.2}-3(0.2)-3=-1.1572$$

We shall tabulate the results as follows

x	0	0.04	0.08	0.12	0.16	0.2
Euler’s y	-1	-1.04	-1.0768	-1.1103	-1.1403	-1.1667
Exact y	-1	-1.0384	-1.0734	-1.1050	-1.1330	-1.1572

we notice that the approximate solution that,s the Euler’s of y is close to the exact solution.

Example 4.2 : solve the differential equation $y' = \sin(x) + 2y$, $y(0) = 1$ using Euler's method with the step size $h = 0.02$ to approximate $y(0.08)$.

Solution

$$y' = \sin(x) + 2y \ , \ y(0) = 1 \qquad (4.32)$$

Here $f(x, y) = \sin(x) + 2y \quad x_0 = 0 \ , y_0 = 1 \ and \ h = 0.02$,

By Euer's method

$$y_{n+1} = y_n + hf\left(x_n, y_n\right), n = 0, 1, 2, 3..$$

Taking $n = 0$ we have,

$$y_1 = y_0 + hf\left(x_0, y_0\right) \qquad (4.33)$$

Substitute for y_0, h and $f\left(x_0, y_0\right)$

$$y_1 = 1 + 0.02f(0, 1)$$

$$y_1 = 1 + 0.02\left[\sin(0) + 2(0)\right]$$

$$y_1 = 1.04 \qquad (4.34)$$

Next we have, $x_n = x_{n-1} + h$

Taking $n = 1$

$$x_1 = x_0 + h \qquad (4.35)$$

Substitute for x_0 and h into equation (4.35)

$$x_1 = 0 + 0.02$$

$$x_1 = 0.02 \qquad (4.36)$$

Therefore $x_1 = 0.02, \ y_1 = 1.04$

Hence taking $n = 1$ into the Euler's formula

$$y_2 = y_1 + hf\left(x_1, y_1\right) \qquad (4.37)$$

$$f\left(x_1, y_1\right)$$

substitute for $y_{1,} h$ and

$y_2 = 1.04 + 0.02 f(0.02, 1.04)$

$y_2 = 1.04 + 0.02[\sin(0.02) + 2(1.04)]$

$y_2 = 1.0816$ (4.38)

Next we have, $x_n = x_{n-1} + h$

taking $n = 2$

$x_2 = x_1 + h$ (4.39)

substitute for x_1 and h into equation

$x_2 = 0.02 + 0.02$

$x_2 = 0.04$ (4.40)

Therefore $x_2 = 0.04$, $y_2 = 1.0816$

Hence taking $n = 2$ into the Euler's formula

$y_3 = y_2 + hf\left(x_2, y_2\right)$ (4.41)

Substitute for y_2, h and $f\left(x_2, y_2\right)$ into equation (4.41)

$y_3 = 1.0816 + 0.02 f(0.04, 1.0816)$

$y_3 = 1.0816 + 0.02[\sin(0.04) + 2(1.0816)]$

$y_3 = 1.1249$ (4.42)

Next we have $x_n = x_{n-1} + h$

taking $n = 3$

$x_3 = x_2 + h$ (4.43)

substitute for x_2 and h into equation (4.44)

$x_3 = 0.04 + 0.02$

$x_3 = 0.06$

(4.44)

Therefore $x_3 = 0.06$, $y_3 = 1.1249$

Hence, taking $n = 3$ into the Euler's formula

$$y_4 = y_3 + hf\left(x_3, y_3\right) \quad (4.45)$$

Substitute for $y_{3,}$ h and $f\left(x_3, y_3\right)$ into equation (4.45)

$$y_4 = 1.1249 + 0.02f\,(0.06, 1.1249)$$

$$y_4 = 1.1249 + 0.02\left[\sin(0.06) + 2(1.1249)\right]$$

$$y_4 = 1.1699 \quad (4.46)$$

Next, we have $x_n = x_{n-1} + h$

taking $n = 4$

$$x_4 = x_3 + h \quad (4.47)$$

Substitute for x_3 and h into equation

$$x_4 = 0.06 + 0.02$$

$$x_4 = 0.08 \quad (4.48)$$

Therefore $x_4 = 0.08$, $y_4 = 1.1699$

i.e $y(0.08) = 1.1699$

Example 4.3: Solve the equation $y'' = x + y$ using the finite difference method with BVP $y(0) = y(1) = 0$.

Solution

$$y'' = x + y \,,\, y(0) = y(1) = 0 \quad (4.49)$$

Write the given ODE as $y\left(x_0\right) = y_0 \,,\, y\left(x_n\right) = y_n$

Hence $x_0=0, x_n=1, y_0=0, y_n=0$, choose n=4 (since n isn't given, we can choose n as we wish)

$$h=\frac{x_n-x_0}{n} \tag{4.50}$$

Substitute x_n, x_0 and n into equation (4.50)

$$h=\frac{1-0}{4}=\frac{1}{4}$$

To get the values of $x_1, x_2, ..x_n$.

taking $n=1,2,3$ and 4 into equation (4.50) and substitute for h

$$x_1=hn+x_0=\frac{1}{4}(1)+0=\frac{1}{4}$$

$$x_2=\frac{1}{4}(2)+0=\frac{1}{2}$$

$$x_3=\frac{1}{4}(3)+0=\frac{3}{4}$$

$$x_4=\frac{1}{4}(4)+0=1$$

note that x_4 gives the x_n values

Therefore $x_1=\frac{1}{4}$, $x_2=\frac{1}{2}$, $x_3=\frac{3}{4}$ and $x_4=1$, there are three internal points x_1, x_2 and x_3. So our aim is to find y values at x_1, x_2 and x_3. This implies that we are to find y_1, y_2 and y_3.

Replace y by y_i and x by x_i into equation (4.49) we have,

$$y''_i=x_i+y_i \tag{4.51}$$

From finite approximation ,

$$y''_i=\frac{1}{h^2}\left[y_{i+1}-2y_i+y_{i-1}\right]$$

Eaution (4.51) becomes

$$\frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right] = x_i + y_i \tag{4.52}$$

Substituting $h = \frac{1}{4}$ into equation (4.52)

$$\frac{1}{4^2}\left[y_{i+1} - 2y_i + y_{i-1}\right] = x_i + y_i$$

$$16\left[y_{i+1} - 2y_i + y_{i-1}\right] = x_i + y_i$$

$$16y_{i+1} - 32y_i + 16y_{i-1} = x_i + y_i$$

$$16y_{i+1} - 32y_i - y_i + 16y_{i-1} = x_i$$

$$16y_{i+1} - 33y_i + 16y_{i-1} = x_i \tag{4.53}$$

since there are three unknown values that's y_1, y_2 and y_3

let $i = 1, 2, 3$

Substitute for $i = 1$ from equation (4.53)

$$16y_2 - 33y_1 + 16y_0 = x_1$$

Recall $y_0 = 0$ and $x_1 = \frac{1}{4}$

$$16y_2 - 33y_1 + 16(0) = \frac{1}{4}$$

$$16y_2 - 33y_1 = \frac{1}{4} \tag{4.54}$$

Substitute for i=2 into equation (4.53)

$$16y_3 - 33y_2 + 16y_1 = x_2$$

Recall $x_2 = \frac{1}{2}$

$$16y_3 - 33y_2 + 16y_1 = \frac{1}{2} \tag{4.55}$$

Substitute for $i=3$ into equation (4.53)

$$16y_4 - 33y_3 + 16y_2 = x_3$$

Recall $y_n=0$ and since we choose $n=4$ then $y_4=0$

$$16(0) - 33y_3 + 16y_2 = \frac{3}{4}$$

$$-33y_3 + 16y_2 = \frac{3}{4} \qquad (4.56)$$

Now, we observe that equation (4.54), (4.55), and (4.56) represents a simultaneous system of linear equations with three unknown y_1, y_2 and y_3

We rearrange to have

$$-3y_1 + 16y_2 = \frac{1}{4}1$$

$$16y_1 - 33y_2 + 16y_3 = \frac{1}{2}$$

$$16y_2 - 33y_3 = \frac{3}{4}$$

Solving this simultaneous system of linear equation by using different method i.e, Substitution method, Matrix method e.t.c

$$y_1 = -0.0348$$

$$y_2 = -0.0563$$

$$y_3 = -0.05004$$

Therefore $y\left(\frac{1}{4}\right) = -0.0348$, $y\left(\frac{1}{2}\right) = -0.0563$ and $y\left(\frac{3}{4}\right) = -0.0500$

Example 4.4: Solve the second order differential equation using the finite difference method $y'' + xy' + y = 3x^2 + 2$ with BVP $y(0)=0,\ y(1)=1$ giving $n=3$

Solution

$$y'' + xy' + y = 3x^2 + 2\ ,\ \ y(0) = 0,\ y(1) = 1 \tag{4.57}$$

Write the given ODE as $y(x_0) = 0,\ y(x_n) = 1$

Hence $x_0 = 0, x_n = 1, y_0 = 0, y_n = 1$ and to find h we have,

$$h = \frac{x_n - x_0}{n} \tag{4.58}$$

Equation (4.58) can also be written as $x_n = hn + x_0$

Substituting for x_1, x_0 and n into equation (4.58)

$$h = \frac{1-0}{3}$$

$$h = \frac{1}{3} \tag{4.59}$$

To get the values for x_1, x_2 and x_3 we substitute for h and $n = 1, 2, 3$ in equation (4.58)

$$x_1 = \frac{1}{3}(1) + 0 = \frac{1}{3}$$

$$x_2 = \frac{1}{3}(2) + 0 = \frac{2}{3}$$

$$x_3 = \frac{1}{3}(3) + 0 = 1$$

Hence $x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3},\ x_3 = 1$. We see that x_3 gives the x_n values, there are two internal points. So our aim is to find y values at x_1, x_2. This implies that we are to find y_1 and y_2.

Replace y by y_i and x by x_i into equation (4.57)

$$y_i'' + x_i y'_i + y_i = 3x_i + 2 \tag{4.60}$$

From finite difference approximation

$$y'_i = \frac{1}{2h}\left[y_{i+1} - y_{i-1}\right]$$

$$y''_i = \frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right]$$

Equation (4.60) becomes

$$\frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right] + \frac{x_i}{2h}\left[y_{i+1} - y_{i-1}\right] + y_i = 3x_i + 2 \qquad (4.61)$$

Substituting for into equation (4.61)

$$9\left[y_{i+1} - 2y_i + y_{i-1}\right] + \frac{3x_i}{2}\left[y_{i+1} - y_{i-1}\right] + y_i = 3x_i^2 + 2$$

$$9y_{i+1} - 18y_i + 9y_{i-1} + \frac{3x_i y_{i+1}}{2} - \frac{3x_i y_{i-1}}{2} + y_i = 3x_i^2 + 2$$

$$9y_{i+1} + \frac{3x_i y_{i+1}}{2} - 18y_i + y_i + 9y_{i-1} - \frac{3x_i y_{i-1}}{2} = 3x_i^2 + 2 \qquad (4.62)$$

Since there are two unknown i.e y_1 and y_2

Let $i = 1, 2$

Substituting for $i = 1$ into equation (4.62)

$$9y_2 + \frac{3x_i y_2}{2} - 17y_1 + 9y_0 - \frac{3x_1 y_0}{2} = 3x_i^2 + 2$$

Recall $x_1 = \frac{1}{3}$ and $y_0 = 0$

$$9y_2 + \frac{3}{2}\left(\frac{1}{3}\right)y_2 - 17y_1 + 9(0) - \frac{3}{2}\left(\frac{1}{3}\right)(0) = 3\left(\frac{1}{3}\right)^2 + 2$$

$$\frac{19y_2}{2} - 17y_1 = \frac{7}{3}$$

Substituting for $i = 2$ into (4.62)

$$9y_3 + \frac{3x_2y_3}{2} - 17y_2 + 9y_1 - 3x_2y_1 = 3x^2{}_2$$

Recall $x_2 = \frac{2}{3}$ since $y_n = 1$ and $n = 3$ then $y_3 = 1$

$$9(1) + \frac{3}{2}\left(\frac{2}{3}\right)(1) - 17y_2 + 9y_1 - \frac{3}{2}\left(\frac{2}{3}\right)y_1 = 3\left(\frac{2}{3}\right)^2 + 2$$

$$10 - 17y_2 + 8y_1 = \frac{10}{3}$$

$$-17y_2 + 8y_1 = \frac{10}{3} - 10$$

$$-17y_2 + 8y_1 = -\frac{20}{3} \quad (4.64)$$

Now, we observe that equation (4.63) and (4.64) represents a simultaneous system of linear equation with two unknown values y_1 and y_2

We rearrange to have

$$-17y_1 + \frac{19y_2}{2} = \frac{7}{3}$$

$$8y_1 - 17y_2 = -\frac{20}{3}$$

Solving this simultaneous system of linear equation by using different method i.e Substitution Method, Matrix Method e.t.c

$y_1 = 0.1111$

$y_2 = 0.444$

Therefore $y\left(\frac{1}{3}\right) = 0.1111$, $y\left(\frac{2}{3}\right) = 0.444$

CHAPTER FIVE
SUMMARY AND CONCLUSION

5.1 Summary

In this study, we introduce numerical methods for solving differential equations. Differential equations are mathematical equations that describe the relationship between a function and its rate of change. They are widely used in science and engineering to model various phenomena. We discuss some basic definitions and two main types of problems: initial value problems (IVPs) and boundary value problems (BVPs). IVPs involve finding a function that satisfies a differential equation with a given starting condition, whereas BVPs involve finding a function that satisfies a differential equation along with specific conditions at two different points. Sometimes it is difficult, if not impossible, to solve a differential problem (that is, find the exact solution). Then iterative methods come in, which provide approximate solutions to a given differential equation. These methods are particularly useful because they offer better accuracy and error control compared to simpler analytical approaches. Different iterative methods, like Euler's method and the finite difference method (FDM), are used to solve IVPs and BVPs.

The literature review dives into the history and details of each method, including Euler's method, the Runge-Kutta method's development by mathematicians, and the advantages and disadvantages of FDM and FEM. It reviews common numerical methods for solving differential equations. It highlights that exact solutions aren't always achievable, and these methods provide approximations. The chapter also discusses two main categories of differential equations: initial value problems (IVPs) and boundary value problems (BVPs). It then covers specific methods for solving each type. Euler's method and Runge-Kutta methods solve for IVPs, and the finite difference method (FDM) and finite element method (FEM) solve for BVPs. Our aim is to use the Euler's method and the finite method to solve differential problems with initial and boundary value problems.

In the methodology chapter, it begins by discussing the existence and uniqueness theorem, which establishes conditions under which solutions to first-order ODEs exist and are unique. Next, we introduce Euler's method, a numerical method for solving initial value problems (IVPs) in ODEs. The derivation of Euler's method is presented, showing how it can be used to approximate the solution of a first-order ODE given initial conditions. Furthermore, the finite difference method (FDM) is introduced as a numerical method for solving boundary value problems (BVPs) in ODEs. The FDM involves approximating derivatives with finite differences, using Taylor series expansions to derive forward, backward, and centered difference approximations. The text outlines the application of the finite difference method to solve second-order linear differential equations subject to boundary conditions, providing a step-by-step approach for implementation.

In chapter four, we solved four numerical examples by using Euler's method to solve differential equations with initial value problems (IVPs) and the finite difference method (FDM) to solve differential equations with boundary value problems (BVPs).

5.2 Conclusion

Iterative methods are numerical methods in which a collection of mathematical methods is repeatedly applied to a differential equation. From this project, it is safe to say that iterative methods are a vital strand of mathematics. They are powerful tools for solving initial and boundary value problems, especially in the cases where exact analytical solutions are difficult or impossible to solve. Only two numerical methods were examined in this project; however, there are several others that are also as important.

One of the most widely used iterative methods for solving initial value problems (IVP) is the Euler's method. The method involves dividing the time interval into small steps and using the slope of the tangent line at the beginning of each step to estimate the value of the solution at the end of the step.

Another commonly used iterative method for solving boundary value problems (BVP) is the finite difference method (FDM). The finite differences method is a numerical method used to approximate solutions to differential equations by approximating derivatives with finite difference approximations. It involves discretizing the domain of the problem and replacing derivatives with finite difference formulas.

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