# THE INVERSE BURR TOPP-LEONE DISTRIBUTION: ITS PROPERTIES AND APPLICATION TO GERMINATION RATE OF OIL PALM SEEDS (*Elaeis guineensis* Jacq.)

**BY**

**STANLEY OSAROGIAGBON UYI**
**M.Sc. Industrial Mathematics (Statistics Opt.) [BENIN]**
**PG/PSC0704430**

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## UNDERTAKING

This thesis was carried out by me, **Stanley Osarogiagbon UYI** with matriculation number **PG/PSC0704430**. I have neither copied nor duplicated the work of any author(s). All work used are duly cited and acknowledged.

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**SIGN/DATE**

**Stanley Osarogiagbon UYI**

## CERTIFICATION

This is to certify that the work reported in this thesis was carried out by **Stanley Osarogiagbon UYI** in the Department of Statistics, University of Benin, Benin City.

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**Dr. OSAGIE S. A.**
**(SUPERVISOR)**

______________________________

**Prof. J. E. OSEMWENKHAE**
**(Co-SUPERVISOR)**

______________________________

**Prof. N. EKHOSUEHI**
**(HEAD OF DEPARTMENT)**

## DEDICATION

This thesis is dedicated to Almighty God.


## ACKNOWLEDGEMENT

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## TABLE OF CONTENTS

## LIST OF TABLES

## LIST OF FIGURES

**Abstract**

Several probability models have been introduced in literature to model data sets arising from different real-world scenarios. Such applications are found in the field of engineering, biological sciences, finance, demography, actuarial sciences, agricultural sciences, etc.

In this study, we develop a hybrid statistical model which is a composition of the inverse Burr and Topp-Leone distribution. We refer to this model as the inverse Burr Topp-Leone (IBTL) distribution. Basic statistical properties of the IBTL distribution are derived. The method of maximum likelihood estimation is employed to obtain the unknown parameters of the IBTL distribution. A Monte Carlo simulation study is carried out to investigate the asymptotic behavior of the maximum likelihood estimates of the parameters of the IBTL distribution.

Finally, two real data sets comprising of the germination rate of oil palm seeds are adapted to illustrate the usefulness of the proposed IBTL distribution in real-life data fittings. For the purpose of model comparison, we considered some existing unit-interval distributions. Result obtained from the analysis revealed that the IBTL distribution outperformed the competitor distributions and thus, becomes a useful alternative to other existing unit-interval distributions in real-life data fittings.

# CHAPTER ONE

# INTRODUCTION

## 1.1 Background of the Study

Oil palm (***Elaeis guineensis* Jacq**) is an oil-producing crop extracted from the fleshy mesocarp of the palm fruit. It is one of the highest oil-producing seed among oil seed crops in the world (Morley, 2015). Thus, there is an increasing market demand globally for several purposes, which range from food (consumption) to production of cosmetics. Before the advent of crude oil, Nigeria was recorded among the largest producers of palm oil globally and have benefited greatly from its export.

In spite of the usefulness of the oil palm, several researches have shown that the germination of oil palm seed is very difficult and time demanding in breaking dormancy, leading to high labour and production cost.

Corrado and Wuidart (1990) suggested that oil palm seeds can be germinated by soaking it in water and treating with fungicide then drying to 19% moisture content, sealing in polyethylene bags and heating at 39°C for 80 days (this procedure is used to break seed dormancy). After this they are soaked in water for another 5 days, treat again with fungicide, dried to 23-25% moisture content and kept in bags at 25-27°C, with regular inspections to remove those which have germinated. From their investigation, germination of oil palm seed reaches 80% at 3-5 weeks after the end of the high temperature treatment and if seeds are stored for at least 6 months before the germination treatment, a rate of 90% is often reached.

Wonkyi-Appiah (1974) reported a heating duration of 70 days and a temperature range of 39-40$^0$C for germinating oil palm seeds in Ghana. Dickson et al. (2021) also conducted a study to evaluate the effects of pre-soaking seeds for 4-7days and employing three heat treatment durations of 60, 65 and 70 days at 38-40$^0$C on the germination rate of three tenera oil palm progenies.

Further studies on the germination of oil palm seeds are found in the works of Koornnef et al. (2002), Norsazwan et al. (2016), Wang et al. (2019), Cui et al. (2020), etc. Figures 1 and 2 show pictorial presentation in the seed production section of the Nigerian Institute for Oil Palm Research (NIFOR), Edo State, where data will be collected for the analysis of this study.

**Figure 1:** Oil Palm Seed Exposed to Heat Temperature to Break Dormancy

**Figure 2:** Selection of sprouted seeds after some weeks of heat exposure

Figure 1 shows the palm oil seeds sealed in polythene sacs in batches exposed to temperature ranging from 39-40$^0$C in order to break seed dormancy. This process can last for up to 80days before sprouting. Also, Figure 2 shows the selection process of seeds that has sprouted after the period of breaking dormancy. This process of seedling selection can be carried out weekly until the sprouting of a considerable number of seeds in the sac.

## 1.2 Basic Mathematical Properties of Lifetime Distribution

### 1.2.1 Probability Density Function and Cumulative Distribution Function

The probability density function (pdf) of a random variable, say $Y$, is a mathematical function that describes the distribution. It is the probability that the variable $Y$ has the value $y$, (i.e., $\Pr(Y = y)$ ). The density function of a random variable can be expressed mathematically or by graphical representation. Various shapes exhibited by the pdf of a distribution include a unimodal (left skewed, right skewed or symmetric), bimodal and exponential decreasing shapes.

Mathematically, the density function of a probability distribution is defined by

$$\int_a^b f(y)\,dy = \Pr[a \le y \le b]. \tag{1.1}$$

Also, the cumulative distribution function (cdf) of a probability distribution is the probability that the variable $Y$ takes a value less than or equal to $y$,(i.e., $\Pr(Y \le y)$ ).

For a continuous distribution, this can be expressed as

$$F(y) = \Pr[Y \leq y] = \int_{-\infty}^{y} f(x)\,dx, \quad 0 \leq y \leq 1. \tag{1.2}$$

### 1.2.2 Survival Function and Hazard Rate Function

The survival function is the probability that the variable *Y* takes a value greater than *y*, (i.e., $\Pr(Y \leq y)$), and it is mathematically defined by

$$S(y) = 1 - F(y). \tag{1.3}$$

Suppose *T* be a non-negative variable which determines the working time of a device, then *T* is called the Lifetime or Survival time. So that the survival time of a unit at time *t* is defined as the probability that the working time of this unit is greater than *t*. On the other hand, the hazard rate function is the ratio of the density function to the survival function. Mathematically, it is defined by

$$h(y) = \frac{f(y)}{1 - F(y)}. \tag{1.4}$$

Equation (1.4) measures the rate of failure of a device after working up to time *T*. Graphical representation of the hazard rate function of a probability distribution has found to exhibit either decreasing, increasing, constant, bathtub-shaped or inverted bathtub-shaped hazard properties.

### 1.2.3 Quantile Function (Inverse Cumulative Distribution Function)

For a given probability in the probability distribution of a random variable, the quantile function specifies the value at which the probability of the random variable is less than or equal to the given probability. It is mathematically defined by

$$Q_Y(u) = F^{-1}(u), \quad 0 < u < 1. \tag{1.5}$$

Generally, the quantile function of a probability distribution is used to generate random samples from the distribution.

### 1.2.4 Moments

A probability distribution can be characterized by its moments. The moment is classified into $r^{th}$-raw moments (about the origin) and $k^{th}$-central moment (about the mean). It gives information about the mean, variance, standard deviation, measures of skewness and kurtosis of a distribution.

Johnson et al. (1994) defined the $r^{th}$-raw moments and $k^{th}$-central moment respectively as

$$\mu_r' = E(Y^r) = \int_{-\infty}^{\infty} y^r f(y)\,dy, \quad r = 1,2,3,4... \tag{1.6}$$

and

$$\mu_r = E(Y-\mu)^k = \int_{-\infty}^{\infty} (y-\mu)^k f(y)\,dy, \quad k = 2,3,4... \tag{1.7}$$

### 1.2.5 Moment generating function

Let $Y$ be a continuous random variable following a known probability distribution with density function $f(y)$, then the moment generating function of $Y$ is defined as

$$M_Y(t) = E\left[e^{ty}\right] = \int_{-\infty}^{\infty} e^{ty} f(y)\,dy. \tag{1.8}$$

By the Maclaurin series expansion of the exponential function, we have

$$e^{ty} = \sum_{n=o}^{\infty} \frac{(ty)^n}{n!},$$

so that equation (1.8) now becomes

$$M_Y(t) = \sum_{n=o}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} y^n f(y)\,dy. \tag{1.9}$$

Generating functions are known to determine the distribution of a random variable, while the moments of a random variable can be obtained from either the derivatives of the generating function, or, the coefficients in the power series expansion of the generating function (See Nduka and Igabari, 2007).

### 1.2.6 Probability Weighted Moments (PWMs)

Suppose $Y$ is a continuous random variable from a known probability distribution with density function $f(y)$, and cumulative distribution function $F(y)$, Greenwood et al. (1979) defined the $(s,k)^{th}$ PWMs of $Y$ as

$$\rho_{s,k} = E\left(Y^k F(y)^s\right) = \int_{-\infty}^{\infty} y^k f(y) F(y)^s\,dy, \tag{1.10}$$

The probability weighted moments (PWMs) are generally used to construct the estimator of the parameters as well as the quantiles of a known statistical distribution whose cdf is invertible.

### 1.2.7 Entropy

Entropy is an imperative concept in probability theory with extensive applications in various areas such as physics, communication, signal processing, etc. An entropy of a random variable $Y$ is defined as the degree of variation or randomness associated with $Y$. Renyi (1961) defined the Renyi entropy of $Y$ as

$$\tau_R(\varpi) = \frac{1}{1-\varpi}\log\int_{-\infty}^{\infty} f^{\varpi}(y)\,dy, \qquad \varpi > 0,\ \varpi \neq 1. \tag{1.11}$$

### 1.2.8 Distribution of order statistics

Let $(Y_1, Y_2, ..., Y_n)$ be random samples of size $n$ from a known probability distribution with density function $f(y)$, and cumulative distribution function $F(y)$. Suppose $Y_{k:n}$ denotes the $k^{th}$ order statistics, then the density function of $Y_{k:n}$ is defined by

$$f_{k:n}(y) = \frac{1}{B(k, n-k+1)}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j f(y)F(y)^{k+j-1}. \tag{1.12}$$

### 1.2.9 Parameter estimation

Parameter estimation is very crucial in every statistical models. It gives an information about how well an estimate of the parameter of a model predicts the true value of the parameter. There are different parameter estimation techniques in literature which include the maximum likelihood, moments, L-moments, least squares, weighted least squares, Anderson-Darling, Cramer-von Mises, etc. This study examines the maximum likelihood estimation technique.

#### 1.2.9.1 Maximum likelihood estimation (MLE)

Let $(Y_1, Y_2, ..., Y_n)$ be random samples of size *n* from a known probability distribution with pdf $f(y)$. the likelihood function of the distribution is obtained as

$$L(y) = \prod_{i=1}^{n} f(y_i), \tag{1.13}$$

By taking the natural logarithm of equation (1.13), the log-likelihood function of the distribution is obtained as

$$\ell(y, \vartheta) = \sum_{i=1}^{n} \log\left(f\left(y_i\right)\right). \tag{1.14}$$

The score function is obtained by taking the first partial derivative of equation (1.14) with respect to the parameter and equating to zero as

$$\frac{\partial \ell(y, \vartheta)}{\partial \vartheta} = 0. \tag{1.15}$$

Where $\vartheta$ is a vector parameter.

### 1.3 Statement of the Problem

Seed germination refers to the physiological and developmental processes that resume in non-dormant seeds when they are exposed to appropriate conditions of water availability, temperature and other physiochemical factors.

Germination experiments are conducted by placing groups of seeds in containers or sac bags, under controlled temperature and light conditions. Seeds are checked for germination on a sequence of observation (days/weeks) over a fixed period of time.

On each observation day, germinated seeds are counted and removed, yielding a sequence of germination numbers. Hence, germination data can be classified as time-to-event data.

Time-to-event analysis can be approached by three different ways: non-parametric (life-table, Kaplan Meier estimator, log-rank test, etc.), semi-parametric (Cox proportional hazard model) and fully parametric (probability distributions).

According to Bhati et al. (2015), in modeling time-to-event data, the monotonicity nature of the hazard rate function is uncertain, thus, the choice of a model becomes challenging. In such situations, one may readily resort to non-parametric model. However, if a parametric model fits the particular data, it is preferable to choose such model which could be a lifetime distribution.

To the best of our knowledge, there is no current literature where germination data are fitted using lifetime distribution. It is in this direction, we attempt to introduce a new lifetime distribution capable of fitting germination data, in particular, oil palm seed germination data and other agricultural data sets.

### 1.4 Aim and Objectives of the Study

The aim of this study is to develop a new lifetime distribution capable of fitting the germination rate of oil palm seeds.

The objectives of the study are to:

i. construct the inverse Burr Topp-Leone distribution;

ii. study useful statistical properties of the inverse Burr Topp-Leone distribution;

iii. present useful graphical plots of the density function and hazard rate function of the inverse Burr Topp-Leone distribution;

iv. obtain the parameter estimate of the inverse Burr Topp-Leone distribution via the maximum likelihood estimation (MLE) method and investigate the asymptotic behavior of the parameter estimates through a Monte-Carlo simulation study;

v. use inverse Burr Topp-Leone distribution alongside with other existing unit-interval lifetime distributions to fit the germination rate of oil palm seeds data.

## 1.5 Organization of the Study

This research study is organized into five (5) chapters. In chapter one, a general background of germination of oil palm seeds, some basic mathematical properties of lifetime distribution, the statement of problem and objectives of the study are established. Chapter two presents a detailed review of literature on statistical distributions derived by adapting the inverse Burr (Burr III) as well as the Topp-Leone distributions as the baseline distribution. Chapter three presents the formulation of the inverse Burr Topp-Leone (IBTL) distribution. Basic mathematical properties of IBTL distribution are derived, while the method of maximum likelihood estimation is employed for model parameter estimation. A Monte Carlo simulation study is conducted to examine the performance and accuracy of the maximum likelihood estimate of the parameters of the IBTL distribution. In chapter four, two real data sets comprising of the germination rate of oil palm seeds is adapted to illustrate the usefulness of the proposed IBTL distribution in real-life data fittings. The fit of IBTL distribution will be compared with the fits of some existing unit-interval lifetime distributions. Finally, chapter five identifies the findings of the study, the contribution to knowledge and suggests possible research area for further studies.

# CHAPTER TWO

# LITERATURE REVIEW

## 2.1 Introduction

Burr (1942) has introduced a family of lifetime distributions (Burr type II, III, X, XII) which have been widely studied and applied to fit lifetime data. Many researchers have shown much interest in the Burr type XII distribution since it could be used to fit almost any form of unimodal (left-skewed, right-skewed or symmetric) data. Silva et al. (2008) developed the log Burr XII regression model, Silva and Cordeiro (2015) introduced the Burr XII power series distribution, Lanjoni et al. (2015) introduced the extended Burr XII regression models, Osatohanmwen et al. (2019) proposed the Gumbel Burr XII distribution, Korkmaz and Chesneau (2021) studied the unit-Burr XII distribution, etc.

Tadikamalla (1980) noted that while the Burr type XII distribution yields a wide range of values of skewness $\left(\sqrt{\beta_1}\right)$ and kurtosis $\left(\beta_2\right)$, the inverse of the Burr type XII distribution referred to as Burr III distribution covers a wider region in the $\left(\sqrt{\beta_1}, \beta_2\right)$ plane than the Burr type XII distribution, the Weibull family, the gamma family, the log-normal family, the normal distribution, the logistic distribution, etc. On this note, Osemwenkhae and Iyenoma (2018) gave a comprehensive study on the mathematical properties of the inverse Burr (Burr III) distribution and illustrated the applicability of the distribution in lifetime data analysis. More study on the Burr III distribution can be found in the works of Burr and Cislak (1968), Johnson et al. (1995), Modi and Gill (2020), etc.

The Burr III (Inverse Burr) distribution is defined by the cumulative distribution function obtain as

$$G(t) = \left(1 + t^{-a}\right)^{-b}, \tag{2.1}$$

with the corresponding density function given as

$$g(t) = abt^{-(a+1)}\left(1 + t^{-a}\right)^{-(b+1)}, \quad t > 0, a, b > 0, \tag{2. 2}$$

where *a* and *b* are shape parameters

Due to the flexibility of the Burr III distribution, several researchers have adopted its distribution function defined in equation (2.1), as the baseline distribution in many developed methodologies. The following sections are dedicated to review of generalized distributions where the Burr III distribution was used as a baseline distribution.

## 2.2 Generalized Distributions with the Burr IIIdistribution as the baseline distribution.

### 2.2.1 The Transmuted Burr III distribution

Shaw and Buckley (2007) suggested a method of generalization by adding extra parameter to the cumulative distribution function of an existing lifetime distribution and called it the transmuted version of the distribution. Suppose a random variable $T$ is defined by the cdf, $G(t,\varphi)$, then the cdf of the transmuted version of $T$ is defined as

$$F(t,\lambda,\varphi) = (1+\lambda)G(t,\varphi) - \lambda\left[G(t,\varphi)\right]^2, \quad t>0, \ |\lambda|\le 1, \tag{2.3}$$

Differentiating equation (2.3), the pdf is obtained as

$$f(t,\lambda,\varphi) = \left[(1+\lambda) - 2\lambda G(t,\varphi)\right]g(t,\varphi), \quad t>0, \ |\lambda|\le 1, \tag{2.4}$$

Abdul-Moniem (2015) utilized the cdf of the Burr III distribution defined in equation (2.1) to introduce the transmuted Burr III (TBIII) distribution with the cdf and pdf, respectively, defined as follows

$$F(t,\lambda,a,b) = \left(1+t^{-a}\right)^{-b}\left[(1+\lambda) - \lambda\left(1+t^{-a}\right)^{-b}\right], \quad t>0, \ |\lambda|\le 1, \tag{2.5}$$

Differentiating equation (2.3), the pdf is obtained as

$$f(t,\lambda,\varphi) = abt^{-(a+1)}\left(1+t^{-a}\right)^{-(b+1)}\left[(1+\lambda) - 2\lambda\left(1+t^{-a}\right)^{-b}\right], \quad t>0, \ |\lambda|\le 1 \tag{2.6}$$

The author obtained other statistical properties such as the survival, hazard rate and reversed hazard rate functions, moments, variance, and coefficient of variation. The maximum likelihood estimation method was employed to estimate the unknown parameters of the TBIII distribution and a real data set comprising the life of fatigue fracture of Kevlar 370/epoxy was used to show the effectiveness of the proposed TBIII distribution over the Burr III distribution. We noticed that some useful statistical properties such as probability weighted moments, entropy (Renyi and Shannon), and stress-strength reliability were not studied, thus limiting the applicability of the distribution in engineering field. Although, the authors obtained the maximum likelihood estimates of the proposed distribution, they failed to examine the performance of the estimates via Monte Carlo simulation study.

### 2.2.2 The Kumaraswamy Burr III distribution

Cordeiro and de Castro (2011) introduced the Kumaraswamy-generated (Kum-G) family of distributions as a generalization of the Kumaraswamy distribution studied by Kumaraswamy (1980). They obtained the cumulative distribution function (cdf) and the probability density function (pdf) of the Kum-G family as

$$F(t,c,k,\varphi) = 1-\left(1-\left[G(t,\varphi)\right]^{c}\right)^{k}, \quad t>0, \quad c,k>0, \tag{2.7}$$

and

$$f(t,c,k,\varphi) = ckg(t,\varphi)\left[G(t,\varphi)\right]^{c-1}\left(1-\left[G(t,\varphi)\right]^{c}\right)^{k-1}, \quad t>0, \quad c,k>0, \tag{2.8}$$

where $c$ and $k$ are shape parameters and $G(t,\varphi)$ is the baseline distribution with vector parameter $\varphi$.

Behairy et al. (2016) considered the Burr III distribution defined in equation (2.1) as the baseline distribution in equation (2.7) to develop the Kumaraswamy Burr III (KBIII) distribution with the cdf obtained as

$$F_{KBIII}\left(t,a,b,c,k\right) = 1-\left(1-\left(1+t^{-a}\right)^{-bc}\right)^{k}, \tag{2.9}$$

and the corresponding pdf defined as

$$f_{KBIII}\left(t,a,b,c,k\right) = abckt^{-(a+1)}\left(1+t^{-a}\right)^{-(bc+1)}\left(1-\left(1+t^{-a}\right)^{-bc}\right)^{k-1}. \tag{2.10}$$

Other statistical properties of the KBIII distribution derived by the authors include the survival and hazard rate functions, quantile function, median, moments, skewness and kurtosis, Renyi entropy and distributions of order statistics. The maximum likelihood estimation method was employed by the authors to estimate the unknown parameters of the KBIII distribution and a Monte Carlo simulation study was conducted to examine the performance of the parameter estimates. Unfortunately, the usefulness of the KBIII distribution in lifetime data analysis was omitted by the authors.

### 2.2.3 The Marshall-Olkin Extended Burr III distribution

The Marshall-Olkin extended family of distributions was originally developed by Marshall and Olkin (1997). They defined the cumulative distribution function of the Marshall-Olkin extended family as

$$F\left(t,\alpha,\varphi\right) = \frac{G\left(t,\varphi\right)}{1-\bar{\alpha}\bar{G}\left(t,\varphi\right)}, \quad t>0, \quad \alpha>0, \tag{2.11}$$

and the pdf defined as

$$f(t,\alpha,\varphi) = \frac{\alpha g(t,\varphi)}{\left[1-\bar{\alpha}\bar{G}(t,\varphi)\right]^2}, \quad t>0, \quad \alpha>0, \; \bar{\alpha}=(1-\alpha) \tag{2.12}$$

By substituting equation (2.1) into equation (2.11), Al-Saiari et al. (2016) proposed the Marshall-Olkin Burr III (MOBIII) distribution with cdf defined as

$$F(t,a,b,\alpha) = \frac{\left(1+t^{-a}\right)^{-b}}{1-(1-\alpha)\left[1-\left(1+t^{-a}\right)^{-b}\right]}, \quad t>0, \quad \alpha>0, \tag{2.13}$$

and

$$f(t,a,b,\alpha) = \frac{ab\alpha t^{-(a+1)}\left(1+t^{-a}\right)^{-(b+1)}}{\left[1-(1-\alpha)\left[1-\left(1+t^{-a}\right)^{-b}\right]\right]^2}, \quad t>0, \quad \alpha>0. \tag{2.14}$$

Various mathematical properties of the MOBIII distribution which include the survival and hazard rate functions, quantile function, median and mode were derived. The maximum likelihood and Bayesian estimation methods were employed to estimate the unknown parameters of the MOBIII distribution. A simulation study was carried out to investigate the asymptotic behavior of the estimates and results obtained revealed that the Bayesian estimators were better than maximum likelihood estimators of the parameters of MOBIII distribution. A real data set which consists of 36 observations on chromium in marine water was used to check the validity of the MOBIII distribution in lifetime data analysis. A likelihood ratio test (LRT) was conducted to test if the MOBIII distribution was statistically better than the Burr III distribution. Result obtained from the data fitting indicated that the MOBIII distribution is statistically better than the Burr III distribution. Although, the authors did excellent work, nevertheless, some useful statistical properties such as the moments, moment generating function, probability weighted moments, entropy (Renyi and Shannon), stress-strength reliability and distribution of order statistics were omitted.

### 2.2.4 The Weibull Burr III distribution

Bourguignon et al. (2014) suggested another form of Weibull generalized family of distributions with cdf and pdf, respectively, defined as follows

$$F(t,\alpha,\beta,\varphi) = 1 - \exp\left(-\alpha\left[\frac{G(t,\varphi)}{1-G(t,\varphi)}\right]^{\beta}\right), \quad t>0, \quad \alpha,\beta>0, \tag{2.15}$$

and

$$f(t,\alpha,\beta,\varphi) = \alpha\beta g(t,\varphi)\frac{\left[G(t,\varphi)\right]^{\beta-1}}{\left[1-G(t,\varphi)\right]^{\beta+1}}\exp\left(-\alpha\left[\frac{G(t,\varphi)}{1-G(t,\varphi)}\right]^{\beta}\right), \quad t>0, \quad \alpha,\beta>0. \tag{2.16}$$

Yakubu and Doguwa (2017) considered the Burr III distribution as the baseline distribution in equation (2.15) to introduce a four-parameter Weibull Burr III (WBIII) distribution. The cdf and pdf of WBIII distribution are, respectively, defined as follows

$$F(t,a,b,\alpha,\beta) = 1 - \exp\left(-\alpha\left[\left(1+t^{-a}\right)^{b}-1\right]^{-\beta}\right), \quad t>0, \tag{2.17}$$

and

$$f(t,a,b,\alpha,\beta) = ab\alpha\beta t^{-(a+1)} \frac{\left(1+t^{-a}\right)^{-(\beta b+1)}}{\left[1-\left(1+t^{-a}\right)^{-b}\right]^{\beta+1}} \exp\left(-\alpha\left[\left(1+t^{-a}\right)^{b}-1\right]^{-\beta}\right), \quad t>0. \quad (2.18)$$

Other statistical properties of the WBIII distribution such as the survival and hazard rate functions, quantile function, median, moments, skewness, kurtosis, Renyi entropy and order statistics. The method of maximum likelihood was employed to estimate the unknown parameters of the WBIII distribution but the authors did not investigate the performance of the estimates through Monte Carlo simulation study. Two data sets consisting of the remission times (in months) of 128 bladder cancer patients and the survival times (in months) of 101 patients with advanced acute myelogenous leukemia were used to show the relevance of the WBIII distribution in lifetime data fittings.

### 2.2.5 The Gamma Burr III distribution

Zografos and Balakrishnan (2009) have proposed a family of distributions by taking the gamma distribution as a generator. They defined the cdf of the gamma-generated family of distributions as

$$F(t,c,\varphi) = \frac{1}{\Gamma(c)}\gamma\left(c,-\log\left[1-G(t,\varphi)\right]\right), \quad t>0, \quad c>0, \quad (2.19)$$

and the corresponding pdf is obtained as

$$f(t,c,\varphi) = \frac{1}{\Gamma(c)}\left(-\log\left[1-G(t,\varphi)\right]\right)^{c-1} g(t,\varphi), \quad t>0, \quad c>0. \quad (2.20)$$

Cordeiro et al. (2017) utilized the cdf of Burr III distribution as the baseline distribution in equation (2.19) to develop the Gamma Burr III (GBIII) distribution with cdf and pdf, respectively, defined as follow

$$F(t,a,b,c) = \frac{1}{\Gamma(c)}\gamma\left(c,-\log\left[1-\left(1+t^{-a}\right)^{-b}\right]\right), \quad t>0, \quad a,b,c>0, \tag{2.21}$$

and

$$f(t,a,b,c) = \frac{ab}{\Gamma(c)}t^{-(a+1)}\left(1+t^{-a}\right)^{-(b+1)}\left(-\log\left[1-\left(1+t^{-a}\right)^{-b}\right]\right)^{c-1}, \quad t>0. \tag{2.22}$$

The authors further derived some statistical properties which include the linear representation of the pdf and cdf, survival and hazard rate functions and quantile function. The maximum likelihood estimation method was adopted for model parameter estimation and the performance of the estimates were investigated through a simulation study. The authors also considered the quantile regression of the GBIII distribution called log-gamma Burr III regression model. A data set representing carbon monoxide measurement made in several brands of cigarettes in 1994 was used to show the flexibility of the GBIII distribution over the Burr III, log-logistic (LL), exponentiated log-logistic (ELL), beta Burr III (BBIII) distributions.

### 2.2.6 The Generalized Marshall-Olkin extended Burr III distribution

Chakraborty et al. (2019) proposed another four-parameter generalization of the Burr III distribution using the generalized Marshall-Olkin methodology. They defined the cdf of the new distribution as

$$F(t,a,b,\alpha,\theta) = 1-\left[\frac{\left(1+t^{-a}\right)^{-b}}{1-(1-\alpha)\left[1-\left(1+t^{-a}\right)^{-b}\right]}\right]^{\theta}, \quad t>0, \tag{2.23}$$

and

$$f(t,a,b,\alpha,\theta) = \frac{ab\theta\alpha^{\theta}\, t^{-(a+1)}\left(1+t^{-a}\right)^{-(b+1)}\left[1-\left(1+t^{-a}\right)^{-b}\right]^{\theta-1}}{\left[1-(1-\alpha)\left[1-\left(1+t^{-a}\right)^{-b}\right]\right]^{\theta+1}}, \quad t>0. \tag{2.24}$$

A random variable *T* having the cdf and pdf in equations (2.23) and (2.24) is said to follow the generalized Marshall-Olkin extended Burr III (GMOBIII) distribution.

Other statistical properties derived by the authors include the survival and hazard rate functions, quantile function, moments, moment generating function, probability weighted moments, Renyi entropy and stochastic ordering. The method of maximum likelihood was employed to estimate the model parameters and a simulation study is conducted to investigate the performance of the parameter estimates. A data set representing the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli was adopted to illustrate the suitability of GMOBIII distribution in fitting lifetime data sets. The likelihood ratio test (LRT) was used to show the flexibility of the GMOBIII distribution over its sub-models (MOBIII and Burr III distributions).

### 2.2.7 The Marshall-Olkin Modified Burr III (MOMBIII) distribution

Haq et al. (2021) developed the Marshall-Olkin modified Burr III (MOMBIII) distribution by utilizing the modified Burr III distribution as the baseline distribution function in the Marshall-Olkin technique in equation (2.11). The cdf of the MOMBIII distribution is thus defined as

$$F(t,a,b,\gamma,\alpha) = \frac{\left(1+\gamma t^{-a}\right)^{-b/\gamma}}{\alpha+(1-\alpha)\left(1+\gamma t^{-a}\right)^{-b/\gamma}}, \quad t>0, \tag{2.25}$$

and the density function is obtained as

$$f(t,a,b,\gamma,\alpha) = \frac{ab\alpha t^{-(a+1)}\left(1+\gamma t^{-a}\right)^{-\left(b/\gamma+1\right)}}{\left[\alpha+(1-\alpha)\left(1+\gamma t^{-a}\right)^{-b/\gamma}\right]^{2}}, \quad t>0. \tag{2.26}$$

Other statistical properties of the MOMBIII distribution such as the linear representation of the cdf and pdf, survival and hazard rate functions, quantile function, moments, moment generating function, probability weighted moments, Renyi entropy and order statistics. The maximum likelihood method of estimation was employed to estimate the unknown parameters of the MOMBIII distribution and a simulation study was adopted to investigate the performance of the parameter estimates. The usefulness of the MOMBIII distribution in lifetime data analysis was illustrated using a data set representing the strength of 1.5cm glass fibers. The authors compared the fit of the MOMBIII distribution to the ones obtained from the modified Burr III (MBIII), beta exponential (BE), exponentiated modified Burr III (EMBIII), transmuted modified Burr III (TMBIII), Gompertz and BIII distributions. Analysis of the results indicated that the MOMBIII distribution performed reasonably better than the competing distributions.

### 2.2.8 The Odd Exponentiated Half-LogisticBurr III (OEHLBIII) Distribution

Afify et al. (2017) introduced the odd exponentiated half-logistic-generated family of distributions with the cdf defined by

$$F(t,\lambda,\theta,\varphi) = \left( \frac{1-\exp\left(\dfrac{-\theta G(t,\varphi)}{1-G(t,\varphi)}\right)}{1+\exp\left(\dfrac{-\theta G(t,\varphi)}{1-G(t,\varphi)}\right)} \right)^{\lambda}, \quad t>0, \tag{2.27}$$

and the associated pdf obtained as

$$f(t,\lambda,\theta,\varphi) = 2\lambda\theta g(t,\varphi)\frac{\exp\left(\dfrac{-\theta G(t,\varphi)}{1-G(t,\varphi)}\right)\left[1-\exp\left(\dfrac{-\theta G(t,\varphi)}{1-G(t,\varphi)}\right)\right]^{\lambda-1}}{\left[1-G(t,\varphi)\right]^{2}\left[1-\exp\left(\dfrac{-\theta G(t,\varphi)}{1-G(t,\varphi)}\right)\right]^{\lambda+1}}, \quad t>0, \tag{2.28}$$

By substituting the cdf of the Burr III distribution into equation (2.27), Okereke and Ohakwe (2021) developed the odd exponentiated half-logistic Burr III (OEHLBIII) distribution with the cdf and pdf defined, respectively, as follow

$$F(t,a,b,\lambda,\theta) = \left( \frac{1-\exp\left( \frac{\theta}{1-\left(1+t^{-a}\right)^{b}} \right)}{1+\exp\left( \frac{\theta}{1-\left(1+t^{-a}\right)^{b}} \right)} \right)^{\lambda}, \quad t>0, \tag{2.29}$$

and

$$f(t,a,b,\lambda,\theta) = \frac{2ab\lambda\theta t^{-(a+1)}\left(1+t^{-a}\right)^{-(b+1)} \exp\left( \frac{\theta}{1-\left(1+t^{-a}\right)^{b}} \right) \left[ 1+\exp\left( \frac{\theta}{1-\left(1+t^{-a}\right)^{b}} \right) \right]^{\lambda-1}}{\left[1-\left(1+t^{-a}\right)^{-b}\right]^{2} \left[ 1+\exp\left( \frac{\theta}{1-\left(1+t^{-a}\right)^{b}} \right) \right]^{\lambda+1}}, \quad t>0. \tag{2.30}$$

Basic statistical properties of the OEHLBIII distribution such as the linear representation of the cdf and pdf, survival and hazard rate functions, quantile function, moments, moment generating function, Renyi entropy and order statistics were derived. The maximum likelihood method of estimation was employed to estimate the unknown parameters of the OEHLBIII distribution but the authors failed to check the performance of the parameter estimates. Two data sets comprising of the annual maximum daily precipitation and fracture toughness from the material alumina were employed to examine the effectiveness of the OEHLBIII distribution.

### 2.2.9 TheAlpha power Kumaraswamy Burr III(APKBIII) Distribution

Suppose $F(x,\zeta)$ is the cumulative distribution function (cdf) of a continuous random variable $X$, Mahdavi and Kundu (2017) defined the alpha-power transformation of $F(x,\zeta)$ as

$$F(t,\alpha,\varphi)=\begin{cases}\dfrac{\alpha^{G(t,\varphi)}-1}{\alpha-1}, & if \quad \alpha>0,\, \alpha\neq 1\\ \\ G(t,\varphi), & if \quad \alpha=1\end{cases}, \tag{2.31}$$

and the pdf associated with equation (2.31) is defined as

$$f(t,\alpha,\varphi)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}g(t,\varphi)\alpha^{G(t,\varphi)}, & if \quad \alpha>0,\, \alpha\neq 1\\ \\ g(t,\varphi), & if \quad \alpha=1\end{cases}. \tag{2.32}$$

Baharith (2022) considered the cdf of the KBIII distribution defined in equation (2.9) as the baseline distribution in equation (2.31) to develop the alpha power Kumaraswamy Burr III (APKBIII) distribution. The cdf and pdf of the APKBIII distribution is defined, respectively, as

$$F(t,a,b,c,k,\alpha)=\begin{cases}\dfrac{\alpha^{\left[1-\left(1-\left(1+t^{-a}\right)^{-bc}\right)^{k}\right]}-1}{\alpha-1}, & if \quad \alpha>0,\, \alpha\neq 1\\ \\ 1-\left(1-\left(1+t^{-a}\right)^{-bc}\right)^{k}, & if \quad \alpha=1\end{cases}, \tag{2.33}$$

and

$$f(t,\alpha,\varphi)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}abckt^{-(a+1)}\left(1+t^{-a}\right)^{-(bc+1)}\left(1-\left(1+t^{-a}\right)^{-bc}\right)^{k-1}\alpha^{\left[1-\left(1-\left(1+t^{-a}\right)^{-bc}\right)^{k}\right]}, & if \quad \alpha>0,\ \alpha\neq 1\\ abckt^{-(a+1)}\left(1+t^{-a}\right)^{-(bc+1)}\left(1-\left(1+t^{-a}\right)^{-bc}\right)^{k-1}, & if \quad \alpha=1\end{cases}. \quad (2.34)$$

Explicit expression for the statistical properties of the APKBIII distribution such as the linear representation of the cdf and pdf, survival and hazard rate functions, quantile function, moments, moment generating function, mean residual life and mean waiting time, Renyi entropy and order statistics. The maximum likelihood method of estimation was employed to estimate the unknown parameters of the APKBIII distribution and a simulation study was used to check the performance of the parameter estimates. The usefulness of the APKBIII distribution was illustrated via two Covid-19 data sets (infected and recovery cases). The fit of the APKBIII distribution for each data set were compared with the ones obtained by Kumaraswamy BIII, Marshall-Olkin Kumaraswamy BIII and BIII distributions. Evidently, the results of the analysis were in favor of the APKBIII distribution.

### 2.2.10 The Inverse Weibull Burr III(IWBIII) Distribution

Ahmad et al. (2022) recently developed the inverse Weibull-generated family of distributions using the *T-X* technique reported in Alzaatreh et al. (2013). They obtained the cdf of the IW-G family as

$$F(t,\lambda,\theta,\varphi) = \exp\left[-\theta^{\lambda}\left(\frac{G(t,\varphi)}{1-G(t,\varphi)}\right)^{-\lambda}\right] \quad t>0, \tag{2.35}$$

and the corresponding pdf obtained as

$$f(t,\lambda,\theta,\varphi) = \lambda\theta^{\lambda}g(t,\varphi)\frac{\left[G(t,\varphi)\right]^{-(\lambda+1)}}{\left[G(t,\varphi)\right]^{-(\lambda-1)}}\exp\left[-\theta^{\lambda}\left(\frac{G(t,\varphi)}{1-G(t,\varphi)}\right)^{-\lambda}\right] \quad t>0. \tag{2.36}$$

By considering the cdf of the Burr III distribution as the baseline cdf in equation (2.35), the authors studied the statistical properties of a sub-model from the proposed IW-G family which they referred to as the inverse Weibull Burr III (IWBIII) distribution. The cdf and pdf of the IWBIII distribution is defined, respectively, as follow

$$F(t,a,b,\lambda,\theta) = \exp\left[-\theta^{\lambda}\left(\left(1+t^{-a}\right)^{b}-1\right)^{\lambda}\right] \quad t>0, \tag{2.37}$$

and

$$f(t,a,b,\lambda,\theta) = ab\lambda\theta^{\lambda}t^{-(a+1)}\left(1+t^{-a}\right)^{b-1}\left(\left(1+t^{-a}\right)^{b}-1\right)^{\lambda-1}\exp\left[-\theta^{\lambda}\left(\left(1+t^{-a}\right)^{b}-1\right)^{\lambda}\right] \quad t>0. \tag{2.38}$$

Other statistical properties of the IWBIII distribution studied by the authors include the linear representation of the cdf and pdf, survival hazard rate and cumulative hazard rate functions, quantile function, moments, moment generating function, mean residual life, Renyi entropy, mean deviation about the mean and median, and order statistics. The maximum likelihood method of estimation was employed to estimate the unknown parameters of the IWBIII distribution and a simulation study was used to check the performance of the parameter estimates. A data set consisting of the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli was adopted to illustrate the effectiveness and flexibility of IWBIII distribution in fitting lifetime data sets.

## 2.3 Generalized Distributions with the Topp-Leone distribution as the baseline distribution.

### 2.3.1 The Type I Half-Logistic Topp-Leone distribution

Cordeiro et al. (2015) introduced the type I half-logistic-generalized family of distributions characterized by the cdf and pdf defined, respectively, as

$$F(t,\lambda,\varphi) = \frac{1-\left[1-G(t,\varphi)\right]^{\lambda}}{1+\left[1-G(t,\varphi)\right]^{\lambda}} , \tag{2.39}$$

and

$$f(t,\lambda,\varphi) = \frac{2\lambda g(t,\varphi)\left[1-G(t,\varphi)\right]^{\lambda-1}}{\left(1+\left[1-G(t,\varphi)\right]^{\lambda}\right)^{2}} \quad 0<t<1. \tag{2.40}$$

Motivated by the technique in equation (2.39), Zein Eldin et al. (2019) considered the one-parameter Topp-Leone distribution reported in Topp and Leone (1955) to develop the type I half-logistic Topp-Leone (TIHLTL) distribution with the cdf defined by

$$F(t,\lambda,\omega) = \frac{1-\left[1-\left[t(2-t)\right]^{\omega}\right]^{\lambda}}{1+\left[1-\left[t(2-t)\right]^{\omega}\right]^{\lambda}}, \qquad (2.41)$$

and

$$f(t,\lambda,\omega) = \frac{4\lambda\omega(1-t)\left[t(2-t)\right]^{\omega-1}\left[1-\left[t(2-t)\right]^{\omega}\right]^{\lambda-1}}{\left(1+\left[1-\left[t(2-t)\right]^{\omega}\right]^{\lambda}\right)^{2}}, \quad 0<t<1. \qquad (2.42)$$

Useful statistical properties of the TIHLTL distribution derived by the authors include the power series expansion for pdf, survival and hazard rate functions, quantile function, moments, moment generating function, stress-strength reliability and order statistics. Seven methods of model parameter estimation which include the maximum likelihood, least squares, weighted least squares, Crammer-von Mises, percentile, Anderson-Darling, right-tail Anderson-Darling estimators were employed to estimate the unknown parameters of the TIHLTL distribution. A Monte Carlo simulation study was conducted to investigate which of the methods best estimate the parameters of TIHLTL distribution. Of course, in all cases, the results were in favor of the maximum likelihood. Two unit-interval data sets characterizing the failure of components and 50 observations on burr (in millimeters), with hole diameter of 12mm were employed to illustrate the applicability of the TIHLTL distribution in lifetime data fittings.

### 2.3.2 The Beta Topp-Leone distribution

Based on the beta-generated family of distributions proposed by Eugene et al. (2002), Zakari and Usman (2020) constructed a generalization of the one-parameter Topp-Leone distribution called the beta Topp-Leone (BTL) distribution. They defined the cdf of the BTL distribution as

$$F\left(t,a,b,\omega\right) = \frac{1}{B\left(a,b\right)}\int_0^{\left[t(2-t)\right]^{\omega}} t^{a-1}\left(1-t\right)^{b-1} dt = \frac{\omega\left(t\left(2-t\right)\right)\left(a,b\right)}{B\left(a,b\right)}, \tag{2.43}$$

and

$$f\left(t,a,b,\omega\right) = \frac{2\omega}{B\left(a,b\right)}\left(1-t\right)\left[t\left(2-t\right)\right]^{\omega a-1}\left[1-\left(t\left(2-t\right)\right)\right]^{b-1}\left(t\left(2-t\right)\right)^{\omega-1},\ 0<t<1. \tag{2.44}$$

The authors also studied some useful statistical properties of the BTL distribution including the survival and hazard rate functions, moments, moment generating function, probability weighted moments, stress-strength reliability and order statistics. The method of maximum likelihood estimation was adopted for model parameter estimation but their performance was not examined through simulation study. The authors did not also check the validity of the BTL distribution in real-world data analysis.

### 2.3.3 The Marshall-Olkin Topp-Leone distribution

Opone and Iwerumor (2021) suggested another generalization of the Topp-Leone distribution using the Marshall-Olkin method defined in equations (2.11) and (2.12). They obtained the cdf and pdf of the Marshall-Olkin extended Topp-Leone (MOETL) distribution, respectively, as

$$F(t,\alpha,\omega) = \frac{\left[t(2-t)\right]^{\omega}}{1-\bar{\alpha}\left[1-\left[t(2-t)\right]^{\omega}\right]}, \tag{2.45}$$

and

$$f(t,\alpha,\varphi) = \frac{2\alpha\omega(1-t)\left[t(2-t)\right]^{\omega-1}}{\left[1-\bar{\alpha}\left[1-\left[t(2-t)\right]^{\omega}\right]\right]^{2}}, \quad 0<t<1. \tag{2.46}$$

Among the statistical properties of the MOETL distribution studied by the authors include the series representation of the pdf, survival and hazard rate functions, quantile function, median, moments, skewness and kurtosis. The maximum likelihood method of estimation was employed to estimate the parameters of the MOETL distribution and a simulation study was carried out to validate the asymptotic behavior of the parameter estimates. Two data sets comprising of the 48 rock samples from a petroleum reservoir and 20 observations of the maximum flood level for Susquehanna River were adopted to prove the usefulness and flexibility of the MOETL distribution in real-life data fittings.

### 2.3.4 The Transmuted Marshall-Olkin Topp-Leone distribution

Opone and Osemwenkhae (2022) studied the composite of the transmuted method defined in equations (2.3) and (2.4), and the Marshall-Olkin Topp-Leone distribution defined in equations (2.45) and (2.46), to introduce the transmuted Marshall-Olkin Topp-Leone (TMOETL) distribution with the cdf defined as

$$F(t,\alpha,\lambda,\omega) = \frac{\left[t(2-t)\right]^{\omega}}{1-\bar{\alpha}\left[1-\left[t(2-t)\right]^{\omega}\right]}\left[1+\lambda-\lambda\left(\frac{\left[t(2-t)\right]^{\omega}}{1-\bar{\alpha}\left[1-\left[t(2-t)\right]^{\omega}\right]}\right)\right], \tag{2.47}$$

and the corresponding pdf is obtained as

$$f(t,\alpha,\lambda,\omega) = \frac{2\alpha\omega(1-t)\left[t(2-t)\right]^{\omega-1}}{\left[1-\bar{\alpha}\left[1-\left[t(2-t)\right]^{\omega}\right]\right]^{2}}\left[1+\lambda-2\lambda\left(\frac{\left[t(2-t)\right]^{\omega}}{1-\bar{\alpha}\left[1-\left[t(2-t)\right]^{\omega}\right]}\right)\right],\ 0<t<1. \tag{2.48}$$

The authors treated some statistical properties of the TMOETL distribution which include the survival, hazard rate and reversed hazard rate functions, quantile function, median, moments, moment generating function, probability generating function and Renyi entropy.The estimation of the unknown parameters of the TMOETL distribution was obtained via the maximum likelihood method but the validity of the parameter estimates via Monte Carlo simulation study was omitted by the authors. A real data set representing 20 observations of the maximum flood level for Susquehanna River were used to show the effectiveness of the TMOETL distribution in real-life data fittings. The analysis of the result shows that the TMOETL distribution is considerably better than Marshall-Olkin extended

Kumaraswamy (MOEK), transmuted Kumaraswamy (TK), Kumaraswamy and beta distributions.

### 2.3.5 The GDUS-Modified Topp-Leone distribution

The generalized-DUS transformation method of generalizing lifetime distributions has been proposed by Maurya et al. (2017). They defined the cdf of the GDUS transformed family as

$$F(t,\alpha,\varphi) = \frac{\left[e^{\left(G^{\alpha}(t,\varphi)\right)} - 1\right]}{e-1}, \tag{2.49}$$

and the associated pdf is given by

$$f(t,\alpha,\varphi) = \frac{\alpha g(t,\varphi)\left[G(t,\varphi)\right]^{\alpha-1} e^{\left(G^{\alpha}(t,\varphi)\right)}}{e-1}, \; 0 < t < 1. \tag{2.50}$$

By inserting the cdf of the one-parameter Topp-Leone distribution into the technique in equation (2.49), Kaushik and Nigam (2022) constructed the cdf of the GDUS-Modified Topp-Leone (GMTL) distribution as

$$F(t,\alpha,\omega) = \frac{\left[e^{\left[t(2-t)\right]^{\omega\alpha}} - 1\right]}{e-1}, \tag{2.51}$$

and the pdf given by

$$f(t,\alpha,\omega) = \frac{2\omega\alpha(1-t)\left[t(2-t)\right]^{\omega-1}\left[\left[t(2-t)\right]^{\omega}\right]^{\alpha-1} e^{\left[t(2-t)\right]^{\omega\alpha}}}{e-1}, \; 0 < t < 1. \tag{2.52}$$

Statistical properties such as the survival, hazard rate and reversed hazard rate functions, quantile function, median, moments, conditional moment,mean deviation, Renyi entropy, stress-strength reliability, stochastic ordering and ordinary differential equations for the density and survival functions were derived.The unknown parameters of the GMTL distribution were estimated via the maximum likelihood and the performance of the parameter estimates was investigated through Monte Carlo simulation study. The authors concluded the paper by analyzing a real data set representing the recovery rates of 239 patients with CD34+ cells after peripheral blood stem cell (PBSC) transplants.

### 2.3.6 The Alpha Power Topp-Leone distribution

Recently, Ehiwario et al. (2023) developed the alpha power Topp-Leone (APTL) distribution using the technique in equations (2.31) and (2.32). The resulting cdf and pdf of the APTL distribution are, respectively, defined as

$$F_{APTL}(t,\alpha,\omega)=\begin{cases}\dfrac{\alpha^{\left[1-(1-t)^2\right]^{\omega}}-1}{\alpha-1}, & if \quad \alpha>0,\, \alpha\neq 1,\\[2ex] \left[1-\left(1-t\right)^2\right]^{\omega}, & if \quad \alpha=1\end{cases} \tag{2.53}$$

and

$$f_{APTL}(t,\alpha,\omega)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}2\omega(1-t)\left[1-(1-t)^2\right]^{\omega-1}\alpha^{\left[1-(1-t)^2\right]^{\omega}}, & if \quad \alpha>0,\ \alpha\neq 1,\\ \\ 2\omega(1-t)\left[1-(1-t)^2\right]^{\omega-1}, & if \quad \alpha=1.\end{cases} \qquad (2.54)$$

Various mathematical properties of the APTL distribution studied include the survival and hazard rate functions, quantiles, moments, moment generating function, probability weighted moments, Renyi entropy and order statistics. The maximum likelihood method was employed to estimate the parameters of the APTL distribution but the estimates were not validated through a Monte Carlo simulation study. Finally, two data sets comprising of the trade share and failure of components were used to illustrate the effectiveness and flexibility over some existing distributions such as unit-Weibull, unit-Gompertz, log-weighted exponential and Topp-Leone distributions.

## 2.4 Review on non-parametric and semi parametric models for analyzing germination data

Onofri et al. (2010) in their study pointed out some useful advantages of survival analysis over some traditional methods of analysis (e.g., ANOVA). In particular, the ANOVA approach do not provide a good description of germination trends and also does not account for censoring (germinated seeds after the experiment), which may be a problem in some situations. They described the time course of germination by using a non-parametric step function (germination function) and the effect of factors and covariates on germination functions was assessed by Accelerated Failure Time (AFT) regression and expressed in terms of time ratios.

For all data sets, germination functions were constructed for each experimental treatment, by using the Kaplan–Meyer (KM) estimators. Comparisons among germination functions were performed by using AFT regression.

The parameters measure how a change in the explanatory variables changes (prolongs/shortens) the time to germination of a seed lot. The authors presented four examples of the application of survival analysis on seed germination and results were compared with those obtained with traditional techniques.

Conclusively, the authors remarked that the Semi parametric AFT models should be considered, even though, at present, they have not yet been widely implemented in statistical software packages.

McNair et al. (2012) conducted a review on three methods of analyzing germination data which include intuition-based germination indexes, classical nonlinear regression analysis and time-to-event analysis (also known as survival analysis) and argues that time-to-event analysis has important advantages over the other methods but has been underutilized to date. The authors also reviewed in detail the types of time-to-event analysis that are most useful in analyzing seed germination data with standard statistical software. These include non-parametric methods (life-table and Kaplan-Meier estimators, and various methods for comparing two or more groups of seeds) and semi-parametric methods (Cox proportional hazards model, which permits inclusion of categorical and quantitative covariates, and fixed and random effects). Each method was illustrated by applying them to a set of real germination data. They noted that the most appropriate method depends on the questions of interest. If one is mainly interested in comparing groups of seeds, then non-parametric methods are appropriate choice. One the other hand, if one is interested in assessing effects of quantitative covariates on germination, such as temperature, duration of seeds storage or duration of stratification (and possibly comparing groups, as well), then the semi-parametric Cox model is appropriate.

Romano and Stevanato (2020) also noted that time-to-event analysis (survival analysis) can be approached by three different ways: non-parametric, semi-parametric and fully parametric. The non-parametric methods make no assumptions about an underlying probability distribution. That is how the event of germination changes over time, based on the probability of seeds development. They pointed out that the Cox's Proportional Hazard model is the most general of the regression models because it is not based on any assumptions concerning the nature of shape of the underlying survival distribution. In fully parametric models, a specific probability distribution of the baseline hazard/survival is assumed, according to a defined probability distribution.

In summary, the authors highlighted useful recommendation of non-parametric and semi parametric models, noting that the Kaplan-Meier (KM) survival curves provide the insight into the shape of the survival function for each treatment/genotype, focusing on not occurring germination event. Its results are graphically intuitive, and it could be a useful method to compare more groups of seeds through the log-rank family tests and in terms of median or quartile of survival times. Moreso, the Cox's PH model provides useful information when one needs to consider a set of covariates that influence germination simultaneously.

# CHAPTER THREE

# RESEARCH METHODOLOGY

## 3.1 Introduction

Alzaatreh et al. (2013) proposed a new method of generating family of continuous distributions were a random variable *X* "the transformer" was used to transform another random variable *T* "the transformed". The resulting family was referred to as the transformed-transformer (*T-X*) family of distributions. Now let *X* be a random variable with pdf, $f(x)$ and cdf, $F(x)$; and also *T* be a random variable with pdf, $r(t)$ and cdf, $R(t)$. The cdf of the *T-X* family of distributions was defined as

$$G(x)=\int_0^{-\log(1-F(x))} r(t)\,dt = R\left[-\log\left(1-F(x)\right)\right], \tag{3.1}$$

and the corresponding density function given by

$$g(x)=\frac{f(x)}{1-F(x)}r\left[-\log\left(1-F(x)\right)\right]. \tag{3.2}$$

By adopting the framework in equations (3.1) and (3.2), we allowed the random variable $X$ to follow the one-parameter Topp-Leone distribution and the random variable $T$ to follow the inverse Burr (Burr III) distribution. The resulting cumulative distribution function of the proposed distribution is thus defined as

$$\begin{aligned} G(x,\lambda,a,b) &= \int_0^{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)} t^{-(a+1)}\left(1+t^{-a}\right)^{-(b+1)} dt \\ &= \left[1+\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right]^{-b}, \quad 0<x<1;\ \lambda,a,b>0, \end{aligned} \tag{3.3}$$

and the density function associated to equation (3.3) is obtained as

$$\begin{aligned} g(x,\lambda,a,b) &= \frac{2ab\lambda(1-x)\left[x(2-x)\right]^{\lambda-1}}{\left[1-\left[x(2-x)\right]^{\lambda}\right]}\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-(a+1)} \\ &\times\left[1+\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{3.4}$$

Equations (3.3) and (3.4) are, respectively, the cdf and pdf of the inverse Burr Topp-Leone (IBTL) distribution.

## 3.2 Mathematical Properties of the Inverse Burr Topp-Leone (IBTL) Distribution

### 3.2.1 Linear Representation of the cumulative distribution function (CDF) and probability density function (PDF)

When dealing with probability distribution with transcendental function, it is important to resolve the function using some special series expansion. This allows for easy derivation of other mathematical properties of the model such as moments, moment generating function, Renyi entropy, etc.

For any real number *s*> 0, Prudnikov et al. (1986) suggested a generalized binomial series expansion

$$(1+x)^{-s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} (-1)^k x^k. \qquad (3.5)$$

Furthermore, for any real parameter $\alpha > 0$, Gradsteyn and Ryzhik, (2000) validates the convergent series

$$\left[-\log(1-x)\right]^{\alpha-1} = x^{\alpha-1} \left[ \sum_{m=0}^{\infty} \binom{\alpha-1}{m} x^m \left( \sum_{s=0}^{\infty} \frac{x^s}{s+2} \right)^m \right], \quad 0 < x < 1. \qquad (3.6)$$

Now, applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left( \sum_{s=0}^{\infty} a_s x^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} x^s,$$

so that

$$\left[-\log(1-x)\right]^{\alpha-1} = \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{\alpha-1}{m} b_{s,m}\, x^{\alpha+m+s-1},$$

where,

$$b_{s,m} = \left(sa_0\right)^{-1} \sum_{q=0}^{s} \left(m\left(q+1\right)-s\right) a_q b_{s-q,m}\,, \quad and \quad b_{0,m} = a_0^m\,.$$

Using these expressions in equation (3.3), we have

$$\left[1+\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-a}\right]^{-b} = \sum_{k=0}^{\infty}\binom{b+k-1}{k}\left(-1\right)^{k}\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-ak}$$

$$\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-ak} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{-ak}{m} b_{s,m}\left[\left[x\left(2-x\right)\right]^{\lambda}\right]^{m+s-ak},$$

combining these results yield,

$$\begin{aligned}\left[1+\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-a}\right]^{-b} &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{b+k-1}{k}\binom{-ak}{m} b_{s,m}\left(-1\right)^{k}\left[\left[x\left(2-x\right)\right]^{\lambda}\right]^{m+s-ak} \\ &= \sum_{k,m,s=0}^{\infty} \vartheta_{m+s-ak} H_{m+s-ak}\left(x\right),\end{aligned} \qquad (3.7)$$

where, $\vartheta_{m+s-ak} = \sum_{k,m,s=0}^{\infty}\binom{b+k-1}{k}\binom{-ak}{m}\left(-1\right)^{k} b_{s,m}$ and $H_{m+s-ak}\left(x\right) = \left[G\left(x\right)\right]^{m+s-ak}$ is the distribution function of the one-parameter Topp-Leone distribution with $m+s-ak$ as the power parameter.

Differentiating (3.7), we obtain its associated density function as

$$g\left(x\right) = \sum_{k,m,s=0}^{\infty} \vartheta_{m+s-ak} h_{m+s-ak+1}\left(x\right), \qquad (3.8)$$

where $h_{m+s-ak+1}\left(x\right) = 2\lambda\left(m+s-ak+1\right)\left(1-x\right)\left[x\left(2-x\right)\right]^{\lambda\left[m+s-ak+1\right]-1}$ is the density functionof the one-parameter Topp-Leone distribution with $m+s-ak+1$ as the power parameter.The graphical plots of the pdf and cdf of the IBTL distribution is shown in Figure 3.1.

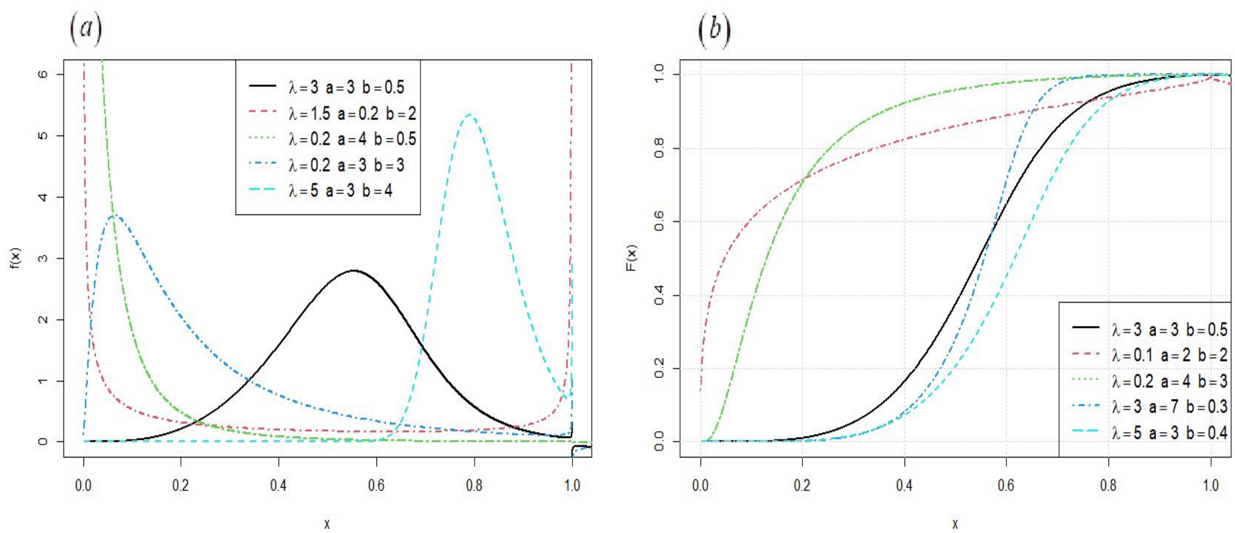


**Figure 3.1:**The graphical plots of the pdf (a) and cdf (b) of the IBTL distribution

Figure 3.1(a) indicates that the pdf of the IBTL distribution accommodates a decreasing (reversed-J), left-skewed, right-skewed, symmetric and bathtub shapes. On the other hand, Figure 3.1(b) shows that for all selected values of the parameters, as the random variable $x$ increases, the cdf of the IBTL distribution tends to one (1). This is one of the unique properties of a cumulative distribution function.

### 3.2.2 The Survival and Hazard Rate Function

Utilizing the cumulative distribution function and the density function in equations (3.3) and (3.4), respectively, we obtain the survival and hazard rate functions of the inverse Burr Topp-Leone (IB-TL) distribution as follows

$$S(x,\lambda,a,b) = 1 - F(x,\lambda,a,b),$$
$$= 1 - \left[1 + \left\{-\log\left(1 - \left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right]^{-b}, \tag{3.9}$$

and

$$h(x,\lambda,a,b) = \frac{f(x,\lambda,a,b)}{S(x,\lambda,a,b)},$$
$$= \frac{2ab\lambda(1-x)\left[x(2-x)\right]^{\lambda-1}\left\{-\log\left[1-\left[x(2-x)\right]^{\lambda}\right]\right\}^{-(a+1)}\left[1+\left\{-\log\left[1-\left[x(2-x)\right]^{\lambda}\right]\right\}^{-a}\right]^{-(b+1)}}{\left[1-\left[x(2-x)\right]^{\lambda}\right]\left[1-\left[1+\left\{-\log\left[1-\left[x(2-x)\right]^{\lambda}\right]\right\}^{-a}\right]^{-b}\right]}. \tag{3.10}$$

The graphical representation of the survival and hazard rate functions in displayed in Figure 3.2.

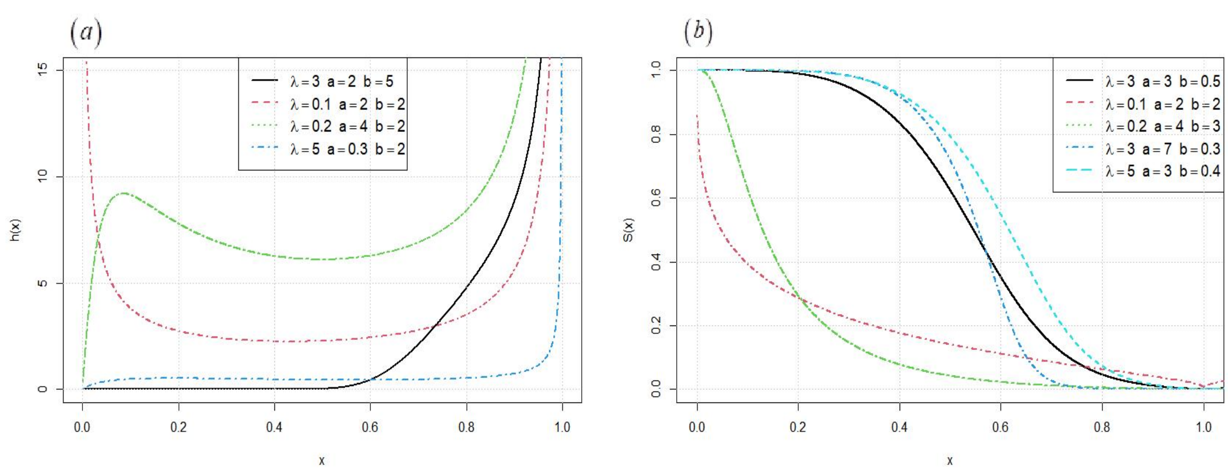


**Figure 3.2:**The graphical plots of the hrf (a) and sf (b) of the IBTL distribution

Evidently, from Figure 3.2(a), the hazard rate function of the IBTL distribution exhibits an increasing, bathtub and inverted bathtub-shaped hazard properties. These features are important in reliability studies.

### 3.2.3 The Quantile Function

The quantile function of the inverse Burr Topp-Leone (IBTL) distribution is obtained by equating the cumulative distribution function defined in equation (3.3) to a uniformly generated random variable *u*, i.e., $F(x,\lambda,a,b)=u$.

$$\left[1+\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right]^{-b} = u,$$

$$\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-a} = u^{-1/b}-1,$$

$$-\log\left(1-\left[x(2-x)\right]^{\lambda}\right) = \left(u^{-1/b}-1\right)^{-1/a}$$

taking the exponent of both sides, we have

$$\left(1-\left[x(2-x)\right]^{\lambda}\right) = e^{-\left(u^{-1/b}-1\right)^{-1/a}}$$

$$\left[1-(1-x)^{2}\right]=\left[1-e^{-\left(u^{-1/b}-1\right)^{-1/a}}\right]^{1/\lambda}$$

$$(1-x)^{2}=1-\left[1-e^{-\left(u^{-1/b}-1\right)^{-1/a}}\right]^{1/\lambda}$$

$$x_u = 1-\sqrt{1-\left[1-e^{-\left(u^{-1/b}-1\right)^{-1/a}}\right]^{1/\lambda}}, \qquad 0<u<1. \tag{3.11}$$

The median of the inverse Burr Topp-Leone (IBTL) distribution is obtained by setting $u = 0.5$ as shown in equation (3.12).

$$x_{0.5} = 1-\sqrt{1-\left[1-e^{-\left((0.5)^{-1/b}-1\right)^{-1/a}}\right]^{1/\lambda}}, \qquad 0<u<1. \tag{3.12}$$

The quantile function in equation (3.11) is useful in generating random samples from the IBTL distribution for simulation purposes. Thus, we generate some quantiles from the IBTL distribution for some selected parameter values as shown in Table 3.1.

**Table 3.1:** Some quantiles of the IBTL Distribution $(a,b,\lambda)$

| $u$ | $(4.0,3.0,0.3)$ | $(2.0,1.5,0.5)$ | $(3.0,2.5,0.7)$ | $(5.0,2.0,4.0)$ |
|---|---|---|---|---|
| 0.05 | 0.0863 | 0.0550 | 0.2281 | 0.6229 |
| 0.25 | 0.1501 | 0.1686 | 0.3389 | 0.6709 |
| 0.29 | 0.1606 | 0.1900 | 0.3552 | 0.6770 |
| 0.34 | 0.1737 | 0.2175 | 0.3752 | 0.6841 |
| 0.39 | 0.1870 | 0.2461 | 0.3950 | 0.6911 |
| 0.44 | 0.2009 | 0.2763 | 0.4151 | 0.6979 |
| 0.49 | 0.2155 | 0.3085 | 0.4357 | 0.7047 |

| | | | | |
|---|---|---|---|---|
| 0.54 | 0.2312 | 0.3433 | 0.4573 | 0.7116 |
| 0.59 | 0.2485 | 0.3813 | 0.4803 | 0.7188 |
| 0.64 | 0.2676 | 0.4235 | 0.5051 | 0.7265 |

Table 3.1 shows some quantiles from the IBTL distribution at different choice of parameter values. We observed that the values are bounded within the unit interval, which agrees with the support of a random variable $X$ following the IBTL distribution.

### 3.2.4 The $r^{th}$ Ordinary Moments

Let $X$ be a continuous random variable with probability density function defined in equation (3.8), we defined the $r^{th}$ ordinary moment of the IBTL distribution as follows

$$\begin{aligned} E\left[X^r\right] &= \int_0^1 x^r g(x)\,dx, \\ &= \sum_{k,m,s=0}^{\infty} \vartheta_{m+s-ak} \int_0^1 x^r h_{m+s-ak+1}(x)\,dx, \\ &= 2\lambda \sum_{k,m,s=0}^{\infty} \vartheta_{m+s-ak}\left(m+s-ak+1\right)\int_0^1 x^r (1-x)\left[x(2-x)\right]^{\lambda[m+s-ak+1]-1}\,dx, \end{aligned} \tag{3.13}$$

further simplification of the integral part of equation (3.13), yields

$$\left[x(2-x)\right]^{\lambda[m+s-ak+1]-1} = \sum_{j=0}^{\infty}\binom{\lambda[m+s-ak+1]-1}{j}(-1)^j(1-x)^{2j},$$

$$(1-x)^{2j+1} = \sum_{q=0}^{2j+1}\binom{2j+1}{q}(-1)^q x^q,$$

substituting these expressions into equation (3.13), we have

$$E\left[X^r\right] = 2\lambda \sum_{k,m,s,j=0}^{\infty} \sum_{q=0}^{2j+1} \vartheta_{m+s-ak}\left(m+s-ak+1\right)\binom{\lambda\left[m+s-ak+1\right]-1}{j}\binom{2j+1}{q}\left(-1\right)^{j+q}\int_0^1 x^{r+q}dx, \tag{3.14}$$

$$\mathrm{M}_r^{'} = 2\lambda \sum_{k,m,s,j=0}^{\infty} \sum_{q=0}^{2j+1} \vartheta_{m+s-ak}\binom{\lambda\left[m+s-ak+1\right]-1}{j}\binom{2j+1}{q}\frac{\left(m+s-ak+1\right)}{r+q+1}\left(-1\right)^{j+q}.$$

The first four ordinary moments are obtained when $r$ = 1, 2, 3, 4. That is,

$$\mathrm{M}_1^{'} = 2\lambda \sum_{k,m,s,j=0}^{\infty} \sum_{q=0}^{2j+1} \vartheta_{m+s-ak}\binom{\lambda\left[m+s-ak+1\right]-1}{j}\binom{2j+1}{q}\frac{\left(m+s-ak+1\right)}{q+2}\left(-1\right)^{j+q};$$

$$\mathrm{M}_2^{'} = 2\lambda \sum_{k,m,s,j=0}^{\infty} \sum_{q=0}^{2j+1} \vartheta_{m+s-ak}\binom{\lambda\left[m+s-ak+1\right]-1}{j}\binom{2j+1}{q}\frac{\left(m+s-ak+1\right)}{q+3}\left(-1\right)^{j+q};$$

$$\mathrm{M}_3^{'} = 2\lambda \sum_{k,m,s,j=0}^{\infty} \sum_{q=0}^{2j+1} \vartheta_{m+s-ak}\binom{\lambda\left[m+s-ak+1\right]-1}{j}\binom{2j+1}{q}\frac{\left(m+s-ak+1\right)}{q+4}\left(-1\right)^{j+q};$$

$$\mathrm{M}_4^{'} = 2\lambda \sum_{k,m,s,j=0}^{\infty} \sum_{q=0}^{2j+1} \vartheta_{m+s-ak}\binom{\lambda\left[m+s-ak+1\right]-1}{j}\binom{2j+1}{q}\frac{\left(m+s-ak+1\right)}{q+5}\left(-1\right)^{j+q}.$$

When $r$ = 1, we obtain the mean of the IBTL distribution.

Furthermore, from the expressions in the first four moments, the variance $\left(\sigma^2\right)$, measures of skewness $(S)$ and kurtosis $(K)$ of the IBTL distribution can be obtained using the following mathematical relationships.

$$\text{variance}\left(\sigma^2\right) = \mathrm{M}_2^{'} - \left(\mathrm{M}_1^{'}\right)^2, \qquad \text{skewness}(S) = \frac{\mathrm{M}_3^{'} - 3\mathrm{M}_2^{'}\mathrm{M}_1^{'} + 2\left(\mathrm{M}_1^{'}\right)^3}{\left(\mathrm{M}_2^{'} - \left(\mathrm{M}_1^{'}\right)^2\right)^{\frac{3}{2}}},$$

$$\text{kurtosis}(K) = \frac{M'_4 - 4M'_3 M'_1 + 6M'_3 (M'_1)^2 - 3(M'_1)^4}{\left(M'_2 - (M'_1)^2\right)^2}.$$

Here, we obtain some numerical computation of moments of the IBTL distribution, including the mean$(M'_1)$, variance$(\sigma^2)$, measures of skewness$(S)$ and kurtosis $(K)$ for some selected parameter values as computed in Table 3.2.

**Table 3.2:** Theoretical moments of the IBTL distribution for $(\lambda = 2)$

| $a$ | $b$ | $M'_1$ | $\sigma^2$ | $S$ | $K$ |
|---|---|---|---|---|---|
| 2.0 | 0.3 | 0.3465 | 0.0468 | 0.4283 | 2.5822 |
| | 0.5 | 0.4394 | 0.0502 | 0.6469 | 2.8611 |
| | 0.8 | 0.5234 | 0.0413 | 0.2789 | 2.5474 |
| 4.0 | 0.3 | 0.4131 | 0.0235 | 0.0334 | 2.8390 |
| | 0.5 | 0.4788 | 0.0177 | 0.0773 | 2.9291 |
| | 0.8 | 0.5309 | 0.0137 | 0.1692 | 2.9257 |
| 6.0 | 0.3 | 0.4473 | 0.0136 | -0.2994 | 3.0557 |
| | 0.5 | 0.4974 | 0.0090 | -0.0935 | 2.4544 |
| | 0.8 | 0.5347 | 0.0066 | -0.0996 | 8.2257 |
| 8.0 | 0.3 | 0.4678 | 0.0090 | -0.6451 | 4.5954 |
| | 0.5 | 0.5082 | 0.0054 | 0.3382 | 5.2197 |
| | 0.8 | 0.5372 | 0.0037 | 0.3723 | 0.3477 |

Observations from the table reveal that IBTL distribution exhibits a left-skewed, right-skewed, approximately symmetric, platykurtic, leptokurtic as well as mesokurtic properties.

### 3.2.5 Moment Generating Function

Let $X$ be a continuous random variable with probability density function defined in equation (3.8), we defined the moment generating function of the IBTL distribution as follows

$$
\begin{aligned}
M_X(t) = E\left[e^{tx}\right] &= \sum_{n=0}^{\infty} \frac{t^n}{n!} E\left[X^n\right] \\
&= \sum_{k,m,s,n=0}^{\infty} \frac{t^n \vartheta_{m+s-ak}}{n!} \int_0^1 x^n h_{m+s-ak+1}(x)\, dx, \\
&= 2\lambda \sum_{k,m,s,n=0}^{\infty} \frac{t^n \vartheta_{m+s-ak}}{n!} (m+s-ak+1) \int_0^1 x^n (1-x) \left[x(2-x)\right]^{\lambda[m+s-ak+1]-1} dx,
\end{aligned}
$$

(3.15)

By applying similar approach used in equation (3.14), we obtain the moment generating function of the IBTL distribution as

$$
\begin{aligned}
M_X(t) &= 2\lambda \sum_{k,m,s,j,n=0}^{\infty} \sum_{q=0}^{2j+1} \vartheta_{m+s-ak} \binom{\lambda[m+s-ak+1]-1}{j} \binom{2j+1}{q} \frac{t^n (m+s-ak+1)}{n!} (-1)^{j+q} \int_0^1 x^{n+q}\, dx \\
&= 2\lambda \sum_{k,m,s,j,n=0}^{\infty} \sum_{q=0}^{2j+1} \vartheta_{m+s-ak} \binom{\lambda[m+s-ak+1]-1}{j} \binom{2j+1}{q} \frac{t^n (m+s-ak+1)}{n!(n+q+1)} (-1)^{j+q}.
\end{aligned}
\qquad (3.16)
$$

### 3.2.6 Probability Weighted Moments (PWMs)

Greenwood et al. (1979) have defined the $(q,r)^{th}$ probability weighted moments of a random variable $X$ with known pdf, $g(x)$ and cdf, $G(x)$ as

$$\rho_{q,r} = E\left[X^r G(x)^q\right] = \int_{-\infty}^{\infty} x^r g(x) G(x)^q \, dx, \tag{3.17}$$

Using the information in equations (3.3) and (3.4), we obtain the $(q,r)^{th}$ PWMs of the IBTL distribution as follows

$$g(x,\lambda,a,b)G(x,\lambda,a,b)^q = \frac{2ab\lambda(1-x)\left[x(2-x)\right]^{\lambda-1}}{\left[1-\left[x(2-x)\right]^{\lambda}\right]}\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-(a+1)} \times\left[1+\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right]^{-(b(1+q)+1)}, \tag{3.18}$$

further evaluation of equation (3.18) yields,

$$\left[1+\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right]^{-(b(1+q)+1)} = \sum_{k=0}^{\infty}\binom{b(1+q)+k}{k}\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-ak},$$

$$\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-[a(k+1)+1]} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{-\left[a(k+1)+1\right]}{m} b_{s,m}\left[\left[x(2-x)\right]^{\lambda}\right]^{m+s-a(k+1)-1},$$

$$\left[1-\left(1-\left[x(2-x)\right]^{\lambda}\right)\right]^{m+s-a(k+1)-1} = \sum_{j=0}^{\infty}\binom{m+s-a(k+1)-1}{j}(-1)^j\left(1-\left[x(2-x)\right]^{\lambda}\right)^j,$$

$$\left(1-\left[x(2-x)\right]^{\lambda}\right)^{j-1} = \sum_{p=0}^{\infty}\binom{j-1}{p}(-1)^p\left[x(2-x)\right]^{\lambda p},$$

$$\left[1-(1-x)^2\right]^{\lambda(p+1)-1} = \sum_{q=0}^{\infty}\binom{\lambda(p+1)-1}{q}(-1)^q(1-x)^{2q},$$

$$(1-x)^{2q+1} = \sum_{l=0}^{\infty}\binom{2q+1}{l}(-1)^l x^l,$$

substituting these expressions into equation (3.17), we have

$$\rho_{q,r} = 2ab\lambda \sum_{k,m,s,j,p,q,l=0}^{\infty} \binom{b(1+q)+k}{k} \binom{-[a(k+1)+1]}{m} \binom{m+s-a(k+1)-1}{j} \binom{j-1}{p} \times \binom{\lambda(p+1)-1}{q} \binom{2q+1}{l} b_{s,m} (-1)^{j+p+q+l} \int_0^1 x^{r+l} dx, \tag{3.19}$$

Evaluating the integral part of equation (3.19) and further simplification yields,

$$\rho_{q,r} = 2ab\lambda \sum_{k,m,s,j,p,q,l=0}^{\infty} \binom{b(1+q)+k}{k} \binom{-[a(k+1)+1]}{m} \binom{m+s-a(k+1)-1}{j} \binom{j-1}{p} \times \binom{\lambda(p+1)-1}{q} \binom{2q+1}{l} \frac{b_{s,m} (-1)^{j+p+q+l}}{r+l+1}. \tag{3.20}$$

### 3.2.7 Renyi Entropy

An entropy of a random variable *X* measures the degree of variation associated with the random variable *X*. Renyi (1961) defined the entropy of a random variable *X* with known pdf, $g(x)$ as

$$\tau_R(\upsilon) = \frac{1}{1-\upsilon} \log \int_{-\infty}^{\infty} g^{\upsilon}(x) dx, \qquad \upsilon > 0, \quad \upsilon \neq 1. \tag{3.21}$$

From equation (3.21), we define the Renyi entropy of a random variable *X* following the IBTL distribution as follows

$$\tau_R(\upsilon) = \frac{1}{1-\upsilon} \log \left[ (2ab\lambda)^{\upsilon} \int_0^1 \frac{(1-x)^{\upsilon} [x(2-x)]^{\upsilon(\lambda-1)}}{\left[1-[x(2-x)]^{\lambda}\right]^{\upsilon}} \left\{-\log\left(1-[x(2-x)]^{\lambda}\right)\right\}^{-\upsilon(a+1)} \times \left[1+\left\{-\log\left(1-[x(2-x)]^{\lambda}\right)\right\}^{-a}\right]^{-\upsilon(b+1)} dx \right], \tag{3.22}$$

further simplification of equation (3.22) yields,

$$\left[1+\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right]^{-\upsilon(b+1)}=\sum_{k=0}^{\infty}\binom{\upsilon(b+1)+k-1}{k}\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-ak},$$

$$\left\{-\log\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-\left[\upsilon(a+1)+ak\right]}=\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{-\left[\upsilon(a+1)+ak\right]}{m}b_{s,m}\left[\left[x(2-x)\right]^{\lambda}\right]^{m+s-\upsilon(a+1)+ak},$$

$$\left[1-\left(1-\left[x(2-x)\right]^{\lambda}\right)\right]^{m+s-\upsilon(a+1)+ak}=\sum_{j=0}^{\infty}\binom{m+s-\upsilon(a+1)+ak}{j}(-1)^{j}\left(1-\left[x(2-x)\right]^{\lambda}\right)^{j},$$

$$\left(1-\left[x(2-x)\right]^{\lambda}\right)^{j-\upsilon}=\sum_{p=0}^{\infty}\binom{j-\upsilon}{p}(-1)^{p}\left[x(2-x)\right]^{\lambda p},$$

$$\left[1-(1-x)^{2}\right]^{\lambda(\upsilon+p)-\upsilon}=\sum_{q=0}^{\infty}\binom{\lambda(\upsilon+p)-\upsilon}{q}(-1)^{q}(1-x)^{2q},$$

$$(1-x)^{2q+\upsilon}=\sum_{l=0}^{\infty}\binom{2q+\upsilon}{l}(-1)^{l}x^{l},$$

inserting these expressions into equation (3.22), we have

$$\tau_R(\upsilon)=\frac{1}{1-\upsilon}\log\left[\begin{array}{l}(2ab\lambda)^{\upsilon}\sum\limits_{k,m,s,j,p,q=0}^{\infty}\binom{\upsilon(b+1)+k-1}{k}\binom{-\left[\upsilon(a+1)+ak\right]}{m}\binom{m+s-\upsilon(a+1)+ak}{j}\\ \times\binom{j-\upsilon}{p}\binom{\lambda(\upsilon+p)-\upsilon}{q}\binom{2q+\upsilon}{l}(-1)^{j+p+q+l}b_{s,m}\int_0^1 x^{l}dx\end{array}\right],\quad(3.23)$$

Evaluating the integral part of equation (3.23), yields

$$\tau_R(\upsilon)=\frac{1}{1-\upsilon}\log\left[\begin{array}{l}(2ab\lambda)^{\upsilon}\sum\limits_{k,m,s,j,p,q=0}^{\infty}\binom{\upsilon(b+1)+k-1}{k}\binom{-\left[\upsilon(a+1)+ak\right]}{m}\binom{m+s-\upsilon(a+1)+ak}{j}\\ \times\binom{j-\upsilon}{p}\binom{\lambda(\upsilon+p)-\upsilon}{q}\binom{2q+\upsilon}{l}\frac{(-1)^{j+p+q+l}b_{s,m}}{l+1}.\end{array}\right],\quad(3.24)$$

### 3.2.8 Distribution of order Statistics

Suppose that $X_1, X_2, ..., X_l$ is a random sample from the inverse Burr Topp-Leone distribution whose cdf and pdf are defined in equations (3.3) and (3.4), respectively. Let $X_{r:l}$ denote the $r^{th}$ order statistic, then the density function of $X_{r:l}$ is obtained as

$$g_{r:l}\left(x,\lambda,a,b\right)=\frac{1}{B\left(r,l-r+1\right)}\sum_{p=0}^{l-r}\binom{l-r}{p}\left(-1\right)^{p} g(x,\lambda,a,b)G^{r+p-1}(x,\lambda,a,b), \qquad (3.25)$$

Substituting the cdf and pdf in equations (3.3) and (3.4) into equation (3.25), we obtain the density function of IBTL$r^{th}$ order statistics as follows.

$$\begin{aligned} g\left(x,\lambda,a,b\right)G\left(x,\lambda,a,b\right)^{r+p-1} &= \frac{2ab\lambda\left(1-x\right)\left[x\left(2-x\right)\right]^{\lambda-1}}{\left[1-\left[x\left(2-x\right)\right]^{\lambda}\right]}\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-(a+1)} \\ &\times\left[1+\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-a}\right]^{-(b(r+p)+1)}, \end{aligned} \qquad (3.26)$$

Again, we simplify equation (3.26) using similar approach in equation (3.22), i.e.,

$$\left[1+\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-a}\right]^{-(b(r+p)+1)} = \sum_{k=0}^{\infty}\binom{b\left(r+p\right)+k}{k}\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-ak},$$

$$\left\{-\log\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right\}^{-\left[a(k+1)+1\right]} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{-\left[a\left(k+1\right)+1\right]}{m}b_{s,m}\left[\left[x\left(2-x\right)\right]^{\lambda}\right]^{m+s-a(k+1)-1},$$

$$\left[1-\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)\right]^{m+s-a(k+1)-1} = \sum_{j=0}^{m+s-a(k+1)-1}\binom{m+s-a\left(k+1\right)-1}{j}\left(-1\right)^{j}\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)^{j},$$

$$\left(1-\left[x\left(2-x\right)\right]^{\lambda}\right)^{j-1} = \sum_{w=0}^{j-1}\binom{j-1}{w}\left(-1\right)^{w}\left[x\left(2-x\right)\right]^{\lambda w},$$

$$\left[1-\left(1-x\right)^{2}\right]^{\lambda(w+1)-1} = \sum_{q=0}^{\lambda(w+1)-1}\binom{\lambda\left(w+1\right)-1}{q}\left(-1\right)^{q}\left(1-x\right)^{2q},$$

$$(1-x)^{2q+1}=\sum_{l=0}^{2q+1}\binom{2q+1}{l}(-1)^l x^l,$$

substituting these expressions into equation (3.25), we have

$$\begin{aligned} g_{r:l}(x,\lambda,a,b)=&\frac{2ab\lambda}{B(r,l-r+1)}\sum_{k,m,s=0}^{\infty}\sum_{p=0}^{l-r}\sum_{j=0}^{m+s-a(k+1)-1}\sum_{w=0}^{j-1}\sum_{q=0}^{\lambda(w+1)-1}\sum_{l=0}^{2q+1}\binom{b(r+p)+k}{k}\binom{-[a(k+1)+1]}{m} \\ &\times\binom{l-r}{p}\binom{m+s-a(k+1)-1}{j}\binom{j-1}{w}\binom{\lambda(w+1)-1}{q}\binom{2q+1}{l}b_{s,m}(-1)^{p+j+w+q+l}x^l. \end{aligned} \quad (3.27)$$

### 3.2.9 Parameter Estimation

#### 3.2.9.1Maximum likelihood estimation

Let $(x_1,x_2,...,x_n)$ be random samples generated from the IBTL distribution. The likelihood function of $X$ is obtained as

$$L(x,\zeta)=\prod_{i=1}^{n}\left[\frac{2ab\lambda(1-x)[x(2-x)]^{\lambda-1}\left\{-\log\left(1-[x(2-x)]^{\lambda}\right)\right\}^{-(a+1)}}{\left[1-[x(2-x)]^{\lambda}\right]\left[1+\left\{-\log\left(1-[x(2-x)]^{\lambda}\right)\right\}^{-a}\right]^{(b+1)}}\right],\ \zeta=(\lambda,a,b)^T \quad (3.28)$$

Taking the natural logarithm of equation (3.28), we have

$$\begin{aligned} \ell(x,\zeta)=&n\ln(2ab\lambda)+\sum_{i=1}^{n}\ln(1-x)+(\lambda-1)\sum_{i=1}^{n}\ln[x(2-x)]-(a+1)\sum_{i=1}^{n}\ln\left\{-\log\left(1-[x(2-x)]^{\lambda}\right)\right\} \\ &-\sum_{i=1}^{n}\ln\left[1-[x(2-x)]^{\lambda}\right]-(b+1)\sum_{i=1}^{n}\ln\left[1+\left\{-\log\left(1-[x(2-x)]^{\lambda}\right)\right\}^{-a}\right], \end{aligned} \quad (3.29)$$

By taking the first derivative of the log-likelihood function in equation (3.29) with respect to the parameters, we have

$$\frac{\partial\ell(x,\zeta)}{\partial a}=\frac{n}{a}-\sum_{i=1}^{n}\ln\left[1-[x(2-x)]^{\lambda}\right]+(b+1)\sum_{i=1}^{n}\frac{\ln\left[-\ln\left(1-[x(2-x)]^{\lambda}\right)\right]\left(-\ln\left(1-[x(2-x)]^{\lambda}\right)\right)^{-a}}{1+\left\{-\ln\left(1-[x(2-x)]^{\lambda}\right)\right\}^{-a}};$$

$$\frac{\partial \ell(x,\zeta)}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n} \ln\left[1 + \left\{-\ln\left(1 - \left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right];$$

$$\frac{\partial \ell(x,\zeta)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \frac{\ln\left[x(2-x)\right]\left[x(2-x)\right]^{\lambda}}{1-\left[x(2-x)\right]^{\lambda}} + (a+1)\sum_{i=1}^{n} \frac{\ln\left[x(2-x)\right]\left[x(2-x)\right]^{\lambda}}{\ln\left[1-\left[x(2-x)\right]^{\lambda}\right]\left(1-\left[x(2-x)\right]^{\lambda}\right)}$$
$$+ \sum_{i=1}^{n} \ln\left[x(2-x)\right] + a(b+1)\sum_{i=1}^{n} \frac{\ln\left[x(2-x)\right]\left[x(2-x)\right]^{\lambda}\left(-\ln\left(1-\left[x(2-x)\right]^{\lambda}\right)\right)^{-(a+1)}}{\left(1+\left\{-\ln\left(1-\left[x(2-x)\right]^{\lambda}\right)\right\}^{-a}\right)\left(1-\left[x(2-x)\right]^{\lambda}\right)}.$$

The maximum likelihood estimates (MLEs) of $\zeta$ say $\hat{\zeta} = \left(\hat{\lambda}, \hat{a}, \hat{b}\right)^{T}$, are obtained from the solution of the score function $U(x_i, \zeta) = \left[\frac{\ell(x,\zeta)}{\partial \lambda}, \frac{\ell(x,\zeta)}{\partial a}, \frac{\ell(x,\zeta)}{\partial b}\right]^{T} = 0.$ These numerical solutions can be obtained using Statistical packages such as ***fitdistrplus*** and ***optim*** in R program.

### 3.2.9.2 Monte Carlo Simulation Study

In this section, we investigate the performance of the parameter estimates of the IBTL distribution via Monte Carlo simulation study. Random samples of size $n = (25, 50, 100, 200, 500)$ are generated from the IBTL distribution at three distinct sets of parameter values $(a = 2.0,\ b = 0.4, \lambda = 0.5)$, $(a = 2.5,\ b = 0.3, \lambda = 0.7)$ and $(a = 3.0,\ b = 0.2, \lambda = 2.0)$. At each case, the simulation is repeated 1000 times and the following quantities are computed:

i) mean estimate $\left(\bar{\zeta}\right) = \frac{1}{N}\sum_{i=1}^{N} \hat{\zeta}_i,$

ii) average bias $= \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\zeta}_i - \bar{\zeta}\right)$,

iii) root mean square error (RMSE) $= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\zeta}_i - \bar{\zeta}\right)^2}$.

iv) Coverage Probability of the 95% confidence interval of the estimates $\hat{\zeta}_i$ given by $CP(\hat{\psi}) = \frac{1}{N}\sum_{i=1}^{N} I\left(\hat{\zeta}_i - Z_{\delta/2}\sqrt{\mathrm{var}\left(\hat{\zeta}\right)} < \zeta_0 < \hat{\zeta}_i + Z_{\delta/2}\sqrt{\mathrm{var}\left(\hat{\zeta}\right)}\right)$.

Where $I(.)$ is an indicator function and $\left(\hat{\zeta}\right)$ is the standard error of the estimate $\zeta_i$.

Tables 3.3 –3.6 display the mean estimate, average bias, root mean square error and coverage probability of the 95% confidence interval of the parameter estimates of the IBTL distribution.

**Table 3.3:** Results for the Mean Estimates of the Parameters of the IBTL distribution

| Parameters | $n$ | Mean$(a)$ | Mean$(b)$ | Mean$(\lambda)$ |
|---|---|---|---|---|
| | 25 | 2.6236 | 1.2405 | 0.4995 |
| $a=2.0$ | 50 | 2.3530 | 0.6628 | 0.5271 |
| $b=0.4$ | 100 | 2.1743 | 0.5744 | 0.5386 |
| $\lambda=0.5$ | 200 | 2.0890 | 0.4898 | 0.5317 |
| | 500 | 2.0350 | 0.4114 | 0.5279 |
| | | | | |
| | 25 | 3.4615 | 0.6818 | 0.7209 |
| $a=2.5$ | 50 | 2.9798 | 0.4156 | 0.7458 |
| $b=0.3$ | 100 | 2.7051 | 0.3524 | 0.7459 |
| $\lambda=0.7$ | 200 | 2.5948 | 0.3112 | 0.7384 |
| | 500 | 2.5227 | 0.3091 | 0.7082 |
| | | | | |
| | 25 | 3.9173 | 0.3282 | 1.8407 |
| $a=3.0$ | 50 | 3.7643 | 0.2670 | 2.0193 |

| | | | | |
|---|---|---|---|---|
| $b=0.2$ | 100 | 3.3766 | 0.2135 | 2.0686 |
| $\lambda=2.0$ | 200 | 3.1830 | 0.2075 | 2.0547 |
| | 500 | 3.0535 | 0.2025 | 2.0185 |

**Table 3.4:** Simulation Results for the Bias of the Parameters of the IBTL distribution

| Parameters | $n$ | Bias$(a)$ | Bias$(b)$ | Bias$(\lambda)$ |
|---|---|---|---|---|
| | 25 | 0.6236 | 0.8405 | -0.0005 |
| $a=2.0$ | 50 | 0.3530 | 0.2628 | 0.0271 |
| $b=0.4$ | 100 | 0.1743 | 0.1744 | 0.0386 |
| $\lambda=0.5$ | 200 | 0.0890 | 0.0898 | 0.0316 |
| | 500 | 0.0350 | 0.0114 | 0.0279 |
| | 25 | 0.9615 | 0.3815 | 0.0209 |
| $a=2.5$ | 50 | 0.4798 | 0.1156 | 0.0458 |
| $b=0.3$ | 100 | 0.2051 | 0.0524 | 0.0459 |
| $\lambda=0.7$ | 200 | 0.0948 | 0.0112 | 0.0384 |
| | 500 | 0.0227 | 0.0091 | 0.0082 |
| | 25 | 0.9173 | 0.1282 | -0.1592 |
| $a=3.0$ | 50 | 0.7643 | 0.0670 | 0.0193 |
| $b=0.2$ | 100 | 0.3766 | 0.0135 | 0.0686 |
| $\lambda=2.0$ | 200 | 0.1830 | 0.0075 | 0.0548 |
| | 500 | 0.0535 | 0.0025 | 0.0185 |

**Table 3.5:** Results for the RMSE of the Parameters of the IBTL distribution

| Parameters | $n$ | RMSE$(a)$ | RMSE$(b)$ | RMSE$(\lambda)$ |
|---|---|---|---|---|
| | 25 | 1.1339 | 2.6137 | 0.3869 |
| $a=2.0$ | 50 | 0.7507 | 0.8872 | 0.3596 |
| $b=0.4$ | 100 | 0.3720 | 0.7533 | 0.3422 |
| $\lambda=0.5$ | 200 | 0.2447 | 0.3679 | 0.2626 |
| | 500 | 0.1437 | 0.1285 | 0.1689 |
| | 25 | 2.0986 | 1.9290 | 0.4556 |
| $a=2.5$ | 50 | 1.2833 | 0.4496 | 0.3948 |

| | | | | |
|---|---|---|---|---|
| $b=0.3$ | 100 | 0.6162 | 0.2365 | 0.3367 |
| $\lambda=0.7$ | 200 | 0.3252 | 0.1188 | 0.2355 |
| | 500 | 0.1701 | 0.0718 | 0.1375 |
| | 25 | 2.0161 | 0.4124 | 0.9454 |
| $a=3.0$ | 50 | 1.1198 | 0.3365 | 0.8416 |
| $b=0.2$ | 100 | 0.9885 | 0.1135 | 0.6522 |
| $\lambda=2.0$ | 200 | 0.5888 | 0.0833 | 0.5267 |
| | 500 | 0.2830 | 0.0419 | 0.3007 |

**Table 3.6:** Results of the CP of 95% CI of the Parameters of the IBTL distribution

| Parameters | $n$ | CP$(a)$ | CP$(b)$ | CP$(\lambda)$ |
|---|---|---|---|---|
| | 25 | 0.913 | 0.912 | 0.826 |
| $a=2.0$ | 50 | 0.875 | 0.904 | 0.842 |
| $b=0.4$ | 100 | 0.892 | 0.920 | 0.866 |
| $\lambda=0.5$ | 200 | 0.927 | 0.938 | 0.888 |
| | 500 | 0.914 | 0.920 | 0.912 |
| | 25 | 0.918 | 0.866 | 0.862 |
| $a=2.5$ | 50 | 0.838 | 0.876 | 0.872 |
| $b=0.3$ | 100 | 0.862 | 0.896 | 0.906 |
| $\lambda=0.7$ | 200 | 0.951 | 0.924 | 0.926 |
| | 500 | 0.917 | 0.940 | 0.934 |
| | 25 | 0.901 | 0.934 | 0.828 |
| $a=3.0$ | 50 | 0.918 | 0.888 | 0.888 |
| $b=0.2$ | 100 | 0.928 | 0.886 | 0.926 |
| $\lambda=2.0$ | 200 | 0.912 | 0.906 | 0.926 |
| | 500 | 0.973 | 0.946 | 0.958 |

Simulation results from Tables 3.3 – 3.6 are discussed as follows:

i) The mean estimates in Table 3.3 approaches the true parameter value as the sample size $n$ increases;

ii) Table 3.4 shows that the parameter estimate $\hat{a}$ and $\hat{b}$ are positively biased while $\hat{\lambda}$ is both negatively and positively biased. Furthermore, the bias of $\hat{a}$ and $\hat{b}$ decrease as the sample size $n$ increases;

iii) From Table 3.5, the root mean square error of the parameter estimates $\hat{a}$ , $\hat{b}$ and $\hat{\lambda}$ decrease as the sample size $n$ increases;

Finally, Table 3.6 shows that the coverage probability of the 95% confidence interval of the estimates are very close to the nominal level of 95%.

# CHAPTER FOUR

# LIFETIME DATA FITTINGS

## 4.1 Introduction

In this chapter, the performance of the inverse Burr Topp-Leone (IBTL) distribution in real-life data fitting is illustrated using two different germination data sets collated from the Nigerian Institute for Oil Palm Research (NIFOR), Edo State. The data sets comprise of the rate of germination of oil palm seeds after the process of breaking dormancy. Two different batches were investigated and the rate of germination were recorded on weekly basis. The duration of the experiment spanned through 23rd May, 2022 – 14th April, 2023.

In other to prove the relevance of the proposed IBTL distribution among existing bounded lifetime distributions, five competing distributions were considered to fit the two germination data sets alongside the proposed IBTL distribution. The probability density function (pdf) of the competing distributions is given as follows:

1. Odd log-logistic Kumaraswamy (OLLK) distribution introduced by Opone et al. (2023);

$$f(x,a,b,\alpha) = \frac{\alpha abx^{a-1}\left(1-x^a\right)^{b\alpha-1}\left[1-\left(1-x^a\right)^b\right]^{\alpha-1}}{\left[\left(1-\left(1-x^a\right)^b\right)^{\alpha}+\left(1-x^a\right)^{b\alpha}\right]^2},$$

2. Unit-Burr XII (UBXII) distribution developed by Korkmaz and Chesneau (2021);

$$f(x,\alpha,\beta) = \alpha\beta x^{-1}\left(-\log x\right)^{\beta-1}\left(1+\left(-\log x\right)^{\beta}\right)^{-(\alpha+1)};$$

3. Unit-Burr III (UBIII) distribution proposed by Modi and Gill (2020);

$$f(x,\lambda,\beta) = \lambda\beta x^{-2}\left(x^{-1}-1\right)^{\beta-1}\left(1+\left(x^{-1}-1\right)^{\beta}\right)^{-(\lambda+1)};$$

4. Beta distribution reported in Opone and Ekhosuehi (2017);

$$f(x,a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)};$$

5. Kumaraswamy distribution developed by Kumaraswamy (1980);

$$f(x,a,b) = abx^{a-1}\left(1-x^{a}\right)^{b-1}.$$

The two germination data sets are given as follow.

**Data Set I**

0.0820, 0.1500, 0.1607, 0.1891, 0.2193, 0.2297, 0.2337, 0.2357, 0.2535, 0.2625, 0.2702, 0.2972, 0.3071, 0.3250, 0.3483, 0.3513, 0.3918, 0.3875, 0.4048, 0.4125, 0.4324, 0.4416, 0.4500, 0.4661, 0.4675, 0.4700, 0.4729, 0.4741, 0.4825, 0.4864, 0.4867, 0.5000, 0.5323, 0.5405, 0.5454, 0.5779, 0.5783, 0.5899, 0.6012, 0.6043, 0.6054, 0.6077, 0.6115, 0.6324, 0.6402, 0.6546, 0.6756, 0.7405, 0.8486

**Data Set II**

0.0885, 0.1656, 0.1906, 0.2420, 0.2524, 0.2573, 0.2611, 0.2640, 0.2802, 0.2865, 0.2929, 0.3111, 0.3234, 0.3508, 0.3636, 0.3703, 0.3426, 0.3764, 0.3949, 0.4039, 0.4152, 0.4343, 0.4433, 0.4491, 0.4554, 0.4662, 0.4820, 0.4983, 0.5289, 0.5475, 0.5863, 0.5914, 0.6011, 0.6070, 0.6229, 0.6250, 0.6367, 0.6524, 0.6931, 0.7191, 0.7292, 0.7386, 0.7404, 0.7516, 0.7727, 0.7954, 0.8340, 0.8681, 0.9363, 0.9568, 0.9681.

The descriptive statistics for the two germination data sets are shown in Table 4.1.

**Table 4.1:** Descriptive Statistics for Data Set I and II

| Min. | 1st Quartile | Median | Mean | 3rd Quartile | Skewness | Max |
|---|---|---|---|---|---|---|
| | | | **DATA I** | | | |
| 0.0820 | 0.3071 | 0.4675 | 0.4434 | 0.5783 | -0.0706 | 0.8486 |
| | | | **DATA II** | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.0885 | 0.3350 | 0.4662 | 0.5091 | 0.6727 | 0.3027 | 0.9681 |

From the Table, we observe that the first germination data set is negatively-skewed, whereas the second germination data set is positively-skewed. Figures 4.1 and 4.2 show the graphical representation of the data sets in Boxplot and Histogram.

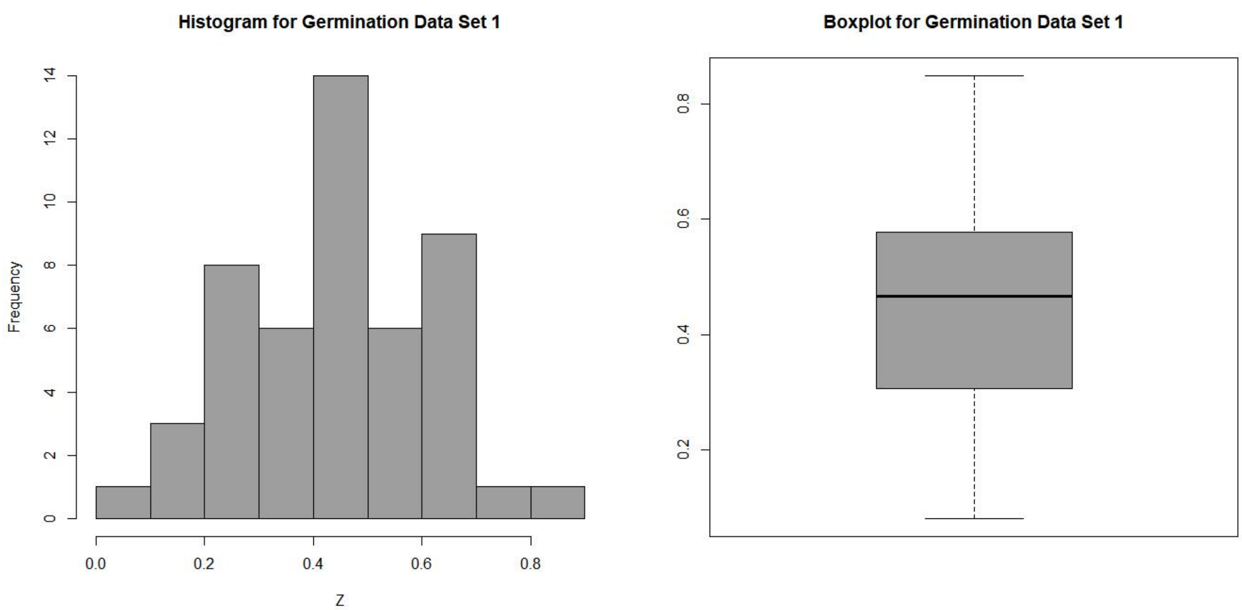


**Figure 4.1:** The Histogram and Boxplot of the Germination Data Set I

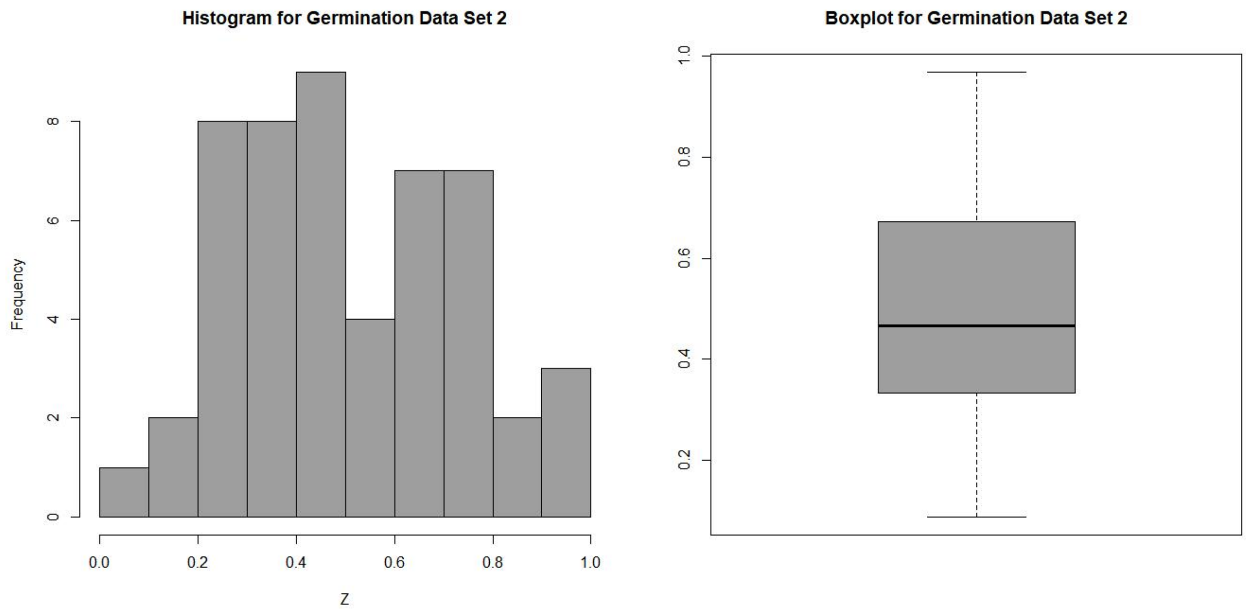


**Figure 4.2:** The Histogram and Boxplot of the Germination Data Set II

From the boxplot in Figure 4.1, we observe that the median line gravitates toward the top of the box which indicates negative Skewness, while in Figure 4.2, the median line gravitates toward the bottom of the box which indicates positive Skewness. Furthermore, we noticed that there is no presence of outliers in both data sets.

For the purpose of model comparison, we considered some model selection criteria such as the maximized log-likelihood (LogL), Akaike Information Criteria (AIC), and some goodness of fit test statistics such as the Komolgorov-Smirnov (K-S), Crammer von Mises ($W^*$) and Anderson Darling ($A^*$) test statistics with their corresponding *p-values.* Tables 4.1 and 4.2 display the summary results of the fit of the distributions for the germination data set 1 and 2, respectively.

**Table 4.2:** Summary Statistics for Germination data set 1

| Models | Parameter estimates | LogL | AIC | *K-S* *(p-value)* | $W^*$ *(p-value)* | $A^*$ *(p-value)* |
|---|---|---|---|---|---|---|
| IBTL | $a = 3.9634$ | 19.7898 | -33.5796 | 0.0674 | 0.0329 | 0.2137 |
| | $b = 0.1909$ | | | **(0.9683)** | **(0.9668)** | **(0.9862)** |
| | $\lambda = 3.1702$ | | | | | |
| OLLK | $a = 2.8196$ | 18.6597 | -31.3194 | 0.0778 | 0.0474 | 0.2981 |
| | $b = 6.4815$ | | | **(0.9056)** | **(0.8942)** | **(0.9393)** |
| | $\alpha = 0.9258$ | | | | | |
| UBXII | $\alpha = 1.5838$ | 18.2078 | -32.4156 | 0.0855 | 0.0597 | 0.3654 |
| | $\beta = 2.9535$ | | | **(0.8363)** | | **(0.8179)** |
| **(0.8817)** | | | | | | |
| UBIII | $\lambda = 0.6746$ | 18.5448 | -33.0896 | 0.0781 | 0.0523 | 0.3238 |
| | $\beta = 2.6951$ | | | **(0.9028)** | **(0.8642)** | **(0.9189)** |
| Beta | $a = 3.3845$ | 18.4266 | -32.8532 | 0.0937 | 0.0681 | 0.3914 |
| | $b = 4.2751$ | | | **(0.7468)** | | **(0.7652)** |
| **(0.8566)** | | | | | | |
| Kumaraswamy | $a = 2.6147$ | 18.6478 | -33.2957 | 0.0801 | 0.0503 | 0.3119 |
| | $b = 5.4606$ | | | **(0.8864)** | **(0.8765)** | |
| **(0.9286)** | | | | | | |

**Table 4.3:** Summary Statistics for Germination data set 2

| Models | Parameter estimates | LogL | AIC | *K-S* *(p-value)* | $W^*$ *(p-value)* | $A^*$ *(p- value)* |
|---|---|---|---|---|---|---|
| IBTL | $a = 2.2091$ | 7.7822 | -9.5646 | 0.0768 | 0.0266 | 0.1872 |
| | $b = 2.2165$ | | | **(0.9010)** | **(0.9867)** | **(0.9936)** |
| | $\lambda = 0.7241$ | | | | | |
| OLLK | $a = 1.0305$ | 6.5373 | -7.0745 | 0.0836 | 0.0626 | 0.4538 |
| | $b = 0.9920$ | | | **(0.8389)** | **(0.8000)** | **(0.7934)** |
| | $\alpha = 1.5706$ | | | | | |
| UBXII | $\alpha = 1.9454$ | 5.1829 | -6.3658 | 0.1076 | 0.1115 | 0.7316 |
| | $\beta = 1.9074$ | | | **(0.5591)** | **(0.5327)** | **(0.5323)** |
| UBIII | $\lambda = 1.1000$ | 6.6694 | -9.3389 | 0.0963 | 0.0852 | 0.5208 |
| | $\beta = 1.5398$ | | | **(0.6952)** | **(0.6644)** | **(0.7248)** |

| | | | | | | |
|---|---|---|---|---|---|---|
| Beta | $a = 2.0795$ <br> $b = 1.9231$ | 6.4680 | -8.9361 | 0.0946 <br> **(0.7156)** | 0.0844 <br> **(0.6684)** | 0.5495 <br> **(0.6961)** |
| Kumaraswamy | $a = 1.8615$ <br> $b = 1.9274$ | 6.1783 | -8.3569 | 0.0966 <br> **(0.6908)** | 0.0925 <br> **(0.6245)** | 0.5983 <br> **(0.6488)** |

Furthermore, we investigate the fit of the competing distributions for the two germination data sets by means of graphical representation such as density fit and probability-probability (p-p) plots.

Consequently, Figures 4.3 – 4.6, respectively, show the density fit as well as the probability-probability (p-p) plots of the distributions for the two data sets.

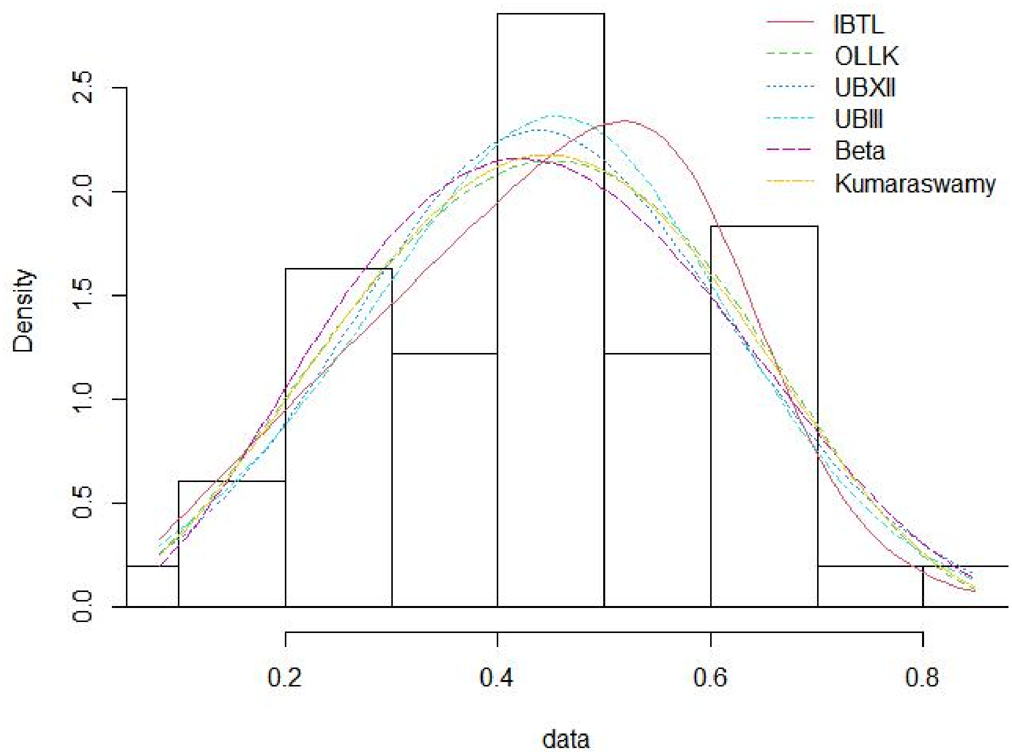


**Figure 4.3:** The density fit of the distributions for the germination data set I

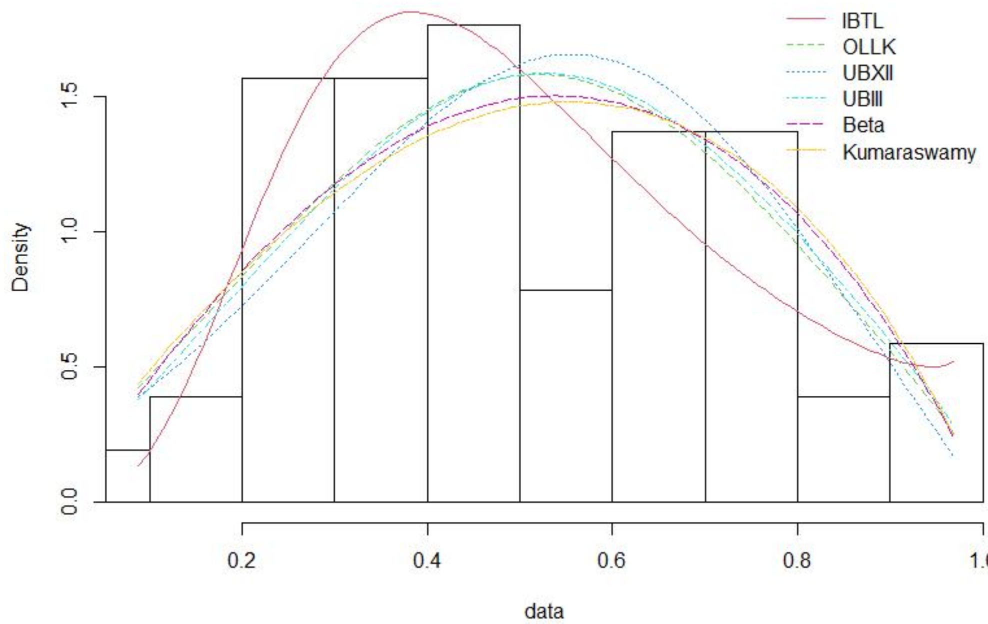


**Figure 4.4:** The density fit of the distributions for the germination data set II

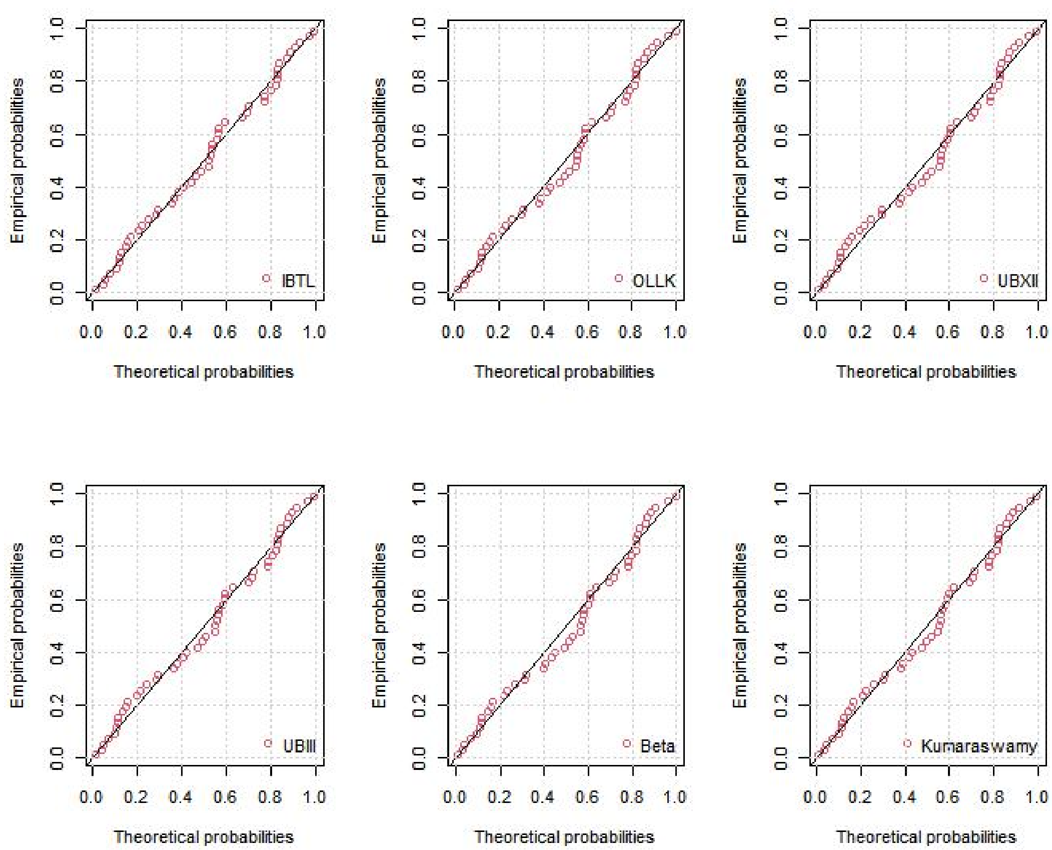


**Figure 4.5:** The (p-p) plots of the distributions for the germination data set I

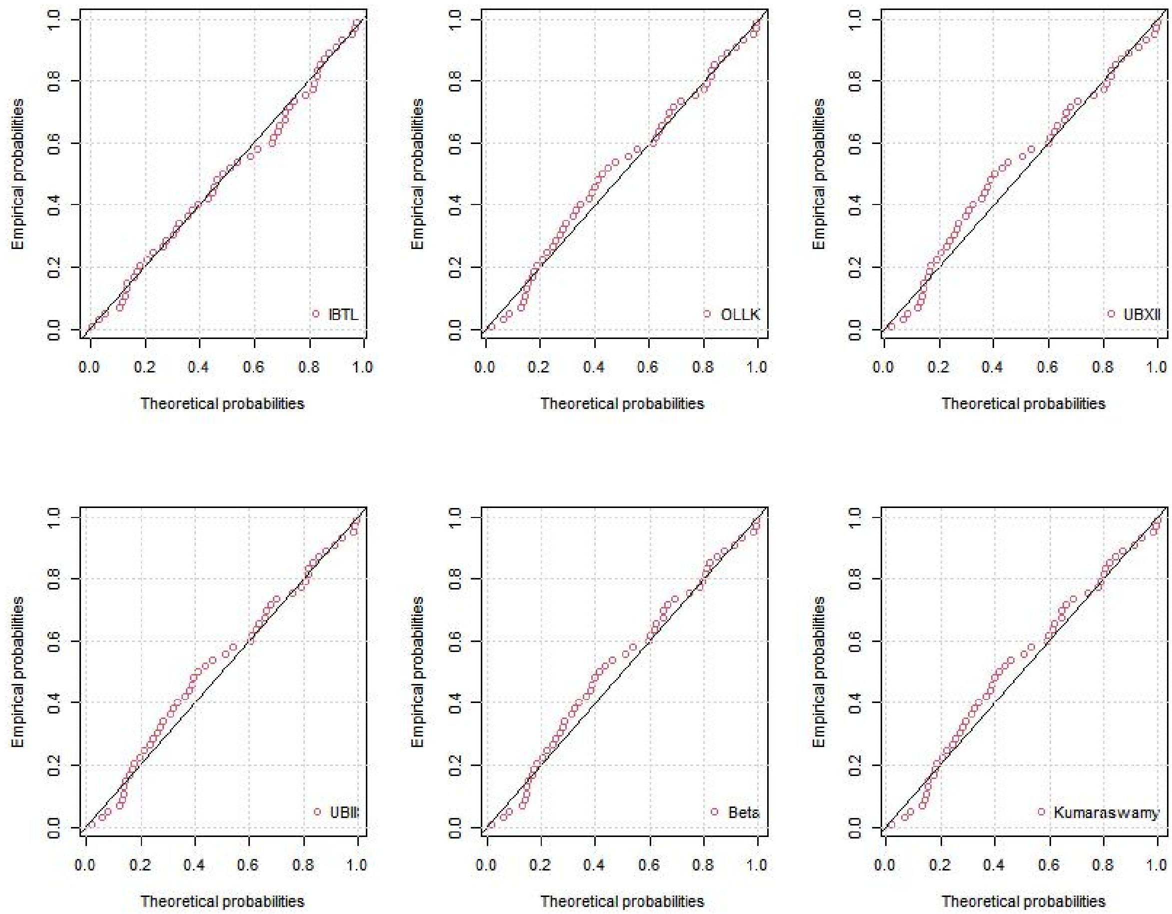


**Figure 4.6:** The (p-p) plots of the distributions for the germination data set II

## 4.2 Discussion of Results

In statistical theory, model comparison is based on some model selection criteria such as log-likelihood (LogL), Akaike Information Criteria (AIC), and some goodness of fit test statistics such as the Komolgorov-Smirnov (K-S), Crammer von Mises ($W^*$) and Anderson Darling ($A^*$) test statistics. More often, the most suitable model that best fit the data set is traceable to the one having the maximized log-likelihood value, and the least value in terms of the AIC, K-S, $W^*$ and $A^*$ test statistics with the corresponding highest *p-value*. Obviously, from Tables 4.2 and 4.3, we observe that the inverse Burr Topp-Leone (IBTL) distribution satisfies the conditions and thus, outperforms the competing distributions in analyzing the two germination data sets under study. The density fit as well as the probability-probability (p-p) plots of the distributions for the two germination data sets were examined in Figures 4.3 – 4.6 to further illustrate the performance of the IBTL distribution over the competing distributions.

# CHAPTER FIVE

# SUMMARY AND CONCLUSION

## 5.1 Findings

The findings of the study are as follows:

1. A new lifetime distribution for modelling germination data is introduced;

2. the shape of the density function of the IBTL distribution can be decreasing, right-skewed or left-skewed unimodal, symmetric shapes, leptokurtic, mesokurtic, platykurtic, etc., whereas, the shape of the hazard function can be increasing, bathtub and inverted bathtub; which means the IBTL distribution can fit any data with these attributes.

3. the average bias and root mean square error for the MLEs of the proposed distribution decrease as the sample size $n$ increases, while the coverage probability approaches the nominal level of 95% as the sample size $n$ increases; which are properties expected of a good estimator.

4. an application of the proposed IBTL distribution to two oil palm seeds germination rate data sets revealed its superiority over some existing competing distributions

## 5.2 Contribution to knowledge

This thesis has contributed to knowledge by developing a new parametric model (Inverse Burr Topp-Leone distribution) to analyze oil palm seeds germination data sets.

## 5.3 Conclusion

Recently, there have been an increase in developing new lifetime distributions to handle different real-world situations. Several methods of generalizing classical lifetime distributions which include the Marshall-Olkin extended family of distributions introduced by Marshall and Olkin (1997), the beta-generated family of distributions proposed by Eugene et al. (2002), the quadratic rank transmutation map (QRTM) developed by Shaw and Buckley (2009), the Kumaraswamy-G family of distributions proposed by Cordeiro and de Castro (2011), the transformed-transformer ( $T-X$ ) family of distributions introduced by Alzaatreh *et al.* (2013), have been established in literature.

In this study, we employed the transformed-transformer ( $T-X$ )framework to develop a new lifetime distribution known as the inverse Burr Topp-Leone (IBTL) distribution. Basic statistical properties of the proposed IBTL distribution such as the density and cumulative distribution functions, survival and hazard rate functions, quantile, moments, moment generating function, probability weighted moments, Renyi entropy and distribution of order statistics were derived. The parameter estimates of the IBTL distribution were obtained via the method of maximum likelihood estimation. The performance of the parameter estimates was investigated through a Monte Carlo simulation study. For data analysis purpose, we adopted two different germination data sets from Nigerian Institute for Oil Palm Research (NIFOR), to illustrate the usefulness of the proposed IBTL distribution in lifetime data fittings. Some existing bounded distributions alongside with the proposed IBTL distribution were used to fit the two germination data sets and their fits were compared based on some model selection criteria and goodness of fit test statistics. Analysis of the results clearly indicated that the IBTL distribution performed reasonably better than the competing distributions and thus, becomes the most appropriate model in fitting the two germination data sets.

## 5.4 Suggestion for further studies

In this research, we have introduced a new lifetime distribution by utilizing the transformed-transformer (*T-X*) framework proposed by Alzaatreh et al. (2013). We refer to this model as the inverse Burr Topp-Leone distribution. Here, the random variable $T$ in the (*T-X*) framework follows the inverse Burr distribution, while the random variable $X$ follows the Topp-Leone distribution.

Subsequently, we suggest that for further research, interested researchers can consider the random variable $X$ to follow the cumulative distribution function of any probability distribution, while retaining $T$ to follow the inverse Burr distribution. By this, we have the inverse Burr generated family of distributions.

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## Appendix

```
#===========PDFof the Inverse Burr-Topp-Leone Distribution
dTLD <- function(x,lambda){2*lambda*(1-x)*(x^(lambda-1))*((2-x)^(lambda-1))}
pTLD <- function(x,lambda){(x*(2-x))^lambda}
dIBTLD <- function(x,a,b,lambda){a0= a*b*dTLD(x,lambda)*((1-pTLD(x,lambda))^(-1)); a1= (-log(1-pTLD(x,lambda)))^(-a-1); a2= ((1+(-log(1-pTLD(x,lambda)))^(-a))^(-b-1)); a3 = a0*a1*a2}
plot(c(0,1), c(0,6), type="n",ylab="f(x)",xlab="x", main="")
x <- seq(0.001,5,0.0001)
lines(x, dIBTLD(x,3,.5,3), col=1, lty=1,lwd=2)
lines(x, dIBTLD(x,.2,2,1.5), col=2, lty=4,lwd=2)
lines(x, dIBTLD(x,4,.5,.2), col=3, lty=6,lwd=2)
lines(x, dIBTLD(x,3,3,.2), col=4, lty=10,lwd=2)
lines(x, dIBTLD(x,3,4,5), col=5, lty=14,lwd=2)
legend("top", legend = c(expression(lambda==3~~a==3~~b==.5),
                    expression(lambda==1.5~~a==.2~~b==2),
                    expression(lambda==0.2~~a==4~~b==.5),
                    expression(lambda==0.2~~a==3~~b==3),
                    expression(lambda==5~~a==3~~b==4)),
     lwd=c(2,2,2,2,2) ,lty=1:5, col=c(1,2,3,4,5), cex = 1.2, box.lwd = 1)
grid()

#=========The CDFOf Inverse Burr-Topp-Leone Distribution
dTLD <- function(x,lambda){2*lambda*(1-x)*(x^(lambda-1))*((2-x)^(lambda-1))}
pTLD <- function(x,lambda){(x*(2-x))^lambda}
pIBTLD <- function(x,a,b,lambda){a2= ((1+(-log(1-pTLD(x,lambda)))^(-a))^(-b))}
plot(c(0,1), c(0,1), type="n",ylab="F(x)",xlab="x", main="")
x <- seq(0.001,5,0.0001)
lines(x, pIBTLD(x,3,.5,3), col=1, lty=1,lwd=2)
lines(x, pIBTLD(x,2,2,.1), col=2, lty=4,lwd=2)
lines(x, pIBTLD(x,4,3,.2), col=3, lty=6,lwd=2)
```

```
lines(x, pIBTLD(x,7,.3,3), col=4, lty=10,lwd=2)
lines(x, pIBTLD(x,3,.4,5), col=5, lty=14,lwd=2)
legend("bottomright", legend = c(expression(lambda==3~~a==3~~b==.5),
                        expression(lambda==.1~~a==2~~b==2),
                        expression(lambda==0.2~~a==4~~b==3),
                        expression(lambda==3~~a==7~~b==.3),
                        expression(lambda==5~~a==3~~b==.4)),
       lwd=c(2,2,2,2,2) ,lty=1:5, col=c(1,2,3,4,5), cex = 1.2, box.lwd = 1)
grid()

#=========The Survival Function of Inverse Burr-Topp-Leone Distribution
dTLD <- function(x,lambda){2*lambda*(1-x)*(x^(lambda-1))*((2-x)^(lambda-1))}
pTLD <- function(x,lambda){(x*(2-x))^lambda}
sIBTLD <- function(x,a,b,lambda){1-pIBTLD(x,a,b,lambda)}
plot(c(0,1), c(0,1), type="n",ylab="S(x)",xlab="x", main="")
x <- seq(0.001,5,0.0001)
lines(x, sIBTLD(x,3,.5,3), col=1, lty=1,lwd=2)
lines(x, sIBTLD(x,2,2,.1), col=2, lty=4,lwd=2)
lines(x, sIBTLD(x,4,3,.2), col=3, lty=6,lwd=2)
lines(x, sIBTLD(x,7,.3,3), col=4, lty=10,lwd=2)
lines(x, sIBTLD(x,3,.4,5), col=5, lty=14,lwd=2)
legend("topright", legend = c(expression(lambda==3~~a==3~~b==.5),
                            expression(lambda==.1~~a==2~~b==2),
                            expression(lambda==0.2~~a==4~~b==3),
                            expression(lambda==3~~a==7~~b==.3),
                            expression(lambda==5~~a==3~~b==.4)),
       lwd=c(2,2,2,2,2) ,lty=1:5, col=c(1,2,3,4,5), cex = 1.2, box.lwd = 1)
grid()

#=========The Hazard Rate Function of Inverse Burr-Topp-Leone Distribution
dTLD <- function(x,lambda){2*lambda*(1-x)*(x^(lambda-1))*((2-x)^(lambda-1))}
pTLD <- function(x,lambda){(x*(2-x))^lambda}
hIBTLD <- function(x,a,b,lambda){dIBTLD(x,a,b,lambda)/sIBTLD(x,a,b,lambda)}
plot(c(0,1), c(0,15), type="n",ylab="h(x)",xlab="x", main="")
x <- seq(0.001,5,0.0001)
lines(x, hIBTLD(x,2,5,3), col=1, lty=1,lwd=2)
lines(x, hIBTLD(x,2,2,.1), col=2, lty=4,lwd=2)
```

```
lines(x, hIBTLD(x,4,2,.2), col=3, lty=6,lwd=2)
lines(x, hIBTLD(x,.3,2,3), col=4, lty=10,lwd=2)
legend("top", legend = c(expression(lambda==3~~a==2~~b==5),
                        expression(lambda==.1~~a==2~~b==2),
                        expression(lambda==0.2~~a==4~~b==2),
                        expression(lambda==5~~a==.3~~b==2)),
      lwd=c(2,2,2,2,2) ,lty=1:5, col=c(1,2,3,4,5), cex = 1.2, box.lwd = 1)
grid()
#===================================================================
#QUANTILE FUNCTION OF THE INVERSE BURR-TOPP-LEONE
        DISTRIBUTION
P <- numeric(10)
Q <- numeric(10)
a <- 5
b <- 2
lambda <- 4
u <- c(0.05, 0.25, 0.29, 0.34, 0.39, 0.44, 0.49, 0.54, 0.59, 0.64)
for (i in 1:10){
 Q[i] <- (1-sqrt(1-(1-exp(-(u[i]^(-1/b)-1)^(-1/a)))^(1/lambda)))
 }
}
print(round(Q,4))

#===================================================================
#THEORETICAL MOMENTS OF THE INVERSE BURR-TOPP-
LEONEDISTRIBUTION
Z <- numeric(4)
r <- c(1,2,3,4)
for (i in 1:4){
 dIBTLD <- function(x,a,b,lambda){a0= 2*lambda*a*b*(1-x)*(x*(2-x))^(lambda-
        1);a1= (-log(1-(x*(2-x))^lambda)); a2=(a1)^(-a-1); a3=(1+(a1)^(-a))^(-b-1);
        a4= (1-(x*(2-x))^lambda);a5= ((a0*a2*a3)/a4)}
 f <- function(x,a,b,lambda,r.){(x^r[i])*dIBTLD(x,a,b,lambda)}
 y <- integrate(f,lower=0,upper=1,subdivisions=1000000,a=8, b=.8, lambda=2,
        r.=r)
 Z[i] <- round(y$value,4)
}
```

```
print(Z)
variance. <- (round(Z[2]-(Z[1])^2,4))
skewness. <- (round((Z[3]-3*Z[1]*Z[2]+2*(Z[1])^3)/((variance.)^(3/2)),4))
kurtosis. <- (round((Z[4]-(4*Z[1]*Z[3])+(6*(Z[1])^2*Z[2])-
    (3*(Z[1])^4))/(variance.)^2,4))
cbind(Z[1], variance., skewness., kurtosis.)

#==========================================================
#APPLICATION OF THE INVERSE BURR-TOPP-LEONE DISTRIBUTION TO
    LIFETIME DATA
#====THE INVERSE BURR-TOPP-LEONE DISTRIBUTION
dTLD <- function(x,lambda){2*lambda*(1-x)*(x^(lambda-1))*((2-x)^(lambda-1))}
pTLD <- function(x,lambda){(x*(2-x))^lambda}
dIBTLD <- function(x,a,b,lambda){a0= a*b*dTLD(x,lambda)*((1-
    pTLD(x,lambda))^(-1)); a1= (-log(1-pTLD(x,lambda)))^(-a-1); a2= ((1+(-
    log(1-pTLD(x,lambda)))^(-a))^(-b-1)); a3 = a0*a1*a2}
pIBTLD <- function(q,a,b,lambda){a2= ((1+(-log(1-pTLD(q,lambda)))^(-a))^(-b))}

#=========THE ODD LOG-LOGISTIC KUMARASWAMY DISTRIBUTION
dOLLKD <- function(x,a,b,alpha){a0= alpha*a*b*x^(a-1); a1= (1-x^a)^(b-1); a2=
    (1-(1-x^a)^b)^(alpha-1); a3= ((1-x^a)^b)^(alpha-1); a4= (a0*a1*a2*a3);
    a5= (1-(1-x^a)^b)^alpha; a6= ((1-x^a)^b)^alpha; a7= (a5+a6)^2; a8= a4/a7}
pOLLKD <- function(q,a,b,alpha){a0=(1-(1-q^a)^b)^alpha; a1= ((1-q^a)^b)^alpha;
    a2= (a0+a1);a3= a0/a2}

#==================== KUMARASWAMY DISTRIBUTION
dKD <- function(x,a,b){e0=(a*b)*(x^(a-1))*((1-x^a)^(b-1))}
pKD <- function(q,a,b){e1=1-((1-q^a)^b)}

#==================== UNIT BURR XII DISTRIBUTION
dUBXIID <- function(x,alpha,beta){e0=alpha*beta*(x^(-1))*(-log(x))^(beta-
    1);e1=(1+(-log(x))^beta)^(-alpha-1);e2=e0*e1}
pUBXIID <- function(q,alpha,beta){e1=(1+(-log(q))^beta)^(-alpha)}
qUBXIID <- function(p,alpha,beta){e1=exp(-(p^(-1/lambda)-1)^(1/beta))}

#==================== UNIT BURR III DISTRIBUTION
```

```
dUBIIID <- function(x,lambda,beta){e0=lambda*beta*(x^(-2))*((1/x)-1)^(beta-
        1);e1=(1+((1/x)-1)^beta)^(-lambda-1);e2=e0*e1}
pUBIIID <- function(q,lambda,beta){e1=(1+((1/q)-1)^beta)^(-lambda)}
qUBIIID <- function(p,lambda,beta){(1+(p^(-1/lambda)-1)^(1/beta))^(-1)}



#Z <- c(0.0820, 0.1500, 0.1607, 0.1891, 0.2193, 0.2297, 0.2337, 0.2357, 0.2535,
0.2625, 0.2702, 0.2972, 0.3071, 0.3250, 0.3483, 0.3513, 0.3918, 0.3875, 0.4048,
0.4125, 0.4324, 0.4416, 0.4500, 0.4661, 0.4675, 0.4700, 0.4729, 0.4741, 0.4825,
0.4864, 0.4867, 0.5000, 0.5323, 0.5405, 0.5454, 0.5779, 0.5783, 0.5899, 0.6012,
0.6043, 0.6054, 0.6077, 0.6115, 0.6324, 0.6402, 0.6546, 0.6756, 0.7405, 0.8486)

Z <- c(0.0885, 0.1656, 0.1906, 0.2420, 0.2524, 0.2573, 0.2611, 0.2640, 0.2802,
0.2865, 0.2929, 0.3111, 0.3234, 0.3508, 0.3636, 0.3703, 0.3426, 0.3764, 0.3949,
0.4039, 0.4152, 0.4343, 0.4433, 0.4491, 0.4554, 0.4662, 0.4820, 0.4983, 0.5289,
0.5475, 0.5863, 0.5914, 0.6011, 0.6070, 0.6229, 0.6250, 0.6367, 0.6524, 0.6931,
0.7191, 0.7292, 0.7386, 0.7404, 0.7516, 0.7727, 0.7954, 0.8340, 0.8681, 0.9363,
0.9568, 0.9681)

length(Z)
summary(Z)
skewness(Z)
kurtosis(Z)
hist(Z, col=8, main="Histogram for Germination Data Set 2")
boxplot(Z, col=8, main="Boxplot for Germination Data Set 2")
M1 <- fitdist(Z,"IBTLD", start=list(a=1,b=1,lambda=1))
M2 <- fitdist(Z,"OLLKD", start=list(alpha=1,a=.1,b=1))
M3 <- fitdist(Z,"UBXIID", start=list(alpha=1,beta=1))
M4 <- fitdist(Z,"UBIIID", start=list(lambda=1,beta=.1))
M5 <- fitdist(Z,"beta", start=list(shape1 =1,shape2=.1))
M6 <- fitdist(Z,"KD", start=list(a=1,b=.1))

summary(M1)
summary(M2)
summary(M3)
summary(M4)
```

```
summary(M5)
summary(M6)

ks.test(Z,"pIBTLD",a=coef(M1)[1],b=coef(M1)[2], lambda=coef(M1)[3])
cvm.test(pIBTLD(Z,a=coef(M1)[1],b=coef(M1)[2], lambda=coef(M1)[3]))
ad.test(pIBTLD(Z,a=coef(M1)[1],b=coef(M1)[2], lambda=coef(M1)[3]))

ks.test(Z,"pOLLKD",alpha=coef(M2)[1],a=coef(M2)[2], b=coef(M2)[3])
cvm.test(pOLLKD(Z,alpha=coef(M2)[1],a=coef(M2)[2], b=coef(M2)[3]))
ad.test(pOLLKD(Z,alpha=coef(M2)[1],a=coef(M2)[2], b=coef(M2)[3]))

ks.test(Z,"pUBXIID",alpha=coef(M3)[1],beta=coef(M3)[2])
cvm.test(pUBXIID(Z,alpha=coef(M3)[1],beta=coef(M3)[2]))
ad.test(pUBXIID(Z,alpha=coef(M3)[1],beta=coef(M3)[2]))

ks.test(Z,"pUBIIID",lambda=coef(M4)[1],beta=coef(M4)[2])
cvm.test(pUBIIID(Z,lambda=coef(M4)[1],beta=coef(M4)[2]))
ad.test(pUBIIID(Z,lambda=coef(M4)[1],beta=coef(M4)[2]))

ks.test(Z,"pbeta",shape1=coef(M5)[1],shape2=coef(M5)[2])
cvm.test(pbeta(Z,shape1=coef(M5)[1],shape2=coef(M5)[2]))
ad.test(pbeta(Z,shape1=coef(M5)[1],shape2=coef(M5)[2]))

ks.test(Z,"pKD",a=coef(M6)[1],b=coef(M6)[2])
cvm.test(pKD(Z,a=coef(M6)[1],b=coef(M6)[2]))
ad.test(pKD(Z,a=coef(M6)[1],b=coef(M6)[2]))

plot.legend <- c("IBTL", "OLLK", "UBXII", "UBIII", "Beta", "Kumaraswamy")

#par(mfrow=c(1,2))
denscomp(list(M1,M2,M3,M4,M5,M6), legendtext=plot.legend[], main = "")
cdfcomp(list(M1,M2,M3,M4,M5,M6), legendtext=plot.legend[], main = "")

par(mfrow=c(2,3))
ppcomp(M1, legendtext=plot.legend[1], main = "")
grid()
ppcomp(M2, legendtext=plot.legend[2], main = "")
```

```
grid()
ppcomp(M3, legendtext=plot.legend[3], main = "")
grid()
ppcomp(M4, legendtext=plot.legend[4], main = "")
grid()
ppcomp(M5, legendtext=plot.legend[5], main = "")
grid()
ppcomp(M6, legendtext=plot.legend[6], main = "")
grid()

#SIMULATION STUDY ON THE INVERSE BURR-TOOP-LEONE DISTRIBUTION
#============================================================
library(Matrix)
library(numDeriv)
library(fBasics)

a=3
b=.2
lambda=2
#Define the log-likelihood function of the IBTLD
IBTLD_LL <- function(par){-(n*log(2) +n*log(par[1]) + n*log(par[2]) +
        n*log(par[3]) + sum(log(1-x)) + (par[3]-1)*sum(log(x*(2-x))) - sum(log(1-
        (x*(2-x))^(par[3]))) - (par[1]+1)*sum(log(-log(1-(x*(2-x))^(par[3])))) -
        (par[2]+1)*sum(log(1+(-log(1-(x*(2-x))^(par[3])))^(-par[1]))))}

#choose sample size
n1<-c(50)
#if you want to check one sample at a time then use n1<-c(sample size)
for (j in 1:length(n1)) {
 # Decide number of simulations
 N=500
 n=n1[j]
 mle_a<-c(rep(0,N))
 mle_b<-c(rep(0,N))
 mle_lambda<-c(rep(0,N))
```

```
LC_a<-c(rep(0,N))
LC_b<-c(rep(0,N))
LC_lambda<-c(rep(0,N))

UC_a<-c(rep(0,N))
UC_b<-c(rep(0,N))
UC_lambda<-c(rep(0,N))

count_a=0
count_b=0
count_lambda=0

temp=1
HH1<-matrix(c(rep(2,9)),nrow=3,ncol=3)
HH2<-matrix(c(rep(2,9)),nrow=3,ncol=3)

for (i in 1:N)
{
  print(i)
  flush.console()
  repeat{
    x<-c(rep(0,n))

    #Generate a random sample from the uniform distribution
    u<-0
    u<-runif(n,min=0,max=1)


    for (k in  1:n){

      x[k]<-((1-sqrt(1-(1-exp(-(u[k]^(-1/b)-1)^(-1/a)))^(1/lambda))))

    }

    #Maximum likelihood estimation
    mle.result<-nlminb(c(a,b,lambda),IBTLD_LL,lower=0,upper=Inf)
```

```
    temp=mle.result$convergence
    if (temp==0){
      temp_a<-mle.result$par[1]
      temp_b<-mle.result$par[2]
      temp_lambda<-mle.result$par[3]


      HH1<-hessian(IBTLD_LL,c(temp_a,temp_b,temp_lambda))
      if ((rcond(HH1)>1e-8) & sum(is.nan(HH1))==0 & (diag(HH1)[1]>0)
& (diag(HH1)[2]>0) & (diag(HH1)[3]>0)){
        HH2<-solve(HH1)
        print(det(HH1))
      }
      else{
        temp=1
      }
    }

    if ((temp==0)& (diag(HH2)[1]>0)&
        (diag(HH2)[2]>0)&(diag(HH1)[3]>0)&(sum(is.nan(HH2))==0)){
      break
    }
  }
  print(temp)
  temp=1
  mle_a[i]<-mle.result$par[1]
  mle_b[i]<-mle.result$par[2]
  mle_lambda[i]<-mle.result$par[3]

  HH<-hessian(IBTLD_LL,c(mle_a[i],mle_b[i],mle_lambda[i]))
  H<-solve(HH)

  LC_a[i]<-mle_a[i]-1.96*sqrt(diag(H)[1])
  UC_a[i]<-mle_a[i]+1.96*sqrt(diag(H)[1])
  if((LC_a[i]<=a)& (a<=UC_a[i])){
    count_a=count_a+1
  }
```

```
  LC_b[i]<-mle_b[i]-1.96*sqrt(diag(H)[2])
  UC_b[i]<-mle_b[i]+1.96*sqrt(diag(H)[2])
  if((LC_b[i]<=b)& (b<=UC_b[i])){
   count_b=count_b+1
  }

  LC_lambda[i]<-mle_lambda[i]-1.96*sqrt(diag(H)[3])
  UC_lambda[i]<-mle_lambda[i]+1.96*sqrt(diag(H)[3])
  if((LC_lambda[i]<=lambda)& (lambda<=UC_lambda[i])){
   count_lambda=count_lambda+1
  }
}
#Calculate Mean Estimate
mean_a <- mean(mle_a)
mean_b <- mean(mle_b)
mean_lambda <- mean(mle_lambda)
print(cbind(mean_a,mean_b,mean_lambda))

#Calculate Average Bias
Bias_a<-sum(mle_a-a)/N
Bias_b<-sum(mle_b-b)/N
Bias_lambda<-sum(mle_lambda-lambda)/N
print(cbind(Bias_a,Bias_b,Bias_lambda))

#Calculate Root Mean Square Error
RSME_a<-sqrt(sum((mle_a-a)^2)/N)
RSME_b<-sqrt(sum((mle_b-b)^2)/N)
RSME_lambda<-sqrt(sum((mle_lambda-lambda)^2)/N)
print(cbind(RSME_a,RSME_b,RSME_lambda))

#Calculate Coverage Probability
CP_a<-count_a/N
CP_b<-count_b/N
CP_lambda<-count_lambda/N
print(cbind(CP_a,CP_b,CP_lambda))
}
```